Shape control tools for periodic Bézier curves

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Abstract

Bézier curves are an essential tool for curve design. Due to their properties, common operations such as translation, rotation, or scaling can be applied to the curve by simply modifying the control polygon of the curve. More flexibility, and thus more diverse types of curves, can be achieved by associating a weight with each control point, that is, by considering rational Bézier curves. As shown by Ramanantoanina and Hormann [7], additional and more direct control over the curve shape can be achieved by exploiting the correspondence between the rational Bézier and the interpolating barycentric form and by exploring the effect of changing the degrees of freedom of the latter (interpolation points, weights, and nodes). In this paper, we explore similar editing possibilities for closed curves, in particular for the rational extension of the periodic Bézier curves that were introduced by Sánchez-Reyes [9]. We show how to convert back and forth between the periodic rational Bézier and the interpolating trigonometric barycentric form, derive a necessary condition to avoid poles of a trigonometric rational interpolant, and devise a general framework to perform degree elevation of periodic rational Bézier curves. We further discuss the editing possibilities given by the trigonometric barycentric form.

1 Introduction

We recall that a Bézier curve $P : [0,1] \to \mathbb{R}^2$, defined by the control points $P_0, \dots, P_n \in \mathbb{R}^2$, is given by

$$P(t) = \sum_{i=0}^{n} B_i^n(t) P_i,$$

where $B_i^n(t) = \binom{n}{i}(1-t)^{n-i} t^i$ denotes the i-th Bernstein polynomial of degree n. The only shape controls induced by this formula are the control points P_i, which can be used to intuitively modify the shape of the curve P. By associating a scalar $\alpha_i \in \mathbb{R}$ with each P_i, we obtain a rational Bézier curve

$$P(t) = \frac{\sum_{i=0}^{n} B_i^n(t) \alpha_i P_i}{\sum_{i=0}^{n} B_i^n(t) \alpha_i}. \tag{1}$$

The additional degrees of freedom $\alpha_0, \dots, \alpha_n$ can be used to induce a push or pull effect towards the control polygon. New shape controls for rational Bézier curves can be achieved by expressing the curve in barycentric form. We recall that a barycentric rational curve $Q : [0,1] \to \mathbb{R}^2$, defined by the distinct nodes $t_0, \dots, t_n \in [0,1]$, the interpolation points $Q_0, \dots, Q_n \in \mathbb{R}^2$, and the non-zero weights $\beta_0, \dots, \beta_n \in \mathbb{R}$, is given by

$$Q(t) = \frac{\sum_{i=0}^{n} \frac{(-1)^i}{t-t_i} \beta_i Q_i}{\sum_{i=0}^{n} \frac{(-1)^i}{t-t_i} \beta_i}. \tag{2}$$

The possibility of converting back and forth between the rational Bézier form (1) and the barycentric form (2) is described by Ramanantoanina and Hormann [7]. Due to the interpolation property $Q(t_i) = Q_i$ for any $i = 0, \dots, n$, the barycentric form offers a more direct control over the curve and can be used to force the curve to pass through specific points, which is much harder to achieve using the rational Bézier form.

Ramanantoanina and Hormann [7] also analyse the effect of changing the other parameters of the barycentric form, namely the nodes t_i and the weights β_i. In particular, they describe how to slide a point Q_k along the curve by modifying the corresponding node t_k and suitably adjusting the weights without changing the shape of the curve, and how to control the flatness of the curve at Q_k by modifying β_k. To this end, they derive the formula for the tangent vector $Q'(t_k)$ from the formula of the derivatives of a barycentric rational interpolant [10] and show that only the length of this vector depends on β_k, but not its direction. Geometrically, this means that decreasing β_k flattens a curve locally around Q_k, while increasing β_k forces
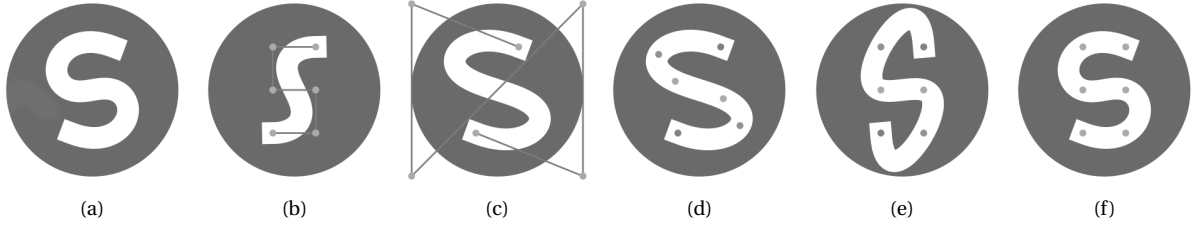
Figure 1: In order to create the curve in (a), a designer would normally use the control points and weights of a rational Bézier curve to control the shape (b,c), which is intuitive, but requires skill and experience to exactly reproduce the shape in (a). Instead, after converting the Bézier curve in (c) to the interpolating barycentric form (d), it is possible to edit the curve by dragging the interpolation points and forcing the curve to pass through a certain set of key points on the desired shape (e) and then adjusting the shape by modifying the flatness at these points (f).
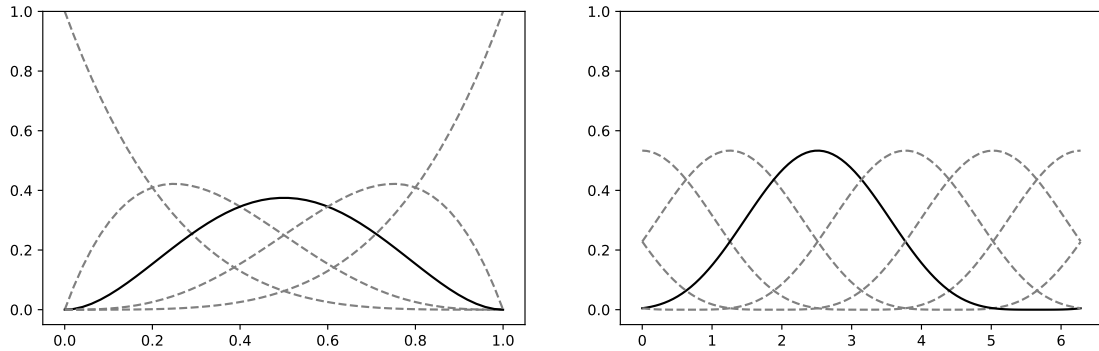
Figure 2: The shape of the Bernstein (left) and the periodic basis functions (right) for $n = 4$.

the curve to bend more tightly at Q_k. They further describe how to insert a new interpolation point without changing the shape of the curve, which effectively raises the degree of the curve from n to $n+1$. Figure 1 showcases the advantages of editing a curve using the barycentric form.

While the aforementioned constructions are only for open curves, the goal of this paper is to derive similar shape control tools for closed curves. One option would be to close the rational Bézier curve in (1) by aligning the end points, that is, by letting $P_0 = P_n$, resulting in C^0-continuity, and higher orders of continuity can be achieved by further constraining control points and weights. For example, C^1-continuity is ensured if $\alpha_1 \alpha_n (P_1 - P_0) = \alpha_0 \alpha_{n-1} (P_n - P_{n-1})$. Another option would be to consider Bézier splines or B-splines, but all these approaches result in curves with a finite order of differentiability.

A third option is to consider *periodic Bézier curves* [9]. These curves are closed and smooth (that is, C^∞) everywhere and inherit many important properties from classical Bézier curves. The main idea behind their construction is to replace the Bernstein polynomials by shifted versions of a periodic function that imitates the shape of a central Bernstein polynomial (see Figure 2). A periodic Bézier curve $P : [0, 2\pi] \to \mathbb{R}^2$ of degree $N \in \mathbb{N}$ with control points $P_0, \dots, P_n \in \mathbb{R}^2$, where $n = 2N$, is defined as

$$P(t) = \sum_{i=0}^{n} B_n(t - \phi_i) P_i,$$

where $B_n(t) = K_n \cos^n \frac{t}{2}$, $K_n = \frac{2^n}{n+1} \binom{n}{N}^{-1}$, and $\phi_i = \frac{2i\pi}{n+1}$ for $i = 0, \dots, n$. Since $\cos^n \frac{t}{2} = (1 + \cos t)^N / 2^N$, it is clear that this is a trigonometric polynomial curve of degree N.

1.1 Contributions

We extend the construction of periodic Bézier curves in the same way that Bézier curves are extended to rational Bézier curves. Namely, we introduce a weight α_i for each control point P_i of a periodic Bézier curve and then project the spatial periodic Bézier curve with control points $(\alpha_i P_i, \alpha_i) \in \mathbb{R}^3$ centrally into the $z = 1$ plane. Hence, a *periodic rational Bézier curve* with control points $P_0, \dots, P_n \in \mathbb{R}^2$ and weights $\alpha_0, \dots, \alpha_n \in \mathbb{R}$ is

given by

$$P(t) = \frac{\sum_{i=0}^{n} B_n(t - \phi_i) \alpha_i P_i}{\sum_{i=0}^{n} B_n(t - \phi_i) \alpha_i}. \tag{3}$$

Notice that $P(t)$ reduces to a trigonometric polynomial curve if all weights α_i are equal.

Recall that our goal is to explore the possibility of converting the periodic rational Bézier form in (3) to an interpolating barycentric form and to investigate the shape control possibilities that the latter offers. In the periodic setting, the equivalent of the barycentric rational curve in (2) turns out to be the *trigonometric barycentric rational curve* $Q: [0, 2\pi] \to \mathbb{R}^2$, defined by the distinct nodes $t_0, \ldots, t_n \in [0, 2\pi)$, the interpolation points $Q_0, \ldots, Q_n$, and the non-zero weights $\beta_0, \ldots, \beta_n \in \mathbb{R}$, and given by

$$Q(t) = \frac{\sum_{i=0}^{n} (-1)^i \csc \frac{t-t_i}{2} \beta_i Q_i}{\sum_{i=0}^{n} (-1)^i \csc \frac{t-t_i}{2} \beta_i}, \tag{4}$$

where $\csc(t) = 1/\sin(t)$.

We first show (Section 2) how to convert back and forth between the periodic rational Bézier form (3) and the trigonometric barycentric form (4). In this context, we also develop a necessary condition on the weights β_i to assure that the curve $Q(t)$ has no poles. We then explore the shape control offered by the trigonometric barycentric form, namely, by modifying the interpolation points, the nodes, or the weights (Section 3). Finally, we present three algorithms for performing degree elevation of a periodic rational Bézier curve (Section 4), using either the discrete Fourier transform as in [9] or a direct conversion of the old to the new control points, or by inserting two new interpolation points into the trigonometric barycentric form such that the shape of the curve remains the same.

2 Equivalence of periodic rational Bézier and trigonometric barycentric form

Let us first recall how to convert a trigonometric polynomial into barycentric form [8, 4, 1]. Consider the trigonometric polynomial $p(t) = a_0 + \sum_{k=1}^{N} (a_k \cos(kt) + b_k \sin(kt))$ of degree N for certain $2N+1$ coefficients $a_0, a_1 \ldots, a_N, b_1, \ldots, b_N$ and $2N+1$ distinct nodes $t_0, \ldots, t_n \in [0, 2\pi)$, where $n = 2N$. It is well known that p can be expressed in terms of the values $p_i = p(t_i)$, $i = 0, \ldots, n$ as

$$p(t) = \sum_{i=0}^{n} \prod_{j=0, j \neq i}^{n} \frac{\sin \frac{t-t_j}{2}}{\sin \frac{t_i-t_j}{2}} p_i,$$

which is commonly known as *Gauss's formula for trigonometric interpolation* [8] and akin to the Lagrange form for classical polynomials. Factoring out the product $\ell(t) = \prod_{j=0}^{n} \sin \frac{t-t_j}{2}$, we get the *first trigonometric barycentric form*

$$p(t) = \ell(t) \sum_{i=0}^{n} W_i \csc \frac{t-t_i}{2} p_i \tag{5}$$

of p, where

$$W_i = \prod_{j=0, j \neq i}^{n} \csc \frac{t_i - t_j}{2}, \qquad i = 0, \ldots, n. \tag{6}$$

Note that in the special case of equidistant nodes $t_i = \frac{2i\pi}{n+1}$, $i = 0, \ldots, n$, Berrut [1] shows that the weights are simply $W_i = (-1)^i W_0$. We can further write p in *second trigonometric barycentric form* after dividing (5) by the constant function 1, expressed in first trigonometric barycentric form as $1 = \ell(t) \sum_{i=0}^{n} W_i \csc \frac{t-t_i}{2}$, and cancelling the factor $\ell(t)$,

$$p(t) = \frac{\sum_{i=0}^{n} W_i \csc \frac{t-t_i}{2} p_i}{\sum_{i=0}^{n} W_i \csc \frac{t-t_i}{2}}.$$

In particular, we can use the first trigonometric barycentric form to convert a periodic rational Bézier curve into trigonometric barycentric form.

Proposition 1. *For any sequence of nodes $0 \le t_0 < \cdots < t_n < 2\pi$, we can express the periodic rational Bézier curve $P(t)$ in (3) in trigonometric barycentric form (4) with $Q_i = P(t_i)$ and $\beta_i = (-1)^i W_i z(t_i)$, where $z(t) = \sum_{i=0}^{n} B_n(t - \phi_i) \alpha_i$ is the denominator of $P(t)$.*

Proof. We first express the denominator $z(t)$ and the numerator of $P(t)$ in first trigonometric barycentric form (5) as

$$z(t) = \sum_{i=0}^{n} B_n(t - \phi_i)\alpha_i = \ell(t)\sum_{i=0}^{n} W_i \csc\frac{t - t_i}{2} z(t_i), \tag{7}$$

$$\sum_{i=0}^{n} B_n(t - \phi_i)\alpha_i P_i = z(t)P(t) = \ell(t)\sum_{i=0}^{n} W_i \csc\frac{t - t_i}{2} z(t_i)P(t_i). \tag{8}$$

We then divide (8) by (7), cancel the factor $\ell(t)$, and substitute $P(t_i) = Q_i$ and $W_i z(t_i) = (-1)^i \beta_i$ to arrive at the trigonometric barycentric form (4). $\qquad\square$

Since each of the $(n + 1)^2$ terms $B_n(t_i - \phi_j)$ can be computed in $O(\log n)$ and each weight W_i in $O(n)$ time, the overall time complexity of this conversion is $O(n^2 \log n)$. Letting $z_i = z(t_i)$, $\hat{P}_i = (\alpha_i P_i, \alpha_i)$ and $\hat{Q}_i = (z_i Q_i, z_i)$ for $i = 0, \ldots, n$, the relations (7) and (8) can be written compactly as $\hat{\boldsymbol{Q}} = \boldsymbol{B}\hat{\boldsymbol{P}}$, where

$$\boldsymbol{B} = \begin{pmatrix} B_n(t_0 - \phi_0) & \cdots & B_n(t_0 - \phi_n) \\ \vdots & \ddots & \vdots \\ B_n(t_n - \phi_0) & \cdots & B_n(t_n - \phi_n) \end{pmatrix}, \qquad \hat{\boldsymbol{P}} = \begin{pmatrix} \hat{P}_0 \\ \vdots \\ \hat{P}_n \end{pmatrix}, \qquad \hat{\boldsymbol{Q}} = \begin{pmatrix} \hat{Q}_0 \\ \vdots \\ \hat{Q}_n \end{pmatrix}.$$

This is useful for showing how to get back from the trigonometric barycentric form to the periodic rational Bézier form.

Proposition 2. *We can express the trigonometric barycentric rational curve $Q(t)$ in* (4) *as a periodic rational Bézier curve* (3) *with control points $P_i = (\hat{x}_i, \hat{y}_i)/\hat{z}_i$ and weights $\alpha_i = \hat{z}_i$, where $(\hat{x}_i, \hat{y}_i, \hat{z}_i)$ is the i-th row of $\hat{\boldsymbol{P}} = \boldsymbol{B}^{-1}\hat{\boldsymbol{Q}}$, as long as all $\hat{z}_i$ are non-zero.*

Proof. Given $Q(t)$, we first use the nodes t_i to compute the W_i as in (6), then use the weights β_i to set $z_i = (-1)^i \beta_i / W_i$, and finally define the i-th row of $\hat{\boldsymbol{Q}}$ as $(z_i Q_i, z_i)$. The statement then follows from the fact that $\hat{\boldsymbol{Q}} = \boldsymbol{B}\hat{\boldsymbol{P}}$, but it remains to show that $\boldsymbol{B}$ is invertible.

To this end, we first recall from [9] that the $n + 1$ functions $B_n(t - \phi_i)$, $i = 0, \ldots, n$ span the $(n + 1)$-dimensional space $T_N = \text{span}\{\cos(kt), \sin(kt) : k = 0, \ldots, N\}$ and are thus linearly independent. Moreover, any non-trivial linear combination of these functions is a trigonometric polynomial of order N and as such has no more than $2N = n$ zeros [6]. Consequently, the functions $B_n(t - \phi_i)$ form a Chebyshev system, which implies that $\boldsymbol{B}$ is non-singular. $\qquad\square$

The time complexity of this conversion is $O(n^3)$, since this is the time that it takes in general to solve the dense linear system $\hat{\boldsymbol{Q}} = \boldsymbol{B}\hat{\boldsymbol{P}}$. If one or more of the $\hat{z}_i$ in Proposition 2 vanish, then this means that the curve cannot be represented in periodic rational Bézier form, unless we extend the definition to allow for infinite control points, like in the case of classical rational Bézier curves [5, 3, 7]. Alternatively, we can apply degree elevation (see Section 4) to represent the curve as a periodic rational Bézier curve with higher degree.

Example 1. Let $N = 1$, so that $n = 2$, and consider the trigonometric barycentric rational curve $Q(t)$ with equidistant nodes $t_0 = 0$, $t_1 = 2\pi/3$, $t_2 = 4\pi/3$, interpolation points $Q_0 = (2, 0)$, $Q_1 = (-1, 1)$, $Q_2 = (-1, -1)$, and weights $\beta_0 = 2/5$, $\beta_1 = \beta_2 = 1$, which turns out to be a non-uniformly parameterized circle with centre $(1/3, 0)$ and radius $5/3$ (see Figure 3.a). Computing $\hat{\boldsymbol{P}}$ as in Proposition 2, we find that $\hat{z}_0 = 0$ and that Q is a periodic rational Bézier curve with normal control points $P_1 = (-4/3, 5/3)$, $P_2 = (-4/3, -5/3)$ and weights $\alpha_1 = \alpha_2 = 9/10$ and an infinite control point P_0 in the direction $(1, 0)$ with homogeneous coordinates $(3/2, 0, 0)$ (see Figure 3.b). After increasing the degree of Q from 1 to 2 by adding two points at $t = \pi/3$ and $t = 5\pi/3$ (see Section 4), resulting in the curve $\tilde{Q}$ with nodes $\tilde{t}_0 = t_0$, $\tilde{t}_1 = \pi/3$, $\tilde{t}_2 = t_1$, $\tilde{t}_3 = t_2$, $\tilde{t}_4 = 5\pi/3$, interpolation points $\tilde{Q}_0 = Q_0$, $\tilde{Q}_1 = Q(\tilde{t}_1) = (1/3, 5/3)$, $\tilde{Q}_2 = Q_1$, $\tilde{Q}_3 = Q_2$, $\tilde{Q}_4 = Q(\tilde{t}_4) = (1/3, -5/3)$, and weights $\tilde{\beta}_0 = 8/5$, $\tilde{\beta}_1 = \tilde{\beta}_4 = 6\sqrt{3}/5$, $\tilde{\beta}_2 = \tilde{\beta}_3 = 2$ (see Figure 3.c), we can apply Proposition 2 to convert the curve into periodic rational Bézier form with control points $\tilde{P}_0 = (7, 0)$, $\tilde{P}_1 \approx (0.254, 2.680)$, $\tilde{P}_2 \approx (-1.444, 0.792)$, $\tilde{P}_3 \approx (-1.444, -0.792)$, $\tilde{P}_4 \approx (0.254, -2.680)$ and weights $\tilde{\alpha}_0 = 3/20$, $\tilde{\alpha}_1 = \tilde{\alpha}_4 \approx 0.461$, $\tilde{\alpha}_2 = \tilde{\alpha}_3 \approx 0.964$ (see Figure 3.d). Note that the control polygon is not regular, because of the non-uniform parameterization of the curve.

Because of the equivalence of the periodic rational Bézier form (3) and the trigonometric barycentric form (4), we will use $P(t)$ and $Q(t)$ interchangeably from now on to refer to the same periodic rational curve, given in either of the two forms.
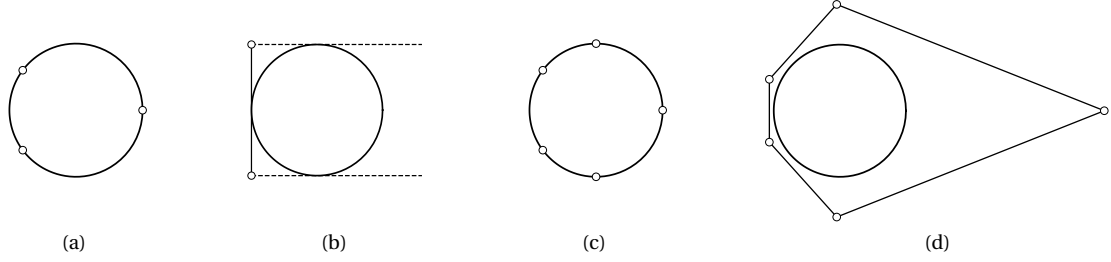
Figure 3: Converting a periodic rational curve from trigonometric barycentric form (a) to periodic rational Bézier form may require infinite control points (b), but after inserting two interpolation points (c), which raises the degree of the curve by one, the curve can be represented as a periodic rational Bézier curve with finite control points (d).

One important aspect when modelling with periodic rational curves it to ensure that they are free of poles. On the one hand, this is guaranteed for $P(t)$ if all weights α_i are positive, since this implies that the denominator $z(t)$ is positive, too, for all $t \in [0, 2\pi]$. On the other hand, it is less clear which choice of weights β_i guarantees that the denominator of $Q(t)$ does not vanish for some $t \in [0, 2\pi]$, because the functions $(-1)^i \csc \frac{t-t_i}{2}$ are neither entirely positive nor entirely negative. However, Berrut [2] shows that $Q(t)$ has no poles if all β_i are equal to some common non-zero value, and we also know that $\beta_i = (-1)^i W_i$ for $i = 0, \ldots, n$ with W_i defined as in (6) is a safe choice, because $Q(t)$ is a trigonometric polynomial curve (in second trigonometric barycentric form) in that case. Since the factors of W_i in (6) are negative for $i < j$ and positive for $i > j$, if we assume the t_i to be ordered and in $[0, 2\pi)$, and $n = 2N$ is even, it is clear that $\text{sign}(W_i) = (-1)^i$ and that the β_i are all positive in this case. This is actually not surprising, because it turns out that a common sign of the β_i is a necessary condition for the absence of poles, exactly as for classical barycentric rational interpolants [10].

Proposition 3. *If $Q(t)$ is a trigonometric barycentric rational curve* (4) *without poles, then the weights $\beta_0, \ldots, \beta_n$ have the same sign.*

Proof. If we follow Proposition 2 and convert $Q(t)$ into the spatial periodic Bézier curve $\hat{P}(t)$ with control points $\hat{P}_i$, without projecting $\hat{P}(t)$ centrally into the $z = 1$ plane to get $P(t)$, then it is clear that the z-component of $\hat{P}(t)$, which is nothing but the denominator $z(t)$ of $P(t)$, does not vanish for any $t \in [0, 2\pi]$, because $Q(t)$ is free of poles. It then follows from Proposition 1 that the $\beta_i = (-1)^i W_i z(t_i)$ are all either positive or negative, because $\text{sign}(W_i) = (-1)^i$, as pointed out above. $\qquad\square$

Since the weights α_i and β_i can always be scaled by a common non-zero factor without changing the curve, we can summarize these considerations crisply as: positive α_i are sufficient, but not necessary for the absence of poles, and positive β_i are necessary, but not sufficient.

3 Shape editing using the trigonometric barycentric form

The equivalence of the periodic rational Bézier and the trigonometric barycentric form enables new editing possibilities for periodic rational Bézier curves, beyond changing the control points P_i and the associated weights α_i, by modifying the parameters of the trigonometric barycentric form, namely the interpolation points Q_i, the nodes t_i, and the weights β_i, similar to how it can be done for classical rational Bézier curves [7].

3.1 Displacing an interpolation point

The interpolation property of the trigonometric barycentric form provides an intuitive means to modify the curve by displacing an interpolation point Q_k (see Figure 4.b). This is particularly useful for forcing the curve to pass through a specific target point, which is much harder to achieve by changing the Bézier control points P_i. However, in contrast to the basis functions $\alpha_i B_n(t - \phi_i)/\sum_{j=0}^{n} \alpha_j B_n(t - \phi_j)$ of the periodic rational Bézier form, the basis functions $(-1)^i \csc \frac{t-t_i}{2} \beta_i / \sum_{j=0}^{n} (-1)^j \csc \frac{t-t_j}{2} \beta_j$ of the trigonometric barycentric form are neither non-negative nor as nicely bell-shaped and a large displacement of some Q_k might therefore induce a less intuitive global deformation of the curve's shape. Hence, while displacing the interpolation points is good for micro-editing the shape, the Bézier control points remain the better handles for global shape design.
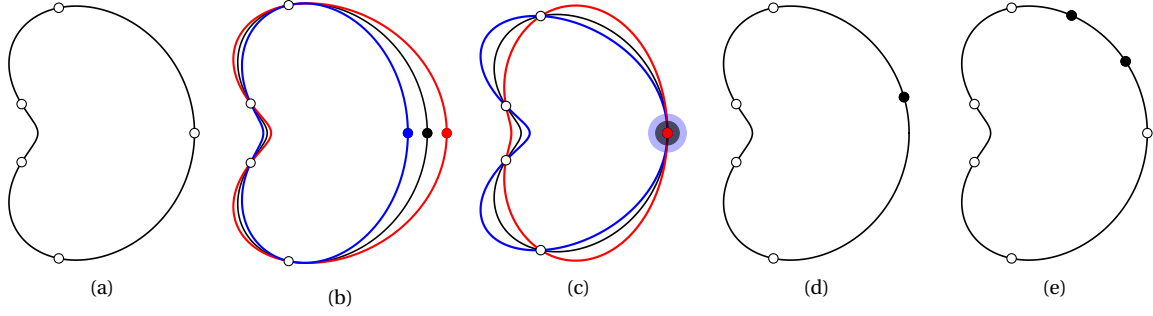
Figure 4: The shape of a periodic rational curve in trigonometric barycentric form (a) can be modified intuitively by displacing an interpolation point (b) or by changing the associated weight (c). The blue and the red curve are obtained from the black curve by moving the indicated interpolation point (b) and by increasing or decreasing its weight, visualized by proportionally sized disks (c), while leaving all other interpolation points and weights unchanged. Moreover, it is possible to slide an interpolation point (d) and to insert new interpolation points (e) without changing the shape of the curve.

3.2   Sliding an interpolation point

For the same reason as before, changing the value of a single node t_k may deform the curve in a non-intuitive way. However, it is possible to update simultaneously the interpolation point Q_k and all the weights β_i, such that changing a node t_k results in sliding Q_k along a curve. This can be achieved in two ways. On the one hand, we can use Proposition 2 to express $Q(t)$ in periodic rational Bézier form using the current sequence of nodes and then convert $P(t)$ back into trigonometric barycentric form with Proposition 1, this time with the new sequence of nodes, where t_k is replaced by some new value $\tilde{t}_k$:

$$
\begin{array}{c}
Q(t) \\
\text{convert to periodic rational Bézier form with respect to } t_0,\dots,t_k,\dots,t_n \Big\downarrow \\
P(t) \\
\text{convert to trigonometric barycentric form with respect to } t_0,\dots,\tilde{t}_k,\dots,t_n \Big\downarrow \\
\tilde{Q}(t)
\end{array}
\tag{9}
$$

On the other hand, we can also compute the parameters of the new curve $\tilde{Q}(t)$ directly.

Proposition 4. *Suppose we change the node t_k for some $k \in \{1,\dots,n-1\}$ to a new value $\tilde{t}_k \in (t_{k-1}, t_{k+1})$. The trigonometric barycentric rational curve $Q(t)$ in (4) can then be expressed in terms of the nodes $t_0,\dots,\tilde{t}_k,\dots,t_n$, the interpolation points $Q_0,\dots,\tilde{Q}_k,\dots Q_n$, and the weights $\tilde{\beta}_i$, where $\tilde{Q}_k = Q(\tilde{t}_k)$ and*

$$
\tilde{\beta}_k = \sin\frac{\tilde{t}_k - t_k}{2} \sum_{j=0}^{n} (-1)^{j+k} \csc\frac{\tilde{t}_k - t_j}{2}\beta_j, \qquad \tilde{\beta}_i = \csc\frac{t_i - \tilde{t}_k}{2}\sin\frac{t_i - t_k}{2}\beta_i, \quad i \neq k.
\tag{10}
$$

Proof. We follow the diagram in (9) and first convert $Q(t)$ to periodic rational Bézier form (3). By Proposition 1, we then know that the weights of the given curve Q can be written with respect to the given nodes $t_0,\dots,t_n$ as $\beta_i = (-1)^i W_i z(t_i)$, where $z(t)$ is the denominator of $P(t)$ and W_i is defined as in (6). It further follows from Proposition 1 that Q can be expressed with respect to the new nodes $\tilde{t}_0,\dots,\tilde{t}_n$, where $\tilde{t}_i = t_i$ for $i \neq k$, using the new interpolation points $\tilde{Q}_i = P(\tilde{t}_i)$ and the new weights $\tilde{\beta}_i = (-1)^i \tilde{W}_i z(\tilde{t}_i)$, where $\tilde{W}_i = \prod_{j=0,\, j\neq i}^{n} \csc\frac{\tilde{t}_i - \tilde{t}_j}{2}$. Clearly, $\tilde{Q}_i = P(t_i) = Q_i$ for $i \neq k$ and $\tilde{Q}_k = P(\tilde{t}_k) = Q(\tilde{t}_k)$. To prove the formulas in (10), we recall that $z(t)$ can be expressed in first trigonometric barycentric form as

$$
z(t) = \ell(t)\sum_{j=0}^{n} W_j \csc\frac{t - t_j}{2} z(t_j) = \ell(t)\sum_{j=0}^{n}(-1)^j \csc\frac{t - t_j}{2}\beta_j,
$$

6

where $\ell(t) = \prod_{j=0}^{n} \sin \frac{t-t_j}{2}$. Since $\ell(\tilde{t}_k) = \prod_{j=0}^{n} \sin \frac{\tilde{t}_k - t_j}{2} = \sin \frac{\tilde{t}_k - t_k}{2} / \tilde{W}_k$, we conclude that

$$\tilde{\beta}_k = (-1)^k \tilde{W}_k z(\tilde{t}_k) = (-1)^k \tilde{W}_k \ell(\tilde{t}_k) \sum_{j=0}^{n} (-1)^j \csc \frac{\tilde{t}_k - t_j}{2} \beta_j = \sin \frac{\tilde{t}_k - t_k}{2} \sum_{j=0}^{n} (-1)^{j+k} \csc \frac{\tilde{t}_k - t_j}{2} \beta_j.$$

For $i \neq k$, we note that

$$\tilde{W}_i = \prod_{j=0, j\neq i}^{n} \csc \frac{\tilde{t}_i - \tilde{t}_j}{2} = \csc \frac{t_i - \tilde{t}_k}{2} \prod_{j=0, j\neq i,k}^{n} \csc \frac{t_i - t_j}{2} = \csc \frac{t_i - \tilde{t}_k}{2} \sin \frac{t_i - t_k}{2} W_i,$$

hence

$$\tilde{\beta}_i = (-1)^i \tilde{W}_i z(\tilde{t}_i) = \csc \frac{t_i - \tilde{t}_k}{2} \sin \frac{t_i - t_k}{2} (-1)^i W_i z(t_i) = \csc \frac{t_i - \tilde{t}_k}{2} \sin \frac{t_i - t_k}{2} \beta_i. \qquad \square$$

Figure 4.d shows that modifying a single node t_k and updating the interpolation points and weights as described in Proposition 4 does not change the shape of the curve, but only the position of Q_k.

Remark 1. We restrict the new value $\tilde{t}_k$ to be in the interval (t_{k-1}, t_{k+1}) in Proposition 4 and exclude the cases $k = 0$ and $k = n$, so as to keep the statement and the proof simple, but we can also deal with the case $\tilde{t}_k \in (t_{l-1}, t_l)$ if $0 < l < k$ or $k+1 < l \leq n$. We just need to make sure that the indices are rearranged such that the new nodes are ordered, and this rearrangement requires to change the signs of some weights, so that they all end up having the same sign. For example, if $0 < l < k$, then we can compute the weights $\tilde{\beta}_i$ as in (10) and then define the correctly ordered nodes $\hat{t}_i$, interpolation points $\hat{Q}_i$, and weights $\hat{\beta}_i$ as

i	0	...	$l-1$	l	$l+1$	...	k	$k+1$	...	n
$\hat{t}_i$	t_0	...	t_{l-1}	$\tilde{t}_k$	t_l	...	t_{k-1}	t_{k+1}	...	t_n
$\hat{Q}_i$	Q_0	...	Q_{l-1}	$\tilde{Q}_k$	Q_l	...	Q_{k-1}	Q_{k+1}	...	Q_n
$\hat{\beta}_i$	$\tilde{\beta}_0$	...	$\tilde{\beta}_{l-1}$	$(-1)^{k+l}\tilde{\beta}_k$	$-\tilde{\beta}_l$	...	$-\tilde{\beta}_{k-1}$	$\tilde{\beta}_{k+1}$	...	$\tilde{\beta}_n$

A similar rearrangement table can be derived if $k+1 < l \leq n$ and also for the cases $\tilde{t}_k \in [0, t_0)$ and $\tilde{t}_k \in (t_n, 2\pi)$.

It is then natural to ask what happens when $\tilde{t}_k$ jumps from one interval to the neighbouring interval, for example, as $\tilde{t}_k$ transitions from (t_{k-1}, t_{k+1}) to (t_{k+1}, t_{k+2}). This can be seen as an elastic collision where Q_k slides towards Q_{k+1}, swaps the order in the moment of collision, and afterwards continues to slide as Q_{k+1}. As long as $h = t_{k+1} - \tilde{t}_k > 0$, Proposition 4 can be used to update all parameters, but it follows from (10) that $\tilde{\beta}_k$ and $\tilde{\beta}_{k+1}$ grow like $O(1/h)$ as h converges to zero. If h is very small, this may lead to numerical instabilities. However, this can be avoided in an application by limiting the maximal zoom factor and not letting the user move Q_k closer than one pixel towards Q_{k+1}. In the moment of collision, when $\tilde{t}_k = t_{k+1}$, the trigonometric barycentric form breaks down as the weights $\tilde{\beta}_k$ and $\tilde{\beta}_{k+1}$ in (10) diverge, but we can still derive a formula for the curve Q in this state.

Proposition 5. *If the node t_k for some $k \in \{0, \ldots, n-1\}$ is set to the new value $\tilde{t}_k = t_{k+1}$, resulting in a double node t_{k+1}, then we can express the trigonometric barycentric rational curve in (4) as*

$$Q(t) = \frac{\sum_{i=0,i\neq k}^{n} (-1)^i \csc \frac{t-t_i}{2} \hat{\beta}_i Q_i + (-1)^{k+1} \csc \frac{t-t_{k+1}}{2} \cot \frac{t-t_{k+1}}{2} \hat{\beta}'_{k+1} \left(Q_{k+1} + 2\tan \frac{t-t_{k+1}}{2} Q'_{k+1} \right)}{\sum_{i=0,i\neq k}^{n} (-1)^i \csc \frac{t-t_i}{2} \hat{\beta}_i + (-1)^{k+1} \csc \frac{t-t_{k+1}}{2} \cot \frac{t-t_{k+1}}{2} \hat{\beta}'_{k+1}}, \qquad (11)$$

where

$$\hat{\beta}_i = \csc \frac{t_i - t_{k+1}}{2} \sin \frac{t_i - t_k}{2} \beta_i, \quad i \neq k, k+1, \qquad (12)$$

$$\hat{\beta}_{k+1} = \sin \frac{t_{k+1} - t_k}{2} \sum_{i=0, i\neq k+1}^{n} (-1)^{k+1+i} \csc \frac{t_{k+1} - t_i}{2} \beta_i + \cos \frac{t_{k+1} - t_k}{2} \beta_{k+1}, \qquad \hat{\beta}'_{k+1} = \sin \frac{t_{k+1} - t_k}{2} \beta_{k+1},$$

and $Q'_{k+1} = Q'(t_{k+1})$.

Proof. If $\tilde{t}_k \in (t_{k-1}, t_{k+1})$, then we know from Proposition 4 that $Q(t) = N(t)/D(t)$, where

$$N(t) = \sum_{i=0, i\neq k, k+1}^{n} (-1)^i \csc\frac{t - t_i}{2}\tilde{\beta}_i Q_i + (-1)^k \csc\frac{t - \tilde{t}_k}{2}\tilde{\beta}_k \tilde{Q}_k + (-1)^{k+1}\csc\frac{t - t_{k+1}}{2}\tilde{\beta}_{k+1}Q_{k+1}$$

and

$$D(t) = \sum_{i=0, i\neq k, k+1}^{n} (-1)^i \csc\frac{t - t_i}{2}\tilde{\beta}_i + (-1)^k \csc\frac{t - \tilde{t}_k}{2}\tilde{\beta}_k + (-1)^{k+1}\csc\frac{t - t_{k+1}}{2}\tilde{\beta}_{k+1},$$

with $\tilde{Q}_k = Q(\tilde{t}_k)$ and the $\tilde{\beta}_i$ as in (10). Our task now is to find the limit of $N(t)$ and $D(t)$ as $\tilde{t}_k$ converges to t_{k+1}.

Let us first focus on the denominator $D(t)$. The sum poses no problem, because the $\tilde{\beta}_i$ for $i \neq k, k+1$ simply converge to the $\hat{\beta}_i$ in (12) as $\tilde{t}_k \to t_{k+1}$, but the remaining two terms need to be analysed more carefully, since $\tilde{\beta}_k$ and $\tilde{\beta}_{k+1}$ diverge. Using their definition in (10), we have

$$\csc\frac{t - \tilde{t}_k}{2}\tilde{\beta}_k - \csc\frac{t - t_{k+1}}{2}\tilde{\beta}_{k+1}$$

$$= \csc\frac{t - \tilde{t}_k}{2}\sin\frac{\tilde{t}_k - t_k}{2}\sum_{i=0}^{n}(-1)^{k+i}\csc\frac{\tilde{t}_k - t_i}{2}\beta_i - \csc\frac{t - t_{k+1}}{2}\csc\frac{t_{k+1} - \tilde{t}_k}{2}\sin\frac{t_{k+1} - t_k}{2}\beta_{k+1}$$

$$= \csc\frac{t - \tilde{t}_k}{2}\sin\frac{\tilde{t}_k - t_k}{2}\sum_{i=0, i\neq k+1}^{n}(-1)^{k+i}\csc\frac{\tilde{t}_k - t_i}{2}\beta_i$$

$$+ \left(\csc\frac{t - \tilde{t}_k}{2}\sin\frac{\tilde{t}_k - t_k}{2} - \csc\frac{t - t_{k+1}}{2}\sin\frac{t_{k+1} - t_k}{2}\right)\csc\frac{t_{k+1} - \tilde{t}_k}{2}\beta_{k+1}.$$

As before, the terms in the sum are uncritical, and for the last term, we can use L'Hôpital's rule to get

$$\lim_{\tilde{t}_k \to t_{k+1}}\left[\left(\csc\frac{t - \tilde{t}_k}{2}\sin\frac{\tilde{t}_k - t_k}{2} - \csc\frac{t - t_{k+1}}{2}\sin\frac{t_{k+1} - t_k}{2}\right)\csc\frac{t_{k+1} - \tilde{t}_k}{2}\right]$$

$$= -\csc\frac{t - t_{k+1}}{2}\cos\frac{t_{k+1} - t_k}{2} - \csc\frac{t - t_{k+1}}{2}\cot\frac{t - t_{k+1}}{2}\sin\frac{t_{k+1} - t_k}{2}.$$

Overall, we conclude that $D(t)$ converges to the denominator in (11) as $\tilde{t}_k \to t_{k+1}$.

Similar arguments can be used to show that $N(t)$ converges to the numerator in (11). We just need to remember that also $\tilde{Q}_k$ depends on $\tilde{t}_k$ when applying L'Hôpital's rule to

$$\lim_{\tilde{t}_k \to t_{k+1}}\left[\left(\csc\frac{t - \tilde{t}_k}{2}\sin\frac{\tilde{t}_k - t_k}{2}\tilde{Q}_k - \csc\frac{t - t_{k+1}}{2}\sin\frac{t_{k+1} - t_k}{2}Q_{k+1}\right)\csc\frac{t_{k+1} - \tilde{t}_k}{2}\right],$$

which eventually leads to the term involving Q'_{k+1}. $\qquad\square$

3.3 Changing a weight

We now investigate the effect of changing a weight β_k. In the case of barycentric rational curves, Raman-antoanina and Hormann [7] observed that only the length of the curve's tangent vector at Q_k depends on β_k, but not its direction, and we observe the same behaviour in the trigonometric setting.

Proposition 6. *For any $k \in \{0, \ldots, n\}$ the tangent vector of the trigonometric rational barycentric curve $Q(t)$ in* (4) *at Q_k is given by*

$$Q'(t_k) = \frac{1}{2\beta_k}\sum_{i=0, i\neq k}^{n}(-1)^{k+i+1}\csc\frac{t_k - t_i}{2}\beta_i(Q_k - Q_i). \tag{13}$$

Proof. Multiplying the numerator and denominator in (4) by $\sin\frac{t - t_k}{2}$, we have $Q(t) = N(t)/D(t)$, where

$$N(t) = \sum_{i=0, i\neq k}^{n}(-1)^i \sin\frac{t - t_k}{2}\csc\frac{t - t_i}{2}\beta_i Q_i + \beta_k Q_k$$

and

$$D(t) = \sum_{i=0, i\neq k}^{n}(-1)^i \sin\frac{t - t_k}{2}\csc\frac{t - t_i}{2}\beta_i + \beta_k.$$

By the product rule,

$$N'(t) = \frac{1}{2} \sum_{i=0,\,i\neq k}^{n} (-1)^i \left(\cos\frac{t-t_k}{2} - \sin\frac{t-t_k}{2} \cot\frac{t-t_i}{2} \right) \csc\frac{t-t_i}{2} \beta_i Q_i$$

and

$$D'(t) = \frac{1}{2} \sum_{i=0,\,i\neq k}^{n} (-1)^i \left(\cos\frac{t-t_k}{2} - \sin\frac{t-t_k}{2} \cot\frac{t-t_i}{2} \right) \csc\frac{t-t_i}{2} \beta_i.$$

For $t = t_k$, these expressions simplify to

$$N(t_k) = (-1)^k \beta_k Q_k, \qquad\qquad N'(t_k) = \frac{1}{2} \sum_{i=0,\,i\neq k}^{n} (-1)^i \csc\frac{t_k-t_i}{2} \beta_i Q_i,$$

$$D(t_k) = (-1)^k \beta_k, \qquad\qquad D'(t_k) = \frac{1}{2} \sum_{i=0,\,i\neq k}^{n} (-1)^i \csc\frac{t_k-t_i}{2} \beta_i.$$

The statement then follows from the quotient rule, which asserts that $Q'(t_k) = \frac{N'(t_k)D(t_k)-N(t_k)D'(t_k)}{D(t_k)^2}$. $\qquad\square$

Since β_k does not appear in the sum in (13), which determines the direction of the tangent vector at Q_k, but only in the denominator of the leading factor, which influences the length of this vector, it follows that decreasing β_k increases the flatness of the curve at Q_k, while increasing β_k lets the curve bend more tightly (see Figure 4.c). However, setting β_k to a very small or a very large value may create poles. Bounds on β_k that prevent this can be derived in the same way as for barycentric rational curves [7, Proposition 6].

4 Degree elevation

4.1 Degree elevation using the discrete Fourier transform

Sánchez-Reyes [9] describes a three-step procedure to increase the degree of a periodic Bézier curve $P(t)$ from N to $N+1$. He first uses the discrete Fourier transform (DFT) to express P as a complex trigonometric polynomial curve with $2N+1$ Fourier coefficients $c_{-N},\dots,c_N \in \mathbb{C}^2$. Degree elevation can then be achieved by simply adding two higher order zero frequencies and converting the new set of $2N+3$ Fourier coefficients $\tilde{c}_i = c_i$, $i = -N,\dots,N$ and $\tilde{c}_{-N-1} = \tilde{c}_{N+1} = (0,0)$ back to the spatial domain with the inverse DFT:

$$
\begin{array}{ccc}
P_N(t) & & P_{N+1}(t) \\
\text{DFT}\downarrow & & \uparrow\text{DFT}^{-1} \\
\{c_j\}_{j=-N}^{N} & \xrightarrow[\text{elevation}]{\text{degree}} & \{\tilde{c}_j\}_{j=-N-1}^{N+1}
\end{array}
$$

This approach can easily be adapted to a periodic rational Bézier curve $P(t)$ with control points P_i and weights α_i, since the latter is just the image of the spatial periodic Bézier curve $\hat{P}(t)$ with control points $(\alpha_i P_i, \alpha_i)$ under the central projection Π into the $z=1$ plane. Hence, the degree of a periodic rational Bézier curve can be raised from N to $N+1$ in five simple steps:

$$
\begin{array}{ccc}
P_N(t) & & P_{N+1}(t) \\
\Pi^{-1}\downarrow & & \uparrow\Pi \\
\hat{P}_N(t) & & \hat{P}_{N+1}(t) \\
\text{DFT}\downarrow & & \uparrow\text{DFT}^{-1} \\
\{c_j\}_{j=-N}^{N} & \xrightarrow[\text{elevation}]{\text{degree}} & \{\tilde{c}_j\}_{j=-N-1}^{N+1}
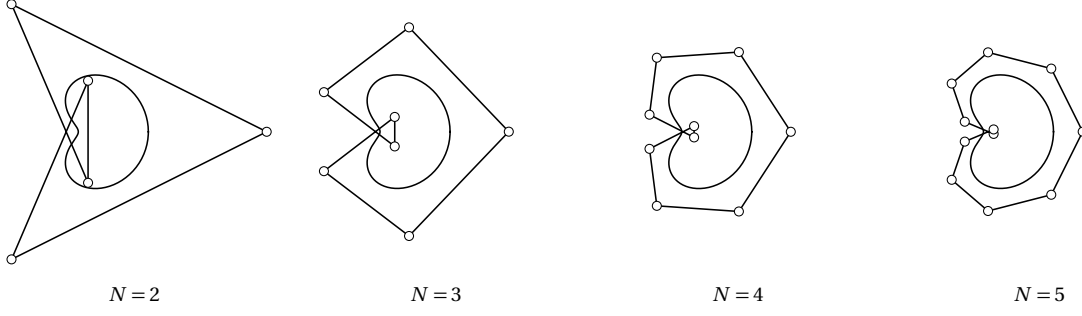\end{array}
$$

Figure 5: Raising the degree (from left to right) of a periodic rational Bézier curve.

4.2 Degree elevation via the trigonometric barycentric form

The equivalence of the periodic rational Bézier and the trigonometric barycentric form offers an alternative approach to degree elevation. Given a periodic rational Bézier curve $P(t)$ of degree N, we first convert P as explained in Proposition 1, but using $n+3$ nodes $0 \le t_0 < \cdots < t_{n+2} < 2\pi$, thus resulting in a trigonometric barycentric rational curve $Q(t)$ of degree $N+1$. We then follow Proposition 2 to convert Q back into periodic rational Bézier form, but now with degree $N+1$:

$$
\begin{array}{r}
P_N(t) \\
\text{convert to trigonometric barycentric form with respect to } t_0,\ldots,t_{n+2} \downarrow \\
Q_{N+1}(t) \\
\text{convert to periodic rational Bézier form with respect to } t_0,\ldots,t_{n+2} \downarrow \\
P_{N+1}(t)
\end{array}
$$

This procedure can be further simplified by using uniformly distributed nodes t_i and recalling the matrix notation of the conversion process. It then follows that we can compute the control points of the degree-raised spatial periodic Bézier curve from the control points of the given spatial periodic Bézier curve as $\hat{\boldsymbol{P}}_{N+1} = \boldsymbol{C}^{-1}\boldsymbol{D}\,\hat{\boldsymbol{P}}_N$, where the entries of the matrices $\boldsymbol{C} \in \mathbb{R}^{(n+3)\times(n+3)}$ and $\boldsymbol{D} \in \mathbb{R}^{(n+3)\times(n+1)}$ are

$$
\boldsymbol{C}_{i,j} = B_{n+2}(\psi_i - \psi_j), \qquad \boldsymbol{D}_{j,k} = B_n(\psi_j - \phi_k), \qquad i,j = 0,\ldots,n+2, \quad k = 0,\ldots,n,
$$

with $\psi_i = \frac{2i\pi}{n+3}$, $i = 0,\ldots,n+2$ and $\phi_i = \frac{2i\pi}{n+1}$, $i = 0,\ldots,n$. Hence, the degree can be raised from N to $N+1$ (see Figure 5) in three simple steps:
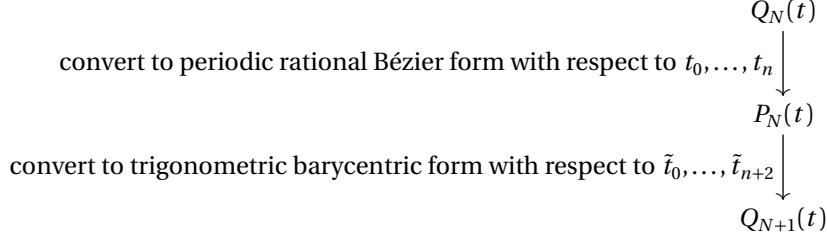
$$
\begin{array}{ccc}
P_N(t) & & P_{N+1}(t) \\
\Pi^{-1} \downarrow & & \uparrow \Pi \\
\hat{P}_N(t) & \xrightarrow[\boldsymbol{C}^{-1}\boldsymbol{D}]{\text{multiply with}} & \hat{P}_{N+1}(t)
\end{array}
$$

Instead of inverting $\boldsymbol{C}$, it is advisable to solve instead the linear system $\boldsymbol{C}\,\hat{\boldsymbol{P}}_{N+1} = \boldsymbol{D}\,\hat{\boldsymbol{P}}_N$, which can be done efficiently in $O(n\log n)$ time using the fast Fourier transform, because $\boldsymbol{C}$ is a symmetric circulant matrix.

4.3 Degree elevation using point insertion

If we prefer to work with the trigonometric barycentric form, then a third variant of degree elevation is the following. Given the trigonometric barycentric rational curve $Q(t)$, its degree can be raised from N to $N+1$ by inserting two new interpolation points. Conceptually, this is achieved by first converting Q with Proposition 2 to periodic rational Bézier form, using the given nodes $t_0,\ldots,t_n$, and then using Proposition 1, but with respect to the new nodes $\tilde{t}_0,\ldots,\tilde{t}_{n+2}$, which are obtained by adding the two parameter values corresponding to the new interpolation points to the sequence of given nodes, to obtain Q as a trigonometric barycentric

rational curve of degree $N+1$:

$$Q_N(t)$$

convert to periodic rational Bézier form with respect to $t_0, \ldots, t_n$ $\Bigg\downarrow$

$$P_N(t)$$

convert to trigonometric barycentric form with respect to $\tilde{t}_0, \ldots, \tilde{t}_{n+2}$ $\Bigg\downarrow$

$$Q_{N+1}(t)$$

As in the previous subsection, the interpolation points and weights of the degree-raised curve can be computed from the interpolation points and weights of the given curve as $\hat{\boldsymbol{Q}}_{N+1} = \boldsymbol{C}\boldsymbol{D}^{-1}\hat{\boldsymbol{Q}}_N$, where the entries of the matrices $\boldsymbol{C} \in \mathbb{R}^{(n+3)\times(n+1)}$ and $\boldsymbol{D} \in \mathbb{R}^{(n+1)\times(n+1)}$ are

$$\boldsymbol{C}_{i,j} = B_n(\tilde{t}_i - \phi_j), \qquad \boldsymbol{D}_{j,k} = B_n(t_j - \phi_k), \qquad i = 0, \ldots, n+2, \quad j,k = 0, \ldots, n,$$

but we can actually avoid matrix multiplication and inversion and compute the parameters of the degree-raised curve directly with simple formulas.

Since we can slide interpolation points to any position along the curve (see Proposition 4 and Remark 1), we assume without loss of generality that the two new interpolation points are inserted at the two parameter values $\tilde{t}_k, \tilde{t}_{k+1} \in (t_{k-1}, t_k)$ for some $k \in \{1, \ldots, n\}$, so that the new nodes $\tilde{t}_0, \ldots, \tilde{t}_{n+2}$ are

i	0	$\ldots$	$k-1$	k	$k+1$	$k+2$	$\ldots$	$n+2$
$\tilde{t}_i$	t_0	$\ldots$	t_{k-1}	$\tilde{t}_k$	$\tilde{t}_{k+1}$	t_k	$\ldots$	t_n

Proposition 7. *The trigonometric barycentric rational curve Q of degree N in* (4) *can be expressed as a trigonometric barycentric rational curve $\tilde{Q}$ of degree $N+1$ with parameters*

$$\tilde{t}_i = \begin{cases} t_i, \\ \tilde{t}_k, \\ \tilde{t}_{k+1}, \\ t_{i-2}, \end{cases} \qquad \tilde{Q}_i = \begin{cases} Q_i, \\ Q(\tilde{t}_k), \\ Q(\tilde{t}_{k+1}), \\ Q_{i-2}, \end{cases} \qquad \tilde{\beta}_i = \begin{cases} \csc\frac{t_i - \tilde{t}_k}{2}\csc\frac{t_i - \tilde{t}_{k+1}}{2}\beta_i, \\ \csc\frac{\tilde{t}_k - \tilde{t}_{k+1}}{2}\sum_{j=0}^{n}(-1)^{j+k}\csc\frac{\tilde{t}_k - t_j}{2}\beta_j, \\ \csc\frac{\tilde{t}_{k+1} - \tilde{t}_k}{2}\sum_{j=0}^{n}(-1)^{j+k+1}\csc\frac{\tilde{t}_{k+1} - t_j}{2}\beta_j, \\ \csc\frac{t_{i-2} - \tilde{t}_k}{2}\csc\frac{t_{i-2} - \tilde{t}_{k+1}}{2}\beta_{i-2} \end{cases} \qquad \text{if} \begin{cases} i < k, \\ i = k, \\ i = k+1, \\ i > k+1. \end{cases}$$

Proof. As in the proof of Proposition 4, the statement is obvious for the nodes $\tilde{t}_i$ and the interpolation points $\tilde{Q}_i$, and we conclude from Proposition 1 that $\beta_i = (-1)^i W_i z(t_i)$ and $\tilde{\beta}_i = (-1)^i \tilde{W}_i z(\tilde{t}_i)$. For $i < k$, we note that

$$\tilde{W}_i = \prod_{j=0, \, j \neq i}^{n+2} \csc\frac{\tilde{t}_i - \tilde{t}_j}{2} = \csc\frac{t_i - \tilde{t}_k}{2}\csc\frac{t_i - \tilde{t}_{k+1}}{2}W_i,$$

and therefore

$$\tilde{\beta}_i = (-1)^i \tilde{W}_i z(\tilde{t}_i) = \csc\frac{t_i - \tilde{t}_k}{2}\csc\frac{t_i - \tilde{t}_{k+1}}{2}(-1)^i W_i z(t_i) = \csc\frac{t_i - \tilde{t}_k}{2}\csc\frac{t_i - \tilde{t}_{k+1}}{2}\beta_i.$$

For $i > k+1$, we can similarly show that

$$\tilde{\beta}_i = \csc\frac{t_{i-2} - \tilde{t}_k}{2}\csc\frac{t_{i-2} - \tilde{t}_{k+1}}{2}\beta_{i-2}.$$

For $i = k$, recall that $z(t) = \ell(t)\sum_{j=0}^{n}(-1)^j \csc\frac{t - t_j}{2}\beta_j$ and since $\ell(\tilde{t}_k) = \csc\frac{\tilde{t}_k - \tilde{t}_{k+1}}{2}/\tilde{W}_k$, we have

$$\tilde{\beta}_k = (-1)^k \tilde{W}_k z(\tilde{t}_k) = (-1)^k \tilde{W}_k \ell(\tilde{t}_k)\sum_{j=0}^{n}(-1)^j \csc\frac{\tilde{t}_k - t_j}{2}\beta_j = \csc\frac{\tilde{t}_k - \tilde{t}_{k+1}}{2}\sum_{j=0}^{n}(-1)^{j+k}\csc\frac{\tilde{t}_k - t_j}{2}\beta_j.$$

For $i = k+1$, a similar reasoning gives

$$\tilde{\beta}_{k+1} = \csc\frac{\tilde{t}_{k+1} - \tilde{t}_k}{2}\sum_{j=0}^{n}(-1)^{j+k+1}\csc\frac{\tilde{t}_{k+1} - t_j}{2}\beta_j.$$

$\square$

Remark 2. In practice, a user would probably prefer to insert one interpolation point at a time. Assuming that the current number of interpolation points is odd, this can be done by simply inserting two new points $\tilde{Q}_k$ and $\tilde{Q}_{k+1}$, following Proposition 7, but showing only $\tilde{Q}_k$ to the user. Once the user decides to insert another interpolation point, $\tilde{Q}_{k+1}$ is revealed and moved to the desired position using Proposition 4.

5 Conclusion

In this paper, we extended the idea of periodic Bézier curves by Sánchez-Reyes [9] to the rational setting and explored the use of the trigonometric barycentric form in the context of curve design. We showed that it offers more direct control over the curve's shape and complements the usual shape control tools given by the periodic rational Bézier form (control points and weights). While the periodic rational Bézier form is more suitable for designing the general shape of a curve, the trigonometric barycentric form is recommended for micro-editing. In particular, the interpolation property can be used to intuitively adjust the shape of a curve by moving the interpolation points and to force a curve to pass through specific points in the plane. If the user wishes to adjust the shape in a region without any interpolation points, there are two possible solutions. One can either slide an existing interpolation point to the desired position or insert a new interpolation point, both without changing the shape of the curve. Moreover, the user can control the curvature of a curve at an interpolation point by adjusting the corresponding weight. Although we restricted our discussion to planar curves, it is clear that the described framework is also valid for 3D curves and even higher dimensional curves, because the construction is independent of the dimension of the control points P_i and the interpolation points Q_i.

Acknowledgements

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