Siggraph Course

Mesh Parameterization: Theory and Practice

Spherical Parameterization

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Why Sphere?

- Why?
- Global Continuity
  - No cross-boundary discontinuities

Traditional: Cut & parameterize in 2D

Alternative: Parameterize on sphere
Spherical Parameterization - Applications

- Remeshing
- Compression
- Surface fitting
- Reconstruction
- Matching

[Praun & Hoppe:03; Losaso:04]
[Zwicker & Gotman:04]
[Gu:04]
Main Approaches

- Projected Gauss-Seidel Iterations [Alexa:01; Gu04]
- Stereographic Projection [Haker:00]
- Spherical barycentric coords [Gotsman:03; Saba:05]
- Coarse-to-fine embedding [Shapiro & Tal:98; Praun & Hoppe:03]
Projected Gauss-Seidel Iterations

- Get initial embedding
- For each vertex
  - Use barycentric formulation to obtain position as function of neighbours
  - Project to sphere
Projected Gauss-Seidel Iterations

- When to stop?
  - Diverges close to solution [Saba:05, Freidel:06]

- Semi-solution 1:
  - stop before convergence
    - stop when residual starts to grow

- Semi-solution 2:
  - use stretch/area component in metric
    - becomes non-linear (Gauss-Seidel might not converge)

- No validity (no flips) guarantee
Stereographic Projection [Haker:00]

- Compute planar map
  - Use one triangle as boundary
  - Any choice of barycentric coordinates
- Project to sphere using stereographic projection
  - Conformal (preserves angles)
Stereographic Projection

- High distortion near pole
- Alternative
  - Compute planar map in two halves [Saba:05]
  - Visible seam, result depends on seam choice
- No validity guarantee (flips)
Barycentric Spherical Embedding

**[Gotsman:03]**

- **Barycentric planar embedding**
  - Embed boundary on convex polygon
  - For interior solve

\[
\begin{align*}
W_x &= b_x \\
W_y &= b_y \\
W_{ij} &= \begin{cases} 
- \sum_{j \neq i} w_{ij} & (i, j) \in E \\
- \sum_{j \neq i} w_{ij} & (i, i) \\
0 & \text{otherwise}
\end{cases}
\]

- **Theorem** [Tutte:63; Floater:01]: barycentric embedding is valid parameterization

- **Extend to sphere?**
Colin de Verdiere (CdV) Matrices

- **Given:**
  - Graph $G = <E, V>$ & family of symmetric matrices $M(G)$:
    - $M_{ij} = \begin{cases} < 0 & (i, j) \in E \\ \text{anything} & i = j \\ 0 & \text{otherwise} \end{cases}$
  - $\lambda(M) = \{\lambda_0, \ldots, \lambda_{n-1}\} = \text{spectrum}(M)$ with corresponding eigenvectors $\{\xi_0, \ldots, \xi_{n-1}\}$
  - Let $r \geq 1$ be maximal integer s. t. $\lambda_1 = \lambda_2 = \ldots = \lambda_r$ over all possible matrices in $M(G)$ & let $M_r$ be the corresponding matrix

- **Theorem [Colin de Verdiere, 90]:** $r \leq 3$ iff $G$ is 3-connected planar graph
Embedding using CdV matrices

• Theorem [Lovasz and Schrijver, 99]: If $G$ is a graph such that $r(G) = 3$, then $\xi_1, \xi_2, \xi_3$ define a valid convex embedding of $G$ in $\mathbb{R}^3$

• Why is this any good?

• Valid convex embedding can be converted to valid spherical embedding by projection

• How to find $M_r$ and $\xi_1, \xi_2, \xi_3$?
Answer

• Format of $M$

$$M_{ij} = \begin{cases} 
< 0 & (i, j) \in E \\
\text{anything} & i = j \\
0 & \text{otherwise}
\end{cases}$$

• Given edge weights $M_{ij} > 0$ finding CdV matrix $M$ is equivalent to solving

$$\begin{align*}
    x_i^2 + y_i^2 + z_i^2 &= 1 & i = 1, \ldots, n \\
    M_{ii}x_i - (Mx)_i &= 0 & i = 1, \ldots, n \\
    M_{ii}y_i - (My)_i &= 0 & i = 1, \ldots, n \\
    M_{ii}z_i - (Mz)_i &= 0 & i = 1, \ldots, n
\end{align*}$$

for $4n$ variables $x_i, y_i, z_i, M_{ii}$
Examples

Tutte

Conformal
Issues

• Guarantee validity.... almost
  – Can have multiple wraps
    • Linked to eigenvector order
    • Resolve by “good” initial guess [Saba:05]
• Not that fast....
  – Speed up [Saba:05]
Conformal Embedding

- Preserve angles but can drastically distort area
- Extra degree of freedom (Mobius transform)
  - Find area optimizing transform [Gu:04]
Coarse to fine Embedding
[Shapiro & Tal:98; Praun & Hoppe:03]

- Simplify model till convex & trivially embed
  - can be embedded on sphere with no flips
**Coarse to fine Embedding**

[Shapiro & Tal:98; Praun & Hoppe:03]

- Introduce vertices back one-by-one
  - Find optimal location
  - Perform *local* smoothing [Praun & Hoppe:03]

- Works well in practice
  - Validity guaranteed
  - Fairly time consuming to get good result
Examples

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Summary

• Gauss-Seidel/Stereo
  – simple & efficient
  – no validity guarantees

• Barycentric
  – Elegant math & validity
  – Efficiency requires complex solver

• Coarse-to-fine
  – valid
  – distortion/speed trade-off

• No free lunch
External Material