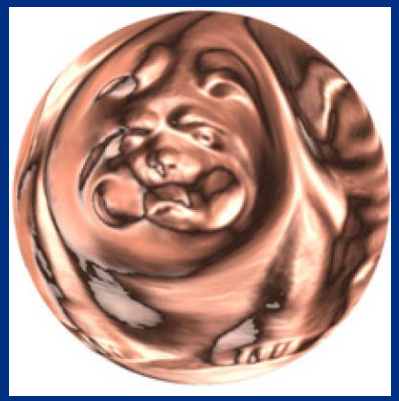
Siggraph Course

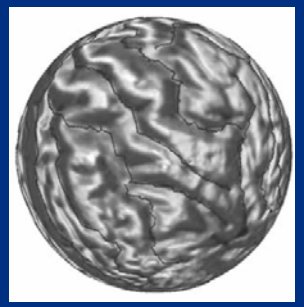
Mesh Parameterization: Theory and Practice

SIGGRAPH2007

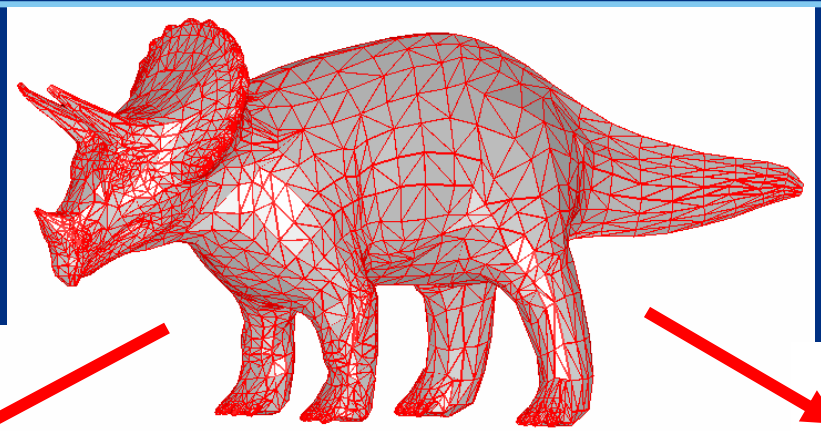
Spherical Parameterization

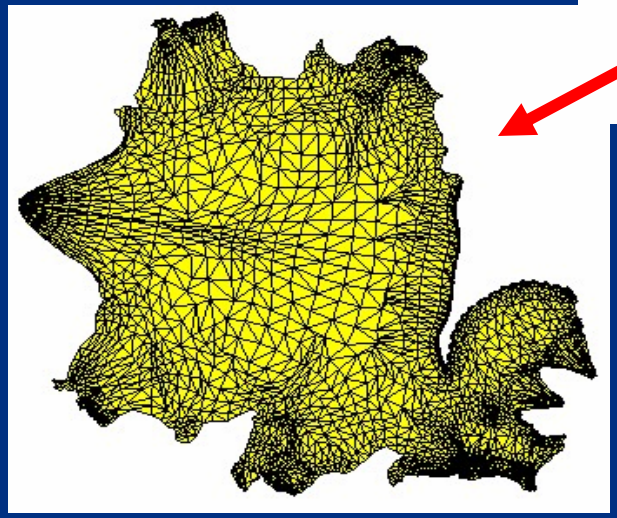
Alla Sheffer





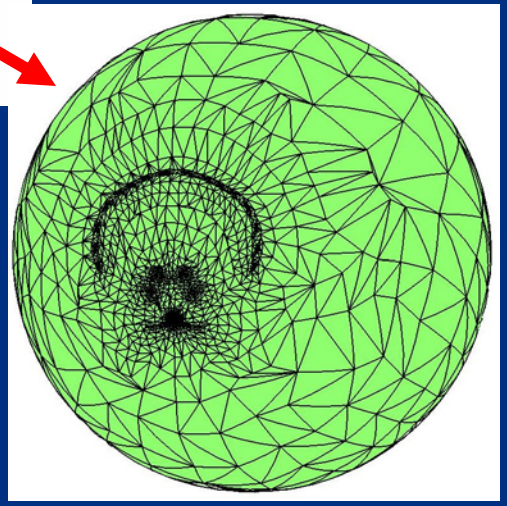
Why Sphere?





Traditional:
Cut & parameterize in 2D

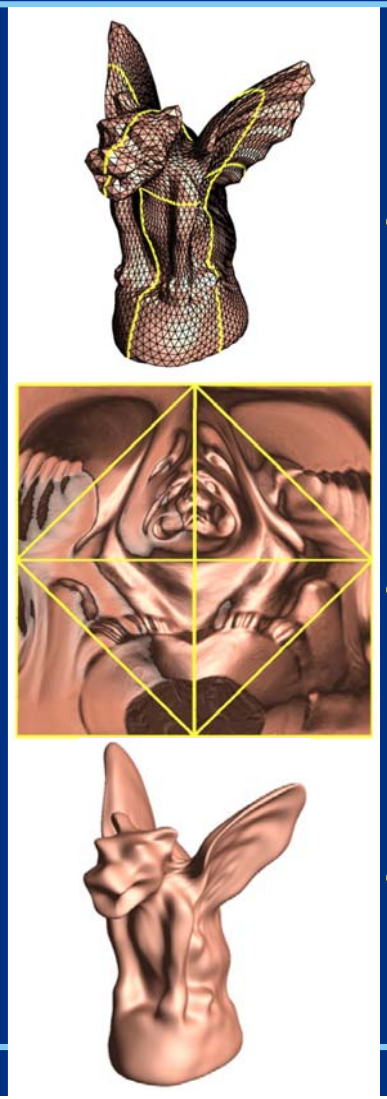
- Why?
- Global Continuity
  - No cross-boundary discontinuities



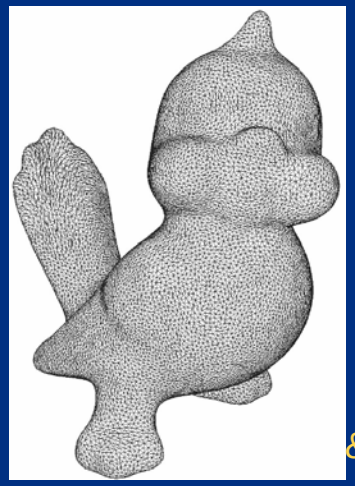
Alternative:
Parameterize on sphere

Spherical Parameterization - Applications

[Praun & Hoppe:03; Losaso:04]

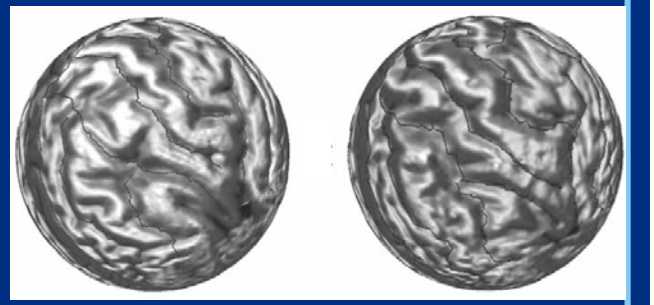


- Remeshing



[Zwicker & Gotman:04]

- Compression
- Reconstruction
- Surface fitting

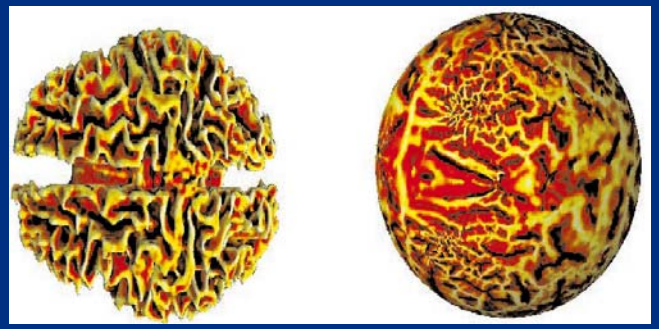


- Matching [Gu:04]

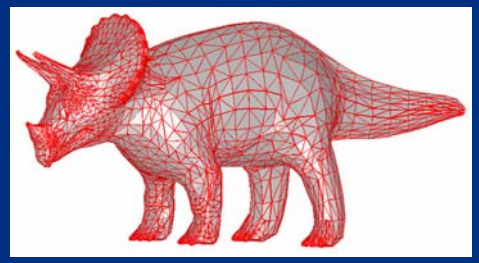
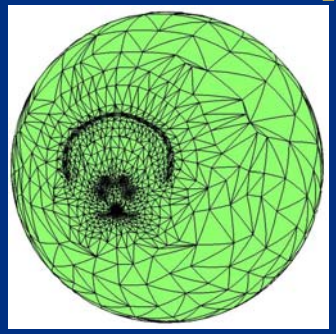
Main Approaches

- Projected Gauss-Seidel Iterations [Alexa:01;Gu04]


- Stereographic Projection [Haker:00]



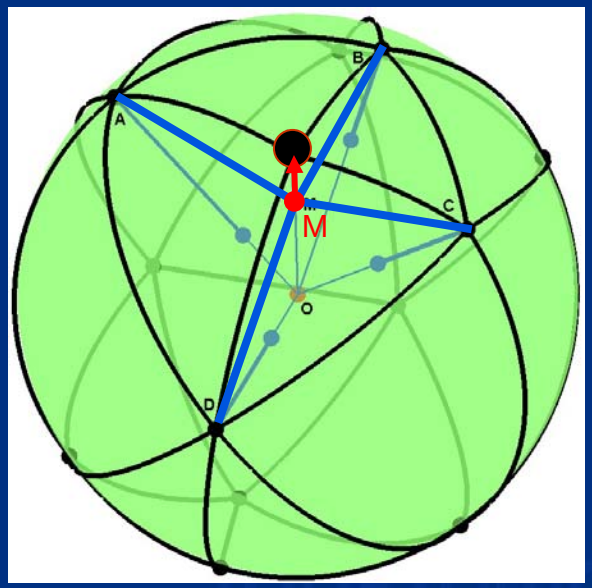
- Spherical barycentric coords [Gotsman:03; Saba:05]

- Coarse-to-fine embedding [Shapiro & Tal:98;Praun & Hoppe:03]


Projected Gauss-Seidel Iterations

- Get initial embedding
- For each vertex
  - Use barycentric formulation to obtain position as function of neighbours
  - Project to sphere

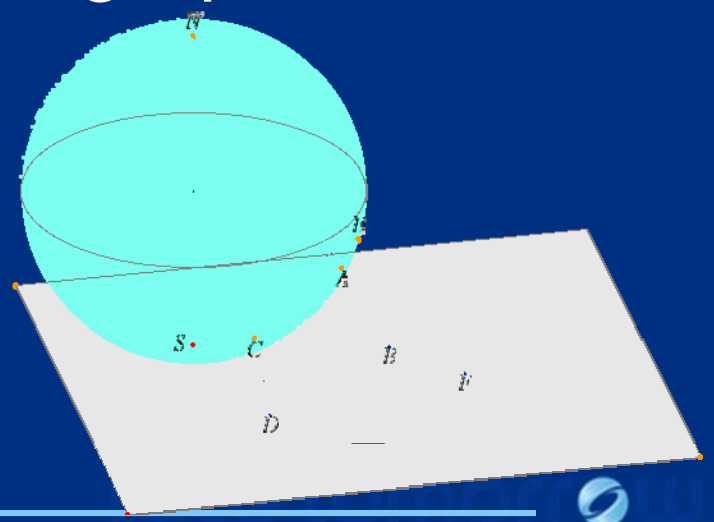


Projected Gauss-Seidel Iterations

- When to stop?
  - Diverges close to solution [Saba:05, Freidel:06]
- Semi-solution 1:
  - stop before convergence
    - stop when residual starts to grow
- Semi-solution 2:
  - use stretch/area component in metric
    - becomes non-linear (Gauss-Seidel might not converge)
- No validity (no flips) guarantee

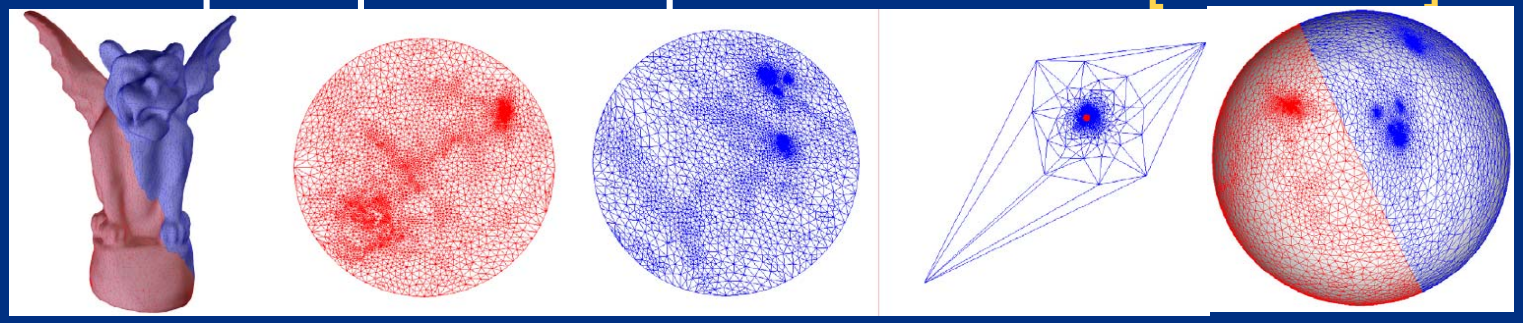
Stereographic Projection [Haker:00]

- Compute planar map
  - Use one triangle as boundary
  - Any choice of barycentric coordinates
- Project to sphere using stereographic projection
  - Conformal (preserves angles)



Stereographic Projection

- High distortion near pole
- Alternative
  - Compute planar map in two haves [Saba:05]



  - Visible seam, result depends on seam choice
- No validity guarantee (flips)

Barycentric Spherical Embedding
[Gotsman:03]

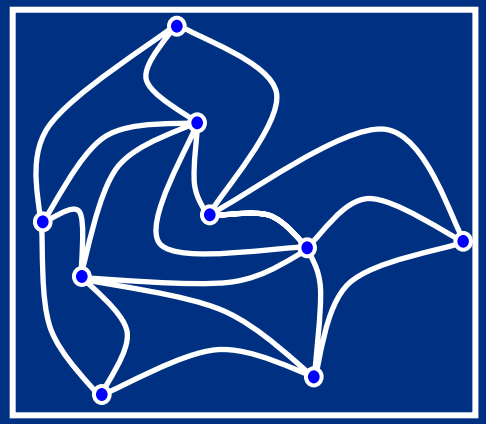
- Barycentric planar embedding
  - Embed boundary on convex polygon
  - For interior solve

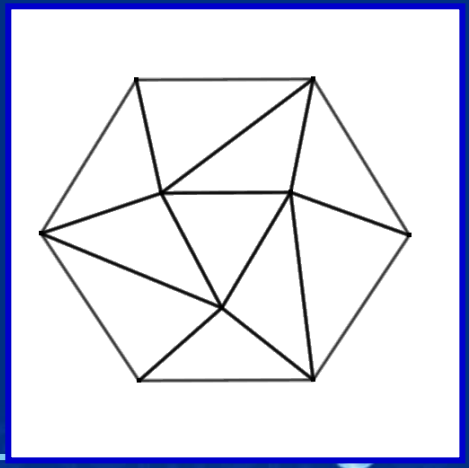
$$Wx = b_x \qquad Wy = b_y$$

$$w_{ij} = \begin{cases} < 0 & (i,j) \in E \\ -\sum_{j \neq i} w_{ij} & (i,i) \\ 0 & otherwise \end{cases}$$

$$W \text{ is } symmetric \ : w_{ij} = w_{ji}$$

  - Theorem [Tutte:63;Floater:01]: barycentric embedding is valid parameterization
- Extend to sphere?





Colin de Verdiere (CdV) Matrices

- Given:
  - Graph $G=<E,V>$ & family of symmetric matrices $M(G)$:

$$M_{ij} = \begin{cases} < 0 & (i,j) \in E \\ \textbf{anything} & i = j \\ 0 & otherwise \end{cases}$$

  - $\lambda(M) = \{\lambda_0, ..., \lambda_{n-1}\} = \text{spectrum}(M)$ with corresponding eigenvectors $\{\xi_0, .., \xi_{n-1}\}$
  - Let $r \geq 1$ be maximal integer s. t. $\lambda_1 = \lambda_2 = ... = \lambda_r$ over all possible matrices in $M(G)$ & let M_r be the corresponding matrix
- **Theorem** [Colin de Verdiere,90]: $r \leq 3$ **iff** G is 3-connected planar graph

Embedding using CdV matrices

- Theorem [Lovasz and Schrijver,99]: If G is a graph such that $r(G) = 3$, then ξ_1, ξ_2, ξ_3 define a valid convex embedding of G in R^3
- Why is this any good?
- Valid convex embedding can be converted to valid spherical embedding by projection
- How to find M_r and ξ_1, ξ_2, ξ_3 ?

Answer

- Format of M

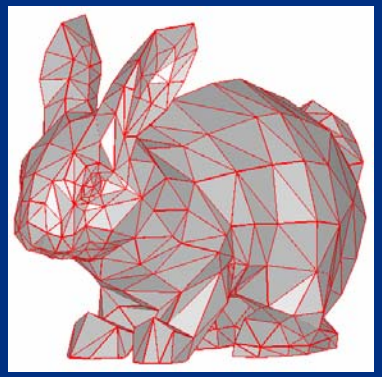
$$M_{ij} = \begin{cases} < 0 & (i,j) \in E \\ \textbf{anything} & i = j \\ 0 & otherwise \end{cases}$$

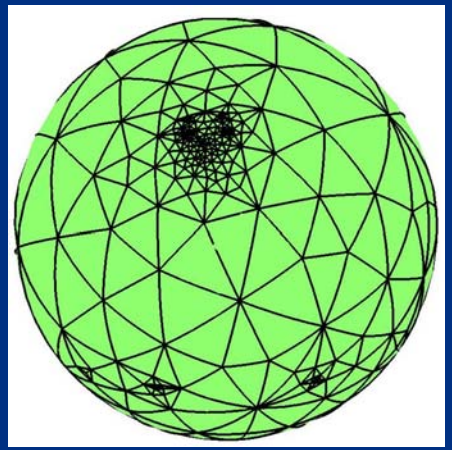
- Given edge weights $M_{ij}>0$ finding CdV matrix M is equivalent to solving

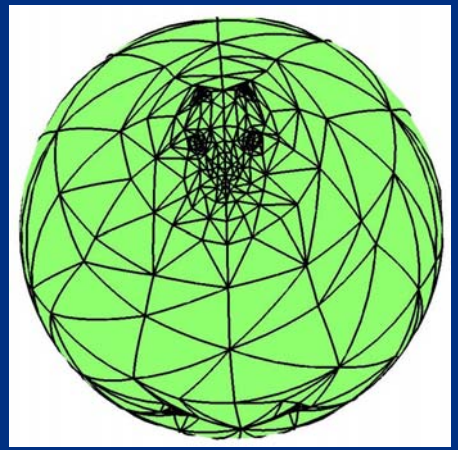
$$\begin{aligned} x_i^2 + y_i^2 + z_i^2 &= 1 & i &= 1,..,n \\ M_{ii} x_i - (Mx)_i &= 0 & i &= 1,..,n \\ M_{ii} y_i - (My)_i &= 0 & i &= 1,..,n \\ M_{ii} z_i - (Mz)_i &= 0 & i &= 1,..,n \end{aligned}$$

for $4n$ variables x_i, y_i, z_i, M_{ii}

Examples

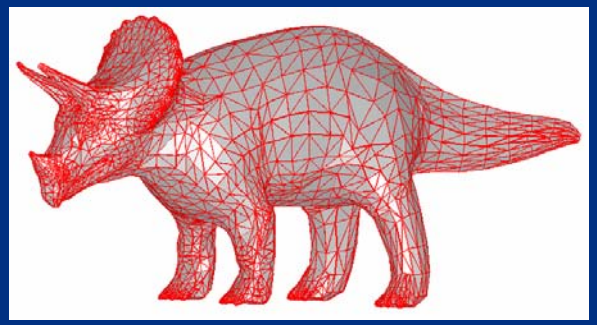


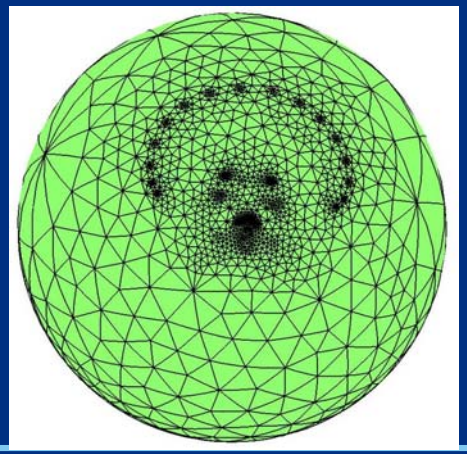


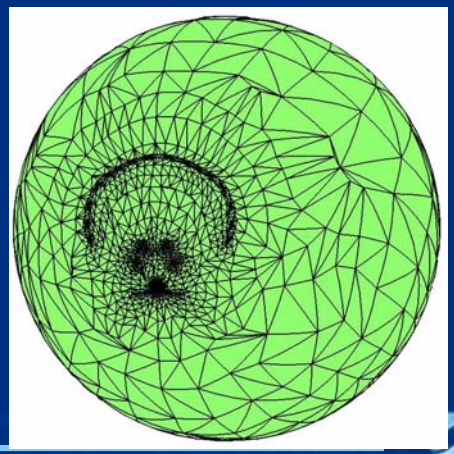


Tutte

Conformal





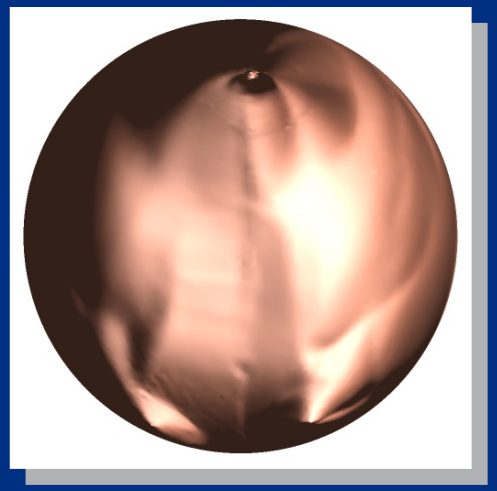


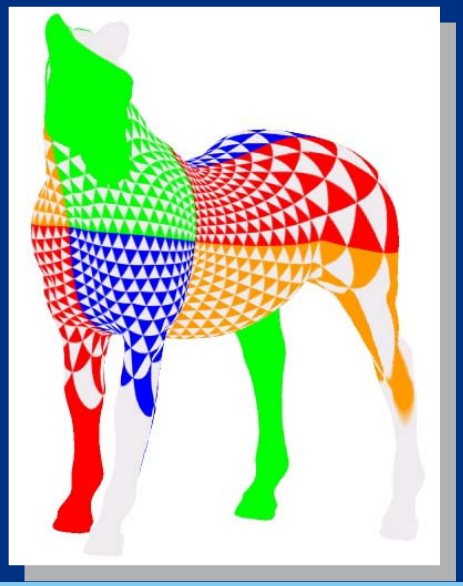
Issues

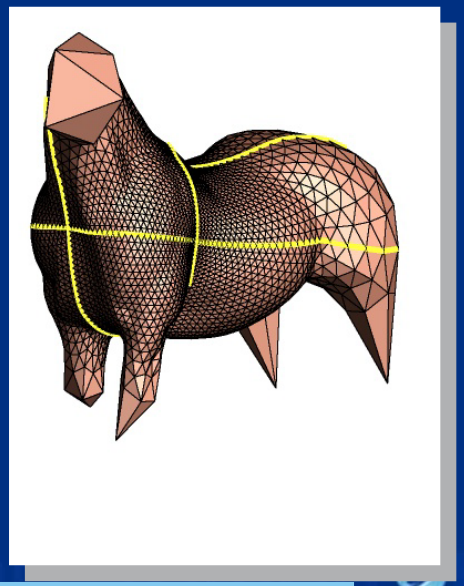
- Guarantee validity.... almost
  - Can have multiple wraps
    - Linked to eigenvector order
    - Resolve by “good” initial guess [Saba:05]
- Not that fast....
  - Speed up [Saba:05]

Conformal Embedding

- Preserve angles but can drastically distort area
- Extra degree of freedom (Mobius transform)
  - Find area optimizing transform [Gu:04]

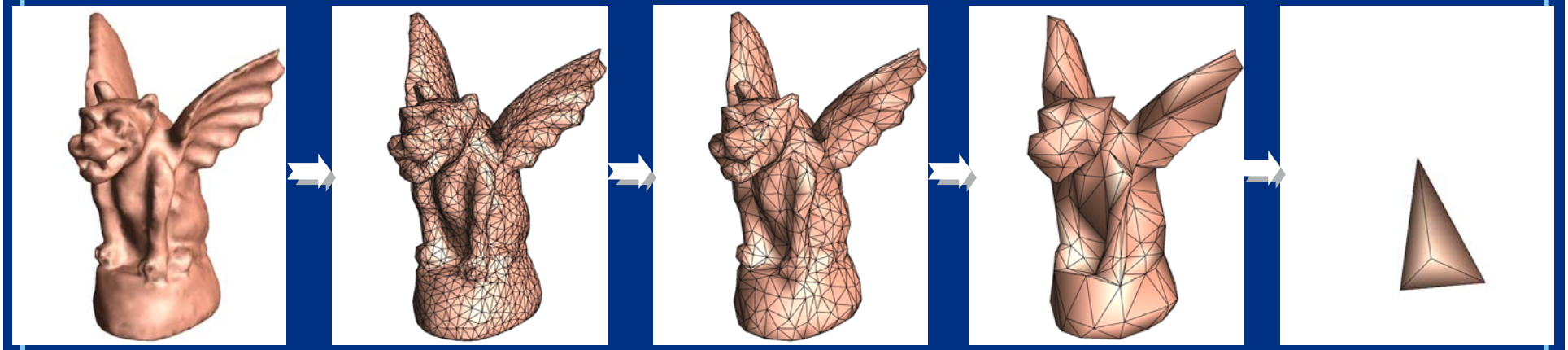






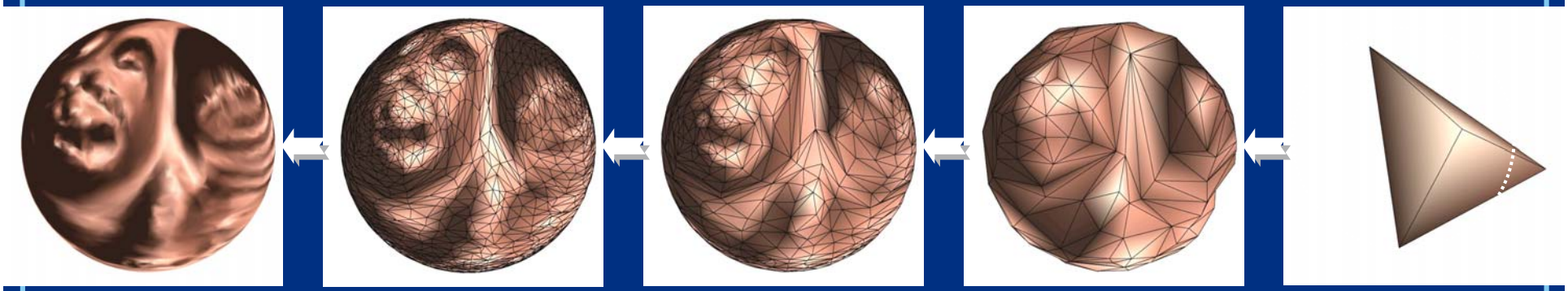
Coarse to fine Embedding
[Shapiro & Tal:98; Praun & Hoppe:03]

- Simplify model till convex & trivially embed
  - can be embedded on sphere with no flips



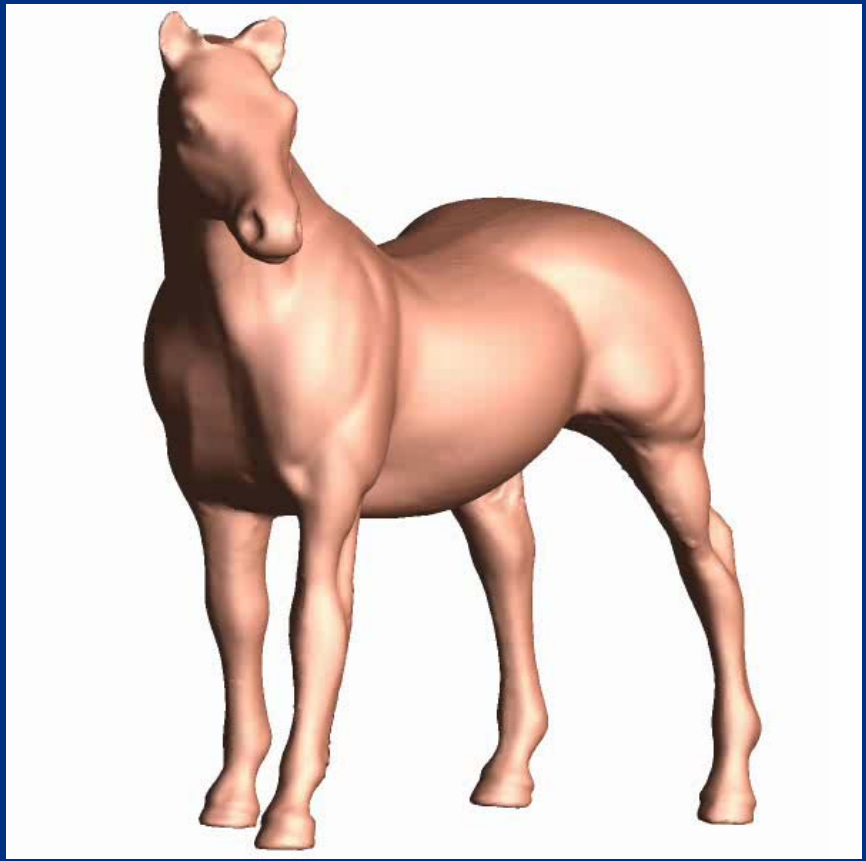
Coarse to fine Embedding
[Shapiro & Tal:98; Praun & Hoppe:03]

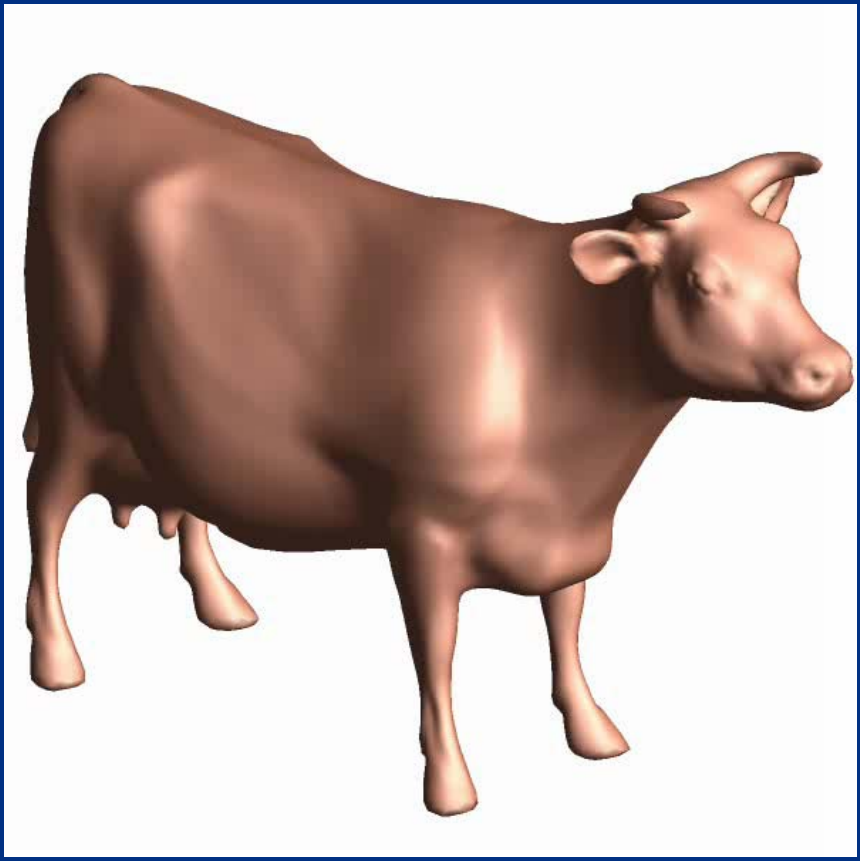
- Introduce vertices back one-by-one
  - Find optimal location
  - Perform *local* smoothing [Praun & Hoppe:03]



- Works well in practice
  - Validity guaranteed
  - Fairly time consuming to get good result

Examples





Summary

- Gauss-Seidel/Stereo
  - simple & efficient
  - no validity guarantees
- Barycentric
  - Elegant math & validity
  - Efficiency requires complex solver
- Coarse-to-fine
  - valid
  - distortion/speed trade-off

- No free lunch

External Material

- Gotsman, C., Gu, X., and Sheffer, A. 2003. Fundamentals of spherical parameterization for 3D meshes. SIGGRAPH 2003, 358-363
- X. Gu, Y. Wang, T. F. Chan, P. M. Thompson and S-T. Yau ,Genus Zero Surface Conformal Mapping and Its Application to Brain Surface Mapping, IEEE Transaction on Medical Imaging, 23(8), 2004, pp. 949-958.
- S. Haker, S. Angenent, A. Tannenbaum, R. Kikinis, G. Sapiro, M. Halle: Conformal Surface Parameterization for Texture Mapping. IEEE Trans. Vis. Comput. Graph. 6(2): 181-189 (2000)
- E. Praun, H. Hoppe, Spherical parametrization and remeshing, SIGGRAPH 2003, 340-349.