Siggraph Course

Mesh Parameterization – Theory and Practice

SIGGRAPH2007

Discrete Exterior Calculus in a nutshell

Background

We've seen plenty of parameterization methods

- planar, spherical, and base mesh domains
- sometimes linear, sometimes non-linear
- using barycentric coordinates and/or differential geometry

A number of unattended issues...

- numerical issues
  - condition number, (a)symmetry of linear systems
- from differential to discrete (and vice-versa)
  - why so many cotangents?
  - can we remove the guesswork of "discretization"?

Talk Overview

Mathematical Framework for Parameterization...

- Discrete Exterior Calculus
  - discrete forms and associated calculus
  - just numbers on mesh elmts and simple operations
- explaining cotangents and other weird formulas
- generalization of traditional finite elements
  - turning a mesh into a computational machine

...and Beyond!

- from geometry processing
- to medical applications
- and even simulation
  - » fluids, elasticity, E&M

Quick Refresher Preamble

Notion of k-simplex

- non-degenerate c... f (k+1) vertices

0-simplex | 1-si... | 3-simplex

- boundary of s...

$\partial \bullet = \emptyset$ $\quad \partial$ / = ... -1 $\quad \partial$ ▲ =

- simplicial com...
  - your typical ... esh

two k-simplices ... a common (k-1)-face

Handling Discrete Geometry

Discrete, yet **differential** quantities:

- they "live" at special places, as *distributions*
  - e.g., Gaussian curvature at vertices ONLY
  - mean curvature at edges ONLY
- they can be handled through integration
  - integration calls for **k-forms** (antisymmetric tensors)
    - objects that beg to be integrated (ex: $\int f(x)\,dx$) — that's a 1-form
  - k-forms are evaluated on k-cells (kD set)
    - point values, line integrals, surface integrals, ...
  - well studied in math, rarely used for computations

Exterior Calculus of Forms

Foundation of calculus on smooth manifolds

- Historically, purpose was to extend div, curl, grad
  - Poincaré, Cartan, Lie, ...
- Basis of differential and integral calculation
  - untangles topological from geometrical structures
  - Helmholtz-Hodge decomposition
    - "fundamental theorem of vector calculus"
  - basis of modern differential geometry
- A hierarchy of basic operators are defined:
  - d, $\star$, $\wedge$, $\flat$, $\sharp$, i_X, $\mathcal{L}_X$
    - » $\nabla f = df^\sharp$, $\Delta = d\star d\star + \star d\star d$,...

See [Abraham, Marsden, Ratiu], ch. 6-7

Discrete Exterior Calculus?

Foundation of computations on meshes

- Basic discrete operators
  - consistently derived, easily computed, noise resilient
  - mostly linear algebra!
- Untangling topology from geometry
  - some things do need a metric; some others don't
  - conservation laws can be preserved *exactly*
- Preserving structures at the discrete level
  - invariants preserved—just like in programming
- Based on coordinate-free measurements
  - fully "intrinsic" representation

Notion of Discrete Forms

Values on simplices $\int_\sigma \omega$

- "measurements" on the mesh
  - "sample" (i.e., integrate) a k-form on k-cells
  - point sample, line integral, flux, density
- invariant to coordinate changes

Math Lingo: Cochains and Chains

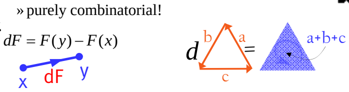
- **chains**: linear combination of simplices
  - discrete k-dimensional sets
- **cochains**: dual notion, i.e., discrete forms!
  - linear mapping from k-dimensional sets to ℝ
  - in CS terms: k-form = array of values on k-cells

Discrete Operators on Forms

Discrete Differential d

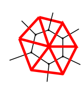
- defined through Stokes' Theorem: $\int_\sigma d\omega = \int_{\partial\sigma} \omega$
  - d is the "dual" of ∂
    - » purely combinatorial!

$\int_x^y dF = F(y) - F(x)$



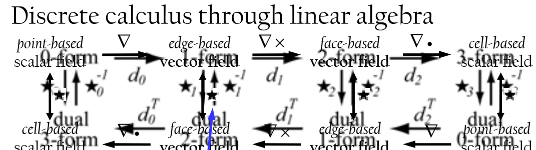
Discrete Hodge Star ⋆

- brings primal values to the dual mesh
- as simple as local rescaling
  - depends on metric (length, area, ...)
  - good enough approximation



Structure-preserving Calculus

Discrete calculus through linear algebra



- simple exercise in matrix assembly
  - cotangent weights in 2D

Number of variants exist:

- FEEC, mimetic methods, discrete Maxwell's house, ...
  - Arnold, Bossavit, Hiptmair, Nicolaides, Bochev, etc...

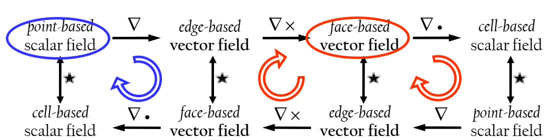
Laplacian Operator

Laplacian needed über alles

- general expression:

$$\Delta = d \star d \star + \star d \star d$$

Try it for 2D 0-forms at home: you'll get the **cot formula**...



Let's Put It to Good Use

Harmonicity for PWL surfaces via Dirichlet

- discrete Dirichlet energy:

$$E_D = \|du\|^2 = \langle du, du \rangle$$
$$= \int_\Omega du \wedge \star du = \sum_i (\cot\alpha_{ij} + \cot\beta_{ij})(u_i - u_j)^2$$

- critical point satisfies:

$$\Delta u = 0 \Leftrightarrow \sum_i (\cot\alpha_{ij} + \cot\beta_{ij})(u_i - u_j) = 0$$

- "preserves angles" in the limit
- anisotropic Laplacian?
  - just change the Hodge star!