Algebraic and geometric characterizations of a class of planar quartic curves with rational offsets

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Abstract

Planar polynomial curves have rational offset curves, if they are either Pythagorean-hodograph (PH) or indirect Pythagorean-hodograph (iPH) curves. In this paper, we derive an algebraic and two geometric characterizations for planar quartic iPH curves. The characterizations are given in terms of quantities related to the Bézier control polygon of the curve, and naturally extend to quartic and cubic PH and quadratic iPH curves.

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1 Introduction

Let $\boldsymbol{r}(t)=(x(t),y(t))$ be a parametric curve in the plane. The *offset curve* to $\boldsymbol{r}$ at some (signed) distance $d\in\mathbb{R}$ can be written as $\boldsymbol{r}_d(t)=\boldsymbol{r}(t)+d\,\boldsymbol{n}(t)$, where $\boldsymbol{n}(t)=(y'(t),-x'(t))/\sigma(t)$ is the *unit normal* of the curve $\boldsymbol{r}(t)$ and $\sigma(t)=\|\boldsymbol{r}'(t)\|=\sqrt{x'(t)^2+y'(t)^2}$ is its *speed*. Offset curves arise in a variety of applications, including CNC machining, railway design, and shape blending [22, 18, 16], but due to the square root in the definition of $\sigma(t)$, they are usually not rational and can thus not be represented exactly in common CAD systems [10, 5]. Therefore, research on conditions for polynomial curves to have rational offsets and methods for constructing such curves has attracted a lot of attention [10, 5, 13, 14], and the theoretical analysis of the rationality of generalized offsets to irreducible hypersurfaces was studied, too [2, 21].

Especially in computer-aided geometric design and manufacturing [4], planar polynomial curves are often expressed in *Bézier form* as $\boldsymbol{r}:[0,1]\to\mathbb{R}^2$,

$$\boldsymbol{r}(t)=\sum_{i=0}^{n}\boldsymbol{p}_i B_i^n(t), \tag{1}$$

where $B_i^n(t)=\binom{n}{i}(1-t)^{n-i}t^i$ are the *Bernstein basis polynomials* of degree $n\in\mathbb{N}$ and $\boldsymbol{p}_0,\dots,\boldsymbol{p}_n\in\mathbb{R}^2$ are the *control points*. Connecting the control points forms the *control polygon* that provides an intuitive approximation and description of the curve. A subset of these curves has exactly representable rational offset curves.

The first class of polynomial curves with rational offsets are *Pythagorean-hodograph* (PH) curves, which were introduced by Farouki and Sakkalis [10]. For these curves, the speed $\sigma(t)$ is a polynomial, so that the two components of the curve's first parametric derivative or *hodograph* $\boldsymbol{r}'(t)=(x'(t),y'(t))$ and $\sigma(t)$ form a polynomial *Pythagorean triple*, that is, $x'(t)^2+y'(t)^2=\sigma(t)^2$. PH curves and their applications have been studied intensively, and we refer the interested reader to the book by Farouki [8] and the references therein.

The second class of polynomial curves that have rational offsets with respect to a properly chosen reparameterization was discovered by Lü [13, 14]. For such a (non-PH) *offset-rational* or *indirect Pythagorean-hodograph* (iPH) curve [15], there exists a suitable rational quadratic parameter transform $t:[0,1]\to[0,1]$ with $t(0)=0$, $t(1)=1$, and $t'(s)>0$ for $s\in[0,1]$, such that the speed $\tilde{\sigma}(s)$ of the reparameterized curve $\tilde{\boldsymbol{r}}(s)=\boldsymbol{r}(t(s))$ and the two components $\tilde{x}'(s)$ and $\tilde{y}'(s)$ of its first derivative form a *rational* Pythagorean triple [14, 15].

In this paper, we study a class of quartic iPH curves, which has been shown to be capable of interpolating first-order Hermite data with up to four different solutions [14], akin to quintic PH curves [5, 9]; see Figure 1. We focus on *properly parameterized* curves, for which the parameter value t and the curve point $\boldsymbol{r}(t)$ are in one-to-one correspondence for all $t\in\mathbb{R}$, except for parameter values corresponding to self-intersections of $\boldsymbol{r}$ [8]. For an improperly parameterized polynomial or rational curve, it is always possible to make it properly parameterized by reparameterization [19, 20]. Following [6], we use the complex representation

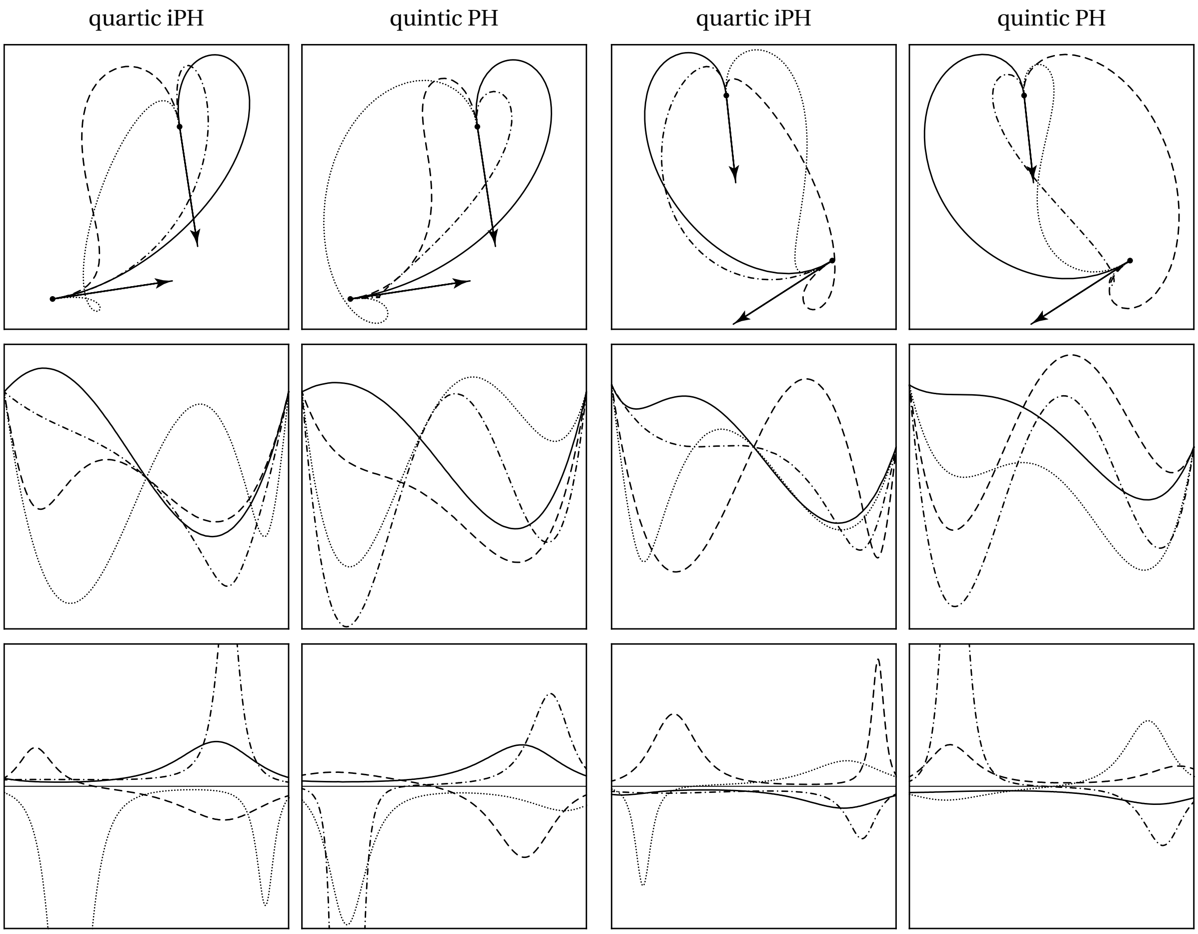

Figure 1: Examples of C^1 Hermite interpolants (top), their speeds (middle), and curvatures (bottom). Note that the prescribed tangents are scaled by a factor of 1/5 to better fit the images.

of planar Bézier curves to analyse the structure of these quartic iPH curves and derive a simple algebraic characterization, given in terms of the complex form of the control edge vectors (see Section 3), which turns out to be useful for the construction of C^1 Hermite interpolants (see Section 3.4). We then investigate two geometric characterizations, where the conditions are stated in terms of quantities related to the control polygon, which can be used to parameterize this class of quartic iPH curves in an intuitive way (see Section 4).

1.1 Related work

Various geometric and algebraic characterizations of lower degree curves with rational offsets have been derived, usually in terms of quantities related to their Bézier control polygons, with most work focussing on PH curves.

We know that a cubic curve is a PH curve, if and only if the two interior angles θ_1 and θ_2 between adjacent edges of the Bézier control polygon (see Figure 2) are the same and the lengths $E_i = \|\boldsymbol{e}_i\|$ of the *control edge vectors* $\boldsymbol{e}_i = \boldsymbol{p}_{i+1} - \boldsymbol{p}_i$ are in geometric progression, that is, $E_1 = \sqrt{E_0 E_2}$ [10], which is equivalent to the condition that the triangles $[\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2]$ and $[\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3]$ are similar [17]. Interpreting the edge vectors $\boldsymbol{e}_i$ as complex numbers, these geometric conditions can be combined to the single complex constraint $\boldsymbol{e}_1^2 = \boldsymbol{e}_0 \boldsymbol{e}_2$ [6], and we derive a similar algebraic condition for quartic iPH curves in Section 3.

According to the analysis in [23], a quartic curve is a PH curve, if there exists a line, which passes through $\boldsymbol{p}_2$ and intersects with the lines $\overline{\boldsymbol{p}_0\boldsymbol{p}_1}$ and $\overline{\boldsymbol{p}_3\boldsymbol{p}_4}$ at $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$, respectively, such that the angles $\sphericalangle(\boldsymbol{p}_1 - \boldsymbol{s}_1, \boldsymbol{p}_2 - \boldsymbol{s}_1)$ and $\sphericalangle(\boldsymbol{p}_2 - \boldsymbol{s}_2, \boldsymbol{p}_3 - \boldsymbol{s}_2)$ are equal (see Figure 2) and the lengths $F_i = \|\boldsymbol{f}_i\|$ of the vectors $\boldsymbol{f}_0 = \boldsymbol{s}_1 - \boldsymbol{p}_1$, $\boldsymbol{f}_1 = \boldsymbol{p}_2 - \boldsymbol{s}_1$, $\boldsymbol{f}_2 = \boldsymbol{s}_2 - \boldsymbol{p}_2$, $\boldsymbol{f}_3 = \boldsymbol{p}_3 - \boldsymbol{s}_2$, together with the lengths E_0 and E_3 satisfy the three constraints $E_0 F_2 = 3 F_0 F_1$, $E_3 F_1 = 3 F_2 F_3$, and $F_1 F_2 = 4 F_0 F_3$. In Section 4, we show that a generalized version of these conditions characterizes quartic iPH curves.

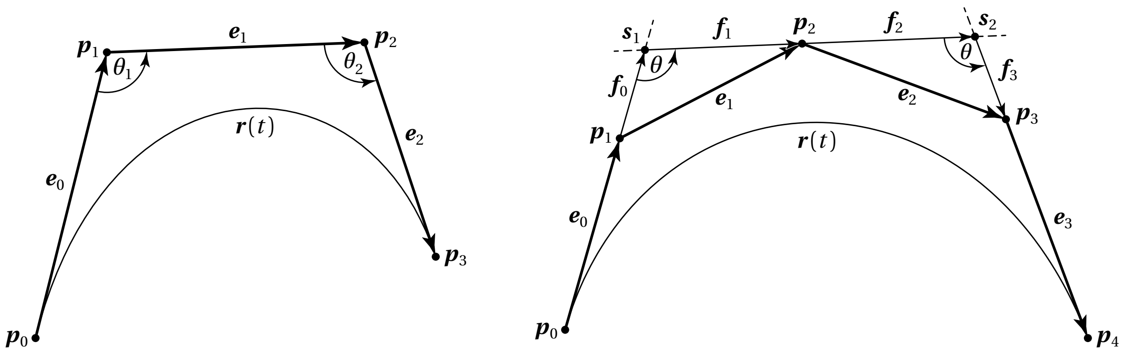

Figure 2: Notation for the characterization of cubic (left) and quartic (right) PH curves in Bézier form.

Further investigations have revealed an additional condition for a quartic PH curve to have monotonic curvature [25], algebraic [6] and geometric [3] characterizations of quintic PH curves, three methods for identifying sextic PH curves [24], as well as geometric properties for PH curves of degree seven [12, 26].

1.2 Contributions

Much less is known for iPH curves, apart from the fact that quadratic curves are iPH, but not PH curves, as long as the control points $\boldsymbol{p}_0$, $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ are not collinear [11, 14], and an in-depth analysis of geometric conditions for properly parameterized cubic iPH curves [15]. To the best of our knowledge, this paper is the first to take a closer look at quartic iPH curves and to derive algebraic as well as geometric characterizations for an important subset of these curves.

2 Preliminaries

For the analysis of PH and iPH curves, it has turned out to be useful to exploit the complex representation of $\mathbb{R}^2$ [6]. We follow this approach and throughout this paper identify the planar point $(x, y) \in \mathbb{R}^2$ with the complex number $x + \mathbf{i}y \in \mathbb{C}$ and likewise for vectors and planar curves. The starting point of our investigations is a necessary and sufficient condition that was discovered by Lü [14].

Theorem 1. *A properly parameterized polynomial curve $\boldsymbol{r}(t)$ has rational offsets, if and only if its hodograph can be written in complex form as*

$$\boldsymbol{r}'(t) = p(t)(1 + \boldsymbol{k}\,t)\boldsymbol{w}(t)^2, \tag{2}$$

where $p(t)$ is a real polynomial, $\boldsymbol{k}$ is a complex constant, and $\boldsymbol{w}(t) = x(t) + \mathbf{i}y(t)$ is a complex polynomial with $x(t)$ and $y(t)$ relatively prime.

Without loss of generality, we can assume the leading coefficient of $p(t)$ in (2) to be equal to 1, since all other cases can be reduced to this situation by multiplying $\boldsymbol{w}(t)$ with the square root of this leading coefficient. Moreover, the curve $\boldsymbol{r}(t)$ is a PH curve, if $\operatorname{Im}(\boldsymbol{k}) = 0$, and an iPH curve, otherwise.

In this paper, we focus on quartic iPH curves. For these curves, we have $\deg(1 + \boldsymbol{k}\,t) = 1$ and consequently $\deg(\boldsymbol{r}) = \deg(\boldsymbol{r}') + 1 = \deg(p) + 2\deg(\boldsymbol{w}) + 2 = 4$, and we distinguish the following cases:

- **Class I:** $\deg(p) = 2$ and $\deg(\boldsymbol{w}) = 0$.
  In this case, we can rewrite the hodograph in (2) as

$$\boldsymbol{r}'(t) = ((t - a)^2 + b)(1 + \boldsymbol{k}\,t)\boldsymbol{w} \tag{3}$$

  for some $a, b \in \mathbb{R}$ and $\boldsymbol{k}, \boldsymbol{w} \in \mathbb{C}$ with $\operatorname{Im}(\boldsymbol{k}) \neq 0$ and $\boldsymbol{w} \neq 0$. These curves have a linearly varying normal [1], and we further recognize three sub-cases:

  - **Class I.0:** $b > 0$. These curves are regular.
  - **Class I.1:** $b = 0$. These curves are singular, but tangent continuous at $t = a$.
  - **Class I.2:** $b < 0$. These curves have two cusps at $t = a \pm \sqrt{-b}$.

- **Class II:** $\deg(p) = 0$ and $\deg(\boldsymbol{w}) = 1$.
  In this case, we can rewrite the hodograph in (2) as

$$\boldsymbol{r}'(t) = (1 + \boldsymbol{k}\,t)(\boldsymbol{w}_0(1-t) + \boldsymbol{w}_1\,t)^2 \tag{4}$$

  for some $\boldsymbol{k}, \boldsymbol{w}_0, \boldsymbol{w}_1 \in \mathbb{C}$ with $\mathrm{Im}(\boldsymbol{k}) \neq 0$, $\boldsymbol{w}_0 \neq 0$, $\boldsymbol{w}_1 \neq 0$, and $\mathrm{Im}(\boldsymbol{w}_1/\boldsymbol{w}_0) \neq 0$. These curves are regular.

In particular, we are interested in class II quartic iPH curves, since they can be used for solving the C^1 Hermite interpolation problem [14], but we will see that class I.1 quartic iPH curves, quartic and cubic PH curves, and quadratic iPH curves play a role in this context, too (see Section 3.4).

3 Algebraic considerations

3.1 Class II quartic iPH curves

Our first observation, which was also mentioned in [14], but without further explanation, is that there exists an alternative representation for the hodograph of class II quartic iPH curves.

Corollary 1. *A properly parameterized quartic curve $\boldsymbol{r}(t)$ is an iPH curve of class II, if and only if its hodograph can be written in complex form as*

$$\boldsymbol{r}'(t) = \left(\boldsymbol{u}_0(1-t) + \frac{\boldsymbol{u}_1}{\boldsymbol{a}^2}t\right)((1-t) + \boldsymbol{a}\,t)^2, \tag{5}$$

where $\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{a} \in \mathbb{C}$ with $\boldsymbol{u}_0 \neq 0$, $\boldsymbol{u}_1 \neq 0$, $\mathrm{Im}(\boldsymbol{a}) \neq 0$, and $\mathrm{Im}(\boldsymbol{a}^2\boldsymbol{u}_0/\boldsymbol{u}_1) \neq 0$.

Proof. If $\boldsymbol{r}'(t)$ is given as in (4), then we can rewrite it as in (5) by letting $\boldsymbol{u}_0 = \boldsymbol{w}_0^2$, $\boldsymbol{u}_1 = (1+\boldsymbol{k})\boldsymbol{w}_1^2$, $\boldsymbol{a} = \boldsymbol{w}_1/\boldsymbol{w}_0$. Vice versa, we can convert a hodograph from the form in (5) to the form in (4) by letting $\boldsymbol{k} = \boldsymbol{u}_1/(\boldsymbol{u}_0\boldsymbol{a}^2) - 1$, $\boldsymbol{w}_0 = \pm\sqrt{\boldsymbol{u}_0}$, $\boldsymbol{w}_1 = \boldsymbol{w}_0\boldsymbol{a}$. Note that the additional conditions on $\boldsymbol{k}$, $\boldsymbol{w}_0$, $\boldsymbol{w}_1$ in (4) and on $\boldsymbol{u}_0$, $\boldsymbol{u}_1$, $\boldsymbol{a}$ in (5) imply each other. □

In order to derive an algebraic characterization of class II quartic iPH curves in terms of the control edges $\boldsymbol{e}_i = \boldsymbol{p}_{i+1} - \boldsymbol{p}_i$ from their Bézier representation, we recall that the hodograph of a quartic curve $\boldsymbol{r}(t)$ can be written, by differentiating its Bézier representation in (1), as

$$\boldsymbol{r}'(t) = 4\big(\boldsymbol{e}_0 B_0^3(t) + \boldsymbol{e}_1 B_1^3(t) + \boldsymbol{e}_2 B_2^3(t) + \boldsymbol{e}_3 B_3^3(t)\big). \tag{6}$$

Comparing the two expressions of $\boldsymbol{r}'(t)$ in (5) and (6) for $t = 0$ and $t = 1$, we observe that

$$\boldsymbol{e}_0 = \frac{1}{4}\boldsymbol{u}_0, \qquad \boldsymbol{e}_3 = \frac{1}{4}\boldsymbol{u}_1, \tag{7}$$

and, by similarly comparing their derivatives, that is, the two forms of $\boldsymbol{r}''(t)$ that they induce,

$$\boldsymbol{e}_1 = \frac{2}{3}\boldsymbol{a}\,\boldsymbol{e}_0 + \frac{1}{3}\boldsymbol{a}^{-2}\boldsymbol{e}_3, \qquad \boldsymbol{e}_2 = \frac{1}{3}\boldsymbol{a}^2\boldsymbol{e}_0 + \frac{2}{3}\boldsymbol{a}^{-1}\boldsymbol{e}_3. \tag{8}$$

Moreover, we need to introduce the concept of non-degenerate Bézier control polygons.

Definition 1. We say that the Bézier control polygon of a quartic curve $\boldsymbol{r}(t)$ is *non-degenerate*, if and only if the first and last control edge do not vanish and the control points are not collinear, that is, $\boldsymbol{e}_0 \neq 0$, $\boldsymbol{e}_3 \neq 0$, and $\mathrm{Im}(\boldsymbol{e}_i/\boldsymbol{e}_0) \neq 0$ for some $i \in \{1,2,3\}$.

This kind of non-degeneracy rules out curves that are singular at either $t = 0$ or $t = 1$ and those that describe a (possibly improperly parameterized) straight line. We are now ready to present our main result.

Theorem 2. *A properly parameterized quartic curve $\boldsymbol{r}(t)$ is an iPH curve of class II, if and only if its Bézier control polygon is non-degenerate and its control edges satisfy*

$$(\boldsymbol{e}_0\boldsymbol{e}_3 - \boldsymbol{e}_1\boldsymbol{e}_2)^2 = 4(\boldsymbol{e}_0\boldsymbol{e}_2 - \boldsymbol{e}_1^2)(\boldsymbol{e}_1\boldsymbol{e}_3 - \boldsymbol{e}_2^2) \tag{9}$$

and either

$$\boldsymbol{e}_0\boldsymbol{e}_3 = \boldsymbol{e}_1\boldsymbol{e}_2 \tag{10}$$

or

$$\mathrm{Im}\left(2\frac{\boldsymbol{e}_1\boldsymbol{e}_3 - \boldsymbol{e}_2^2}{\boldsymbol{e}_0\boldsymbol{e}_3 - \boldsymbol{e}_1\boldsymbol{e}_2}\right) \neq 0 \qquad \text{and} \qquad \mathrm{Im}\left(4\left(\frac{\boldsymbol{e}_1\boldsymbol{e}_3 - \boldsymbol{e}_2^2}{\boldsymbol{e}_0\boldsymbol{e}_3 - \boldsymbol{e}_1\boldsymbol{e}_2}\right)^2\frac{\boldsymbol{e}_0}{\boldsymbol{e}_3}\right) \neq 0. \tag{11}$$

Proof. Let us start by assuming that $\boldsymbol{r}(t)$ is a properly parameterized class II quartic iPH curve. To show that the Bézier control polygon of $\boldsymbol{r}(t)$ is non-degenerate, we first recall from Corollary 1 that $\boldsymbol{u}_0 \neq 0$, $\boldsymbol{u}_1 \neq 0$, and $\operatorname{Im}(\boldsymbol{a}) \neq 0$, hence $\boldsymbol{a} \neq 0$. It then follows from (7) that $\boldsymbol{e}_0 \neq 0$ and $\boldsymbol{e}_3 \neq 0$. Now assume that all $\boldsymbol{e}_i$ are parallel, that is, $\boldsymbol{e}_i = \lambda_i \boldsymbol{e}_0$ for $i = 1,2,3$ and some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \setminus \{0\}$. By (8), we then have $\lambda_2 = \boldsymbol{a}(2\lambda_1 - \boldsymbol{a})$ and $\lambda_3 = \boldsymbol{a}^2(3\lambda_1 - 2\boldsymbol{a})$. Since $\operatorname{Im}(\boldsymbol{a}(2\lambda_1 - \boldsymbol{a})) = 2\operatorname{Im}(\boldsymbol{a})(\lambda_1 - \operatorname{Re}(\boldsymbol{a}))$ and $\operatorname{Im}(\boldsymbol{a}) \neq 0$, we conclude that $\lambda_1 = \operatorname{Re}(\boldsymbol{a})$, because λ_2 would otherwise not be a real number. With this, however, we find that $\operatorname{Im}(\boldsymbol{a}^2(3\lambda_1 - 2\boldsymbol{a})) = 2\operatorname{Im}(\boldsymbol{a})^3 \neq 0$, which contradicts the assumption that $\lambda_3 \in \mathbb{R}$. We now proceed to prove the algebraic conditions. It follows from (8) that

$$\boldsymbol{e}_0\boldsymbol{e}_3 - \boldsymbol{e}_1\boldsymbol{e}_2 = -\frac{2}{9}\boldsymbol{a}(\boldsymbol{a}\boldsymbol{e}_0 - \boldsymbol{a}^{-2}\boldsymbol{e}_3)^2, \tag{12a}$$

$$\boldsymbol{e}_0\boldsymbol{e}_2 - \boldsymbol{e}_1^2 = -\frac{1}{9}(\boldsymbol{a}\boldsymbol{e}_0 - \boldsymbol{a}^{-2}\boldsymbol{e}_3)^2, \tag{12b}$$

$$\boldsymbol{e}_1\boldsymbol{e}_3 - \boldsymbol{e}_2^2 = -\frac{1}{9}(\boldsymbol{a}^2\boldsymbol{e}_0 - \boldsymbol{a}^{-1}\boldsymbol{e}_3)^2, \tag{12c}$$

which immediately implies (9). By (12a), condition (10) is equivalent to $\boldsymbol{a}^3\boldsymbol{e}_0 = \boldsymbol{e}_3$, because $\boldsymbol{a} \neq 0$, and using (8), we see that more generally $\boldsymbol{e}_i = \boldsymbol{a}\boldsymbol{e}_{i-1} = \boldsymbol{a}^i\boldsymbol{e}_0$ for $i = 1,2,3$ in this special case, which means that the control edges are in geometric progression, just like the control edges of a cubic PH curve. Otherwise, $\boldsymbol{e}_0\boldsymbol{e}_3 \neq \boldsymbol{e}_1\boldsymbol{e}_2$, and it follows from (12a) and (12c) that

$$2\frac{\boldsymbol{e}_1\boldsymbol{e}_3 - \boldsymbol{e}_2^2}{\boldsymbol{e}_0\boldsymbol{e}_3 - \boldsymbol{e}_1\boldsymbol{e}_2} = \boldsymbol{a}.$$

By (7), we further find that $\boldsymbol{e}_0/\boldsymbol{e}_3 = \boldsymbol{u}_0/\boldsymbol{u}_1$, and the conditions in (11) then follow from Corollary 1, which guarantees that $\operatorname{Im}(\boldsymbol{a}) \neq 0$ and $\operatorname{Im}(\boldsymbol{a}^2\boldsymbol{u}_0/\boldsymbol{u}_1) \neq 0$.

Let us now assume that $\boldsymbol{r}(t)$ is a properly parameterized quartic curve with a non-degenerate Bézier control polygon and control edges $\boldsymbol{e}_i$ that satisfy conditions (9) and (10). Then, either $\boldsymbol{e}_0\boldsymbol{e}_2 = \boldsymbol{e}_1^2$ or $\boldsymbol{e}_1\boldsymbol{e}_3 = \boldsymbol{e}_2^2$, but as both identities actually imply each other, by (9) and under the non-degeneracy assumption of the control polygon, which guarantees $\boldsymbol{e}_0 \neq 0$ and $\boldsymbol{e}_3 \neq 0$, we can safely assume both of them to be true. Consequently, $\boldsymbol{e}_2 = \boldsymbol{e}_1^2/\boldsymbol{e}_0$ and $\boldsymbol{e}_3 = \boldsymbol{e}_1^3/\boldsymbol{e}_0^2$, which means that the control edges are in geometric progression, and it is not hard to verify that both conditions in (8) hold for $\boldsymbol{a} = \boldsymbol{e}_1/\boldsymbol{e}_0$. Consequently, and in view of (7), we can write the hodograph of $\boldsymbol{r}(t)$ as in (5) with $\boldsymbol{u}_0 = 4\boldsymbol{e}_0 \neq 0$ and $\boldsymbol{u}_1 = 4\boldsymbol{e}_3 \neq 0$, and it actually simplifies to $\boldsymbol{r}'(t) = \boldsymbol{u}_0((1-t) + \boldsymbol{a}\,t)^3$ in this case. To show that $\operatorname{Im}(\boldsymbol{a}) \neq 0$, let us assume the opposite, that is, $\operatorname{Im}(\boldsymbol{e}_1/\boldsymbol{e}_0) = 0$. We then have $\operatorname{Im}(\boldsymbol{e}_2/\boldsymbol{e}_0) = \operatorname{Im}(\boldsymbol{a}^2) = 0$ and $\operatorname{Im}(\boldsymbol{e}_3/\boldsymbol{e}_0) = \operatorname{Im}(\boldsymbol{a}^3) = 0$, thus contradicting the assumption that the $\boldsymbol{e}_i$ are not all parallel to each other. Therefore, $\operatorname{Im}(\boldsymbol{a}) \neq 0$, which also implies $\operatorname{Im}(\boldsymbol{a}^2\boldsymbol{u}_0/\boldsymbol{u}_1) = \operatorname{Im}(\boldsymbol{a}^{-1}) \neq 0$.

Finally, let us assume that (11) holds instead of (10), and note that $\boldsymbol{e}_0\boldsymbol{e}_3 \neq \boldsymbol{e}_1\boldsymbol{e}_2$, by (9), implies $\boldsymbol{e}_0\boldsymbol{e}_2 \neq \boldsymbol{e}_1^2$. Letting

$$\boldsymbol{a}_1 = \frac{\boldsymbol{e}_0\boldsymbol{e}_3 - \boldsymbol{e}_1\boldsymbol{e}_2}{2(\boldsymbol{e}_0\boldsymbol{e}_2 - \boldsymbol{e}_1^2)}, \qquad \boldsymbol{a}_2 = \frac{2(\boldsymbol{e}_1\boldsymbol{e}_3 - \boldsymbol{e}_2^2)}{\boldsymbol{e}_0\boldsymbol{e}_3 - \boldsymbol{e}_1\boldsymbol{e}_2},$$

it then follows from (9) that $\boldsymbol{a}_1 = \boldsymbol{a}_2$, and it can further be verified that

$$4\boldsymbol{a}_1^2\boldsymbol{e}_1 = \frac{(\boldsymbol{e}_0\boldsymbol{e}_3 - \boldsymbol{e}_1\boldsymbol{e}_2)^2}{(\boldsymbol{e}_0\boldsymbol{e}_2 - \boldsymbol{e}_1^2)^2}\boldsymbol{e}_1 = \frac{\boldsymbol{e}_0^2\boldsymbol{e}_1\boldsymbol{e}_3^2 - 2\boldsymbol{e}_0\boldsymbol{e}_1^2\boldsymbol{e}_2\boldsymbol{e}_3 + \boldsymbol{e}_1^3\boldsymbol{e}_2^2}{(\boldsymbol{e}_0\boldsymbol{e}_2 - \boldsymbol{e}_1^2)^2},$$

$$2\boldsymbol{a}_1^2\boldsymbol{a}_2\boldsymbol{e}_0 = \frac{(\boldsymbol{e}_0\boldsymbol{e}_3 - \boldsymbol{e}_1\boldsymbol{e}_2)(\boldsymbol{e}_1\boldsymbol{e}_3 - \boldsymbol{e}_2^2)}{(\boldsymbol{e}_0\boldsymbol{e}_2 - \boldsymbol{e}_1^2)^2}\boldsymbol{e}_0 = \frac{\boldsymbol{e}_0^2\boldsymbol{e}_1\boldsymbol{e}_3^2 - \boldsymbol{e}_0^2\boldsymbol{e}_2^2\boldsymbol{e}_3 - \boldsymbol{e}_0\boldsymbol{e}_1^2\boldsymbol{e}_2\boldsymbol{e}_3 + \boldsymbol{e}_0\boldsymbol{e}_1\boldsymbol{e}_2^3}{(\boldsymbol{e}_0\boldsymbol{e}_2 - \boldsymbol{e}_1^2)^2},$$

$$\boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{e}_1 = \frac{(\boldsymbol{e}_0\boldsymbol{e}_2 - \boldsymbol{e}_1^2)(\boldsymbol{e}_1\boldsymbol{e}_3 - \boldsymbol{e}_2^2)}{(\boldsymbol{e}_0\boldsymbol{e}_2 - \boldsymbol{e}_1^2)^2}\boldsymbol{e}_1 = \frac{\boldsymbol{e}_0\boldsymbol{e}_1^2\boldsymbol{e}_2\boldsymbol{e}_3 - \boldsymbol{e}_0\boldsymbol{e}_1\boldsymbol{e}_2^3 - \boldsymbol{e}_1^4\boldsymbol{e}_3 + \boldsymbol{e}_1^3\boldsymbol{e}_2^2)}{(\boldsymbol{e}_0\boldsymbol{e}_2 - \boldsymbol{e}_1^2)^2}.$$

Therefore, $4\boldsymbol{a}_1^2\boldsymbol{e}_1 = 2\boldsymbol{a}_1^2\boldsymbol{a}_2\boldsymbol{e}_0 + \boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{e}_1 + \boldsymbol{e}_3$, and we conclude that the condition for $\boldsymbol{e}_1$ in (8) holds for $\boldsymbol{a} = \boldsymbol{a}_1 = \boldsymbol{a}_2$. The condition for $\boldsymbol{e}_2$ can be checked similarly. As in the previous case, we can then write the hodograph of $\boldsymbol{r}(t)$ as in (5) with $\boldsymbol{u}_0 = 4\boldsymbol{e}_0 \neq 0$ and $\boldsymbol{u}_1 = 4\boldsymbol{e}_3 \neq 0$, and the conditions $\operatorname{Im}(\boldsymbol{a}) \neq 0$ and $\operatorname{Im}(\boldsymbol{a}^2\boldsymbol{u}_0/\boldsymbol{u}_1) \neq 0$ follow directly from (11). □

The key ingredient to the characterization of class II quartic iPH curves in Theorem 2 certainly is condition (9), and we will now show that this condition is also valid for other quartic Bézier curves with rational offsets, which are not iPH of class II.

3.2 Cubic PH and quadratic iPH curves

If $\boldsymbol{r}(t)$ is a cubic Bézier curve with control edges $\boldsymbol{d}_0$, $\boldsymbol{d}_1$, $\boldsymbol{d}_2$, then the control edges after degree elevation are

$$\boldsymbol{e}_0 = \frac{3}{4}\boldsymbol{d}_0, \qquad \boldsymbol{e}_1 = \frac{1}{4}\boldsymbol{d}_0 + \frac{1}{2}\boldsymbol{d}_1, \qquad \boldsymbol{e}_2 = \frac{1}{2}\boldsymbol{d}_1 + \frac{1}{4}\boldsymbol{d}_2, \qquad \boldsymbol{e}_3 = \frac{3}{4}\boldsymbol{d}_2, \tag{13}$$

and a straightforward calculation reveals that

$$(\boldsymbol{e}_0\boldsymbol{e}_3 - \boldsymbol{e}_1\boldsymbol{e}_2)^2 - 4(\boldsymbol{e}_0\boldsymbol{e}_2 - \boldsymbol{e}_1^2)(\boldsymbol{e}_1\boldsymbol{e}_3 - \boldsymbol{e}_2^2) = \frac{3}{64}(\boldsymbol{d}_0\boldsymbol{d}_2 - \boldsymbol{d}_1^2)(\boldsymbol{d}_0 - 2\boldsymbol{d}_1 + \boldsymbol{d}_2)^2.$$

Therefore, condition (9) is true, if $\boldsymbol{d}_1^2 = \boldsymbol{d}_0\boldsymbol{d}_2$ or $\boldsymbol{d}_1 = \frac{1}{2}(\boldsymbol{d}_0 + \boldsymbol{d}_2)$, that is, $\boldsymbol{d}_1$ must be the *geometric* or the *arithmetic* mean of $\boldsymbol{d}_0$ and $\boldsymbol{d}_2$. Note that the first case holds, if $\boldsymbol{r}(t)$ is a cubic PH curve, while the second case occurs, if and only if $\boldsymbol{r}(t)$ is a degree-raised quadratic curve. This suggests that there might be characterizations of cubic PH and quadratic iPH curves that are similar to the one in Theorem 2, but before we get to the additional algebraic conditions that are needed for this purpose, let us state the equivalents of Corollary 1.

Corollary 2. *A properly parameterized cubic curve $\boldsymbol{r}(t)$ is a PH curve, if and only if its hodograph can be written in complex form as in (5) with $\boldsymbol{u}_0 \neq 0$, $\boldsymbol{u}_1 \neq 0$, $\operatorname{Im}(\boldsymbol{a}) \neq 0$, and $\boldsymbol{a}^2\boldsymbol{u}_0 = \boldsymbol{u}_1$.*

Proof. We first recall from [10] that the hodograph of a properly parameterized cubic PH curve can written as in (2) with $\boldsymbol{k} = 0$, $\deg(p) = 0$, and $\deg(\boldsymbol{w}) = 1$, that is, as

$$\boldsymbol{r}'(t) = (\boldsymbol{w}_0(1-t) + \boldsymbol{w}_1 t)^2 \tag{14}$$

for some $\boldsymbol{w}_0, \boldsymbol{w}_1 \in \mathbb{C}$ with $\boldsymbol{w}_0 \neq 0$, $\boldsymbol{w}_1 \neq 0$, and $\operatorname{Im}(\boldsymbol{w}_1/\boldsymbol{w}_0) \neq 0$. It follows directly, that we can then rewrite $\boldsymbol{r}'(t)$ in the form (5) by letting $\boldsymbol{u}_0 = \boldsymbol{w}_0^2$, $\boldsymbol{u}_1 = \boldsymbol{w}_1^2$, and $\boldsymbol{a} = \boldsymbol{w}_1/\boldsymbol{w}_0$. Vice versa, if $\boldsymbol{r}'(t)$ is given as in (5), then we get back to the form in (14) by letting $\boldsymbol{w}_0 = \pm\sqrt{\boldsymbol{u}_0}$ and $\boldsymbol{w}_1 = \boldsymbol{w}_0\boldsymbol{a}$. Note that the additional conditions on $\boldsymbol{w}_0$, $\boldsymbol{w}_1$ in (14) and on $\boldsymbol{u}_0$, $\boldsymbol{u}_1$, $\boldsymbol{a}$ in the statement imply each other. □

Corollary 3. *A properly parameterized quadratic curve $\boldsymbol{r}(t)$ is an iPH curve, if and only if its hodograph can be written in complex form as in (5) with $\boldsymbol{u}_0 \neq 0$, $\boldsymbol{u}_1 \neq 0$, $\boldsymbol{a} = 1$, and $\operatorname{Im}(\boldsymbol{a}^2\boldsymbol{u}_0/\boldsymbol{u}_1) = \operatorname{Im}(\boldsymbol{u}_0/\boldsymbol{u}_1) \neq 0$.*

Proof. Properly parameterized quadratic iPH curves have a hodograph as in (2) with $\operatorname{Im}(\boldsymbol{k}) \neq 0$, $\deg(p) = 0$, and $\deg(\boldsymbol{w}) = 0$, that is, $\boldsymbol{w}(t)^2 \equiv \boldsymbol{u}_0$ for some $\boldsymbol{u}_0 \in \mathbb{C}$ with $\boldsymbol{u}_0 \neq 0$, which can be rewritten as in (5) by letting $\boldsymbol{u}_1 = (1+\boldsymbol{k})\boldsymbol{u}_0 \neq 0$ and $\boldsymbol{a} = 1$, and vice versa. □

In view of Corollaries 2 and 3, our previous discovery is actually not surprising, because the identities in (12), which imply (9), follow only from the representation of the hodograph in (5), but do not depend on the specific conditions on $\boldsymbol{u}_0$, $\boldsymbol{u}_1$, and $\boldsymbol{a}$. Let us now present the analogues of Theorem 2.

Theorem 3. *A properly parameterized cubic curve $\boldsymbol{r}(t)$ is a PH curve, if and only if its degree-raised, quartic Bézier control polygon is non-degenerate and its control edges satisfy (9), $\boldsymbol{e}_0\boldsymbol{e}_3 \neq \boldsymbol{e}_1\boldsymbol{e}_2$, and*

$$\operatorname{Im}\left(2\frac{\boldsymbol{e}_1\boldsymbol{e}_3 - \boldsymbol{e}_2^2}{\boldsymbol{e}_0\boldsymbol{e}_3 - \boldsymbol{e}_1\boldsymbol{e}_2}\right) \neq 0 \qquad \textit{and} \qquad 4\left(\frac{\boldsymbol{e}_1\boldsymbol{e}_3 - \boldsymbol{e}_2^2}{\boldsymbol{e}_0\boldsymbol{e}_3 - \boldsymbol{e}_1\boldsymbol{e}_2}\right)^2 \frac{\boldsymbol{e}_0}{\boldsymbol{e}_3} = 1. \tag{15}$$

Proof. To prove the necessity of the conditions, let us assume that $\boldsymbol{r}(t)$ is a cubic PH curve with a hodograph as stated in Corollary 2. The non-degeneracy of the degree-raised, quartic Bézier control polygon then follows as in the proof of Theorem 2, because we still have $\boldsymbol{u}_0 \neq 0$, $\boldsymbol{u}_1 \neq 0$, and $\operatorname{Im}(\boldsymbol{a}) \neq 0$, and the validity of (9) was derived above. Now, if $\boldsymbol{e}_0\boldsymbol{e}_3$ were equal to $\boldsymbol{e}_1\boldsymbol{e}_2$, then $\boldsymbol{a}^3\boldsymbol{e}_0/\boldsymbol{e}_3 = 1$, by (12a), but it also follows from (7) and Corollary 2 that $\boldsymbol{a}^2\boldsymbol{e}_0/\boldsymbol{e}_3 = \boldsymbol{a}^2\boldsymbol{u}_0/\boldsymbol{u}_1 = 1$, which implies $\boldsymbol{a} = 1$, thus contradicting the fact that $\operatorname{Im}(\boldsymbol{a}) \neq 0$. Expressing the $\boldsymbol{e}_i$ in terms of the cubic Bézier control edges $\boldsymbol{d}_i$ as in (13) and substituting $\boldsymbol{d}_2 = \boldsymbol{d}_1^2/\boldsymbol{d}_0$, a straightforward calculation shows that

$$2\frac{\boldsymbol{e}_1\boldsymbol{e}_3 - \boldsymbol{e}_2^2}{\boldsymbol{e}_0\boldsymbol{e}_3 - \boldsymbol{e}_1\boldsymbol{e}_2} = \frac{\boldsymbol{d}_1}{\boldsymbol{d}_0} = \frac{\boldsymbol{d}_2}{\boldsymbol{d}_1}, \tag{16}$$

which has a non-vanishing imaginary part, if and only if all $\boldsymbol{d}_i$ and thus also all $\boldsymbol{e}_i$ are parallel, but we just showed that the latter cannot happen. The remaining identity in (15) follows directly from (16), because $\boldsymbol{e}_0/\boldsymbol{e}_3 = \boldsymbol{d}_0/\boldsymbol{d}_2$. The sufficiency of the conditions can be shown with the same arguments as in the proof of Theorem 2. □

Theorem 4. *A properly parameterized quadratic curve $\mathbf{r}(t)$ is an iPH curve, if and only if its twice degree-raised, quartic Bézier control polygon is non-degenerate and its control edges satisfy* (9)*,* $\boldsymbol{e}_0\boldsymbol{e}_3 \neq \boldsymbol{e}_1\boldsymbol{e}_2$*, and*

$$2\frac{\boldsymbol{e}_1\boldsymbol{e}_3-\boldsymbol{e}_2^2}{\boldsymbol{e}_0\boldsymbol{e}_3-\boldsymbol{e}_1\boldsymbol{e}_2}=1 \qquad \textit{and} \qquad \operatorname{Im}\left(4\left(\frac{\boldsymbol{e}_1\boldsymbol{e}_3-\boldsymbol{e}_2^2}{\boldsymbol{e}_0\boldsymbol{e}_3-\boldsymbol{e}_1\boldsymbol{e}_2}\right)^2\frac{\boldsymbol{e}_0}{\boldsymbol{e}_3}\right)=\operatorname{Im}\left(\frac{\boldsymbol{e}_0}{\boldsymbol{e}_3}\right)\neq 0. \tag{17}$$

Proof. Let us first assume that $\boldsymbol{r}(t)$ is a quadratic iPH curve with a hodograph as stated in Corollary 3 and quadratic Bézier control edges $\boldsymbol{c}_0$, $\boldsymbol{c}_1$. After degree elevation, the control edges of the quartic Bézier control polygon are

$$\boldsymbol{e}_0=\frac{1}{2}\boldsymbol{c}_0, \qquad \boldsymbol{e}_1=\frac{1}{3}\boldsymbol{c}_0+\frac{1}{6}\boldsymbol{c}_1, \qquad \boldsymbol{e}_2=\frac{1}{6}\boldsymbol{c}_0+\frac{1}{3}\boldsymbol{c}_1, \qquad \boldsymbol{e}_3=\frac{1}{2}\boldsymbol{c}_1. \tag{18}$$

As before, it is clear that $\boldsymbol{e}_0\neq 0$ and $\boldsymbol{e}_3\neq 0$. Moreover, since $\operatorname{Im}(\boldsymbol{e}_0/\boldsymbol{e}_3)=\operatorname{Im}(\boldsymbol{u}_0/\boldsymbol{u}_1)\neq 0$ by (7) and Corollary 3, we conclude that at least $\boldsymbol{e}_0$ and $\boldsymbol{e}_3$ are not parallel, hence the quartic control polygon is non-degenerate. We already discussed above that (9) holds; to show the remaining conditions, we use (18) to get

$$\boldsymbol{e}_0\boldsymbol{e}_3-\boldsymbol{e}_1\boldsymbol{e}_2=2(\boldsymbol{e}_0\boldsymbol{e}_2-\boldsymbol{e}_1^2)=2(\boldsymbol{e}_1\boldsymbol{e}_3-\boldsymbol{e}_2^2)=-\frac{1}{18}(\boldsymbol{c}_1-\boldsymbol{c}_0)^2$$

and recall $\operatorname{Im}(\boldsymbol{e}_0/\boldsymbol{e}_3)=\operatorname{Im}(\boldsymbol{c}_0/\boldsymbol{c}_1)\neq 0$, which implies $\boldsymbol{c}_0\neq\boldsymbol{c}_1$ and further $\boldsymbol{e}_0\boldsymbol{e}_3\neq\boldsymbol{e}_1\boldsymbol{e}_2$, as well as (17). The sufficiency of the conditions can again be shown with the same arguments as in the proof of Theorem 2. □

3.3 Quartic PH and class I.1 quartic iPH curves

Let us now extend the results above to quartic PH and class I.1 quartic iPH curves. We first observe that Corollary 1 can be extended to both kinds of curves, if we exclude the occurrence of singularities at $t=0$ and $t=1$.

Corollary 4. *A properly parameterized quartic curve $\mathbf{r}(t)$ is a PH curve with $\mathbf{r}'(0)\neq 0$ and $\mathbf{r}'(1)\neq 0$, if and only if its hodograph can be written in complex form as in* (5) *with $\boldsymbol{u}_0\neq 0$, $\boldsymbol{u}_1\neq 0$, $\operatorname{Im}(\boldsymbol{a})\neq 0$, $\operatorname{Im}(\boldsymbol{a}^2\boldsymbol{u}_0/\boldsymbol{u}_1)=0$, and $\boldsymbol{a}^2\boldsymbol{u}_0\neq\boldsymbol{u}_1$.*

Proof. We first recall from [10] that the hodograph of a properly parameterized quartic PH curve can be written as in (2) with either $\boldsymbol{k}=0$, $\deg(p)=1$ or $\boldsymbol{k}\neq 0$, $\operatorname{Im}(\boldsymbol{k})=0$, $\deg(p)=0$, and $\deg(w)=1$, that is, as

$$\boldsymbol{r}'(t)=(t-a)(\boldsymbol{w}_0(1-t)+\boldsymbol{w}_1 t)^2 \tag{19}$$

for some $a\in\mathbb{R}$ and $\boldsymbol{w}_0,\boldsymbol{w}_1\in\mathbb{C}$ with $\boldsymbol{w}_0\neq 0$, $\boldsymbol{w}_1\neq 0$, and $\operatorname{Im}(\boldsymbol{w}_1/\boldsymbol{w}_0)\neq 0$. This representation shows that quartic PH curves have a cusp at $t=a$, hence $a\notin\{0,1\}$, because of the restrictions stated above. It then follows that we can rewrite $\boldsymbol{r}'(t)$ in the form (5) by letting $\boldsymbol{u}_0=-a\boldsymbol{w}_0^2$, $\boldsymbol{u}_1=(1-a)\boldsymbol{w}_1^2$, and $\boldsymbol{a}=\boldsymbol{w}_1/\boldsymbol{w}_0$. Vice versa, if $\boldsymbol{r}'(t)$ is given as in (5), then we get back to the form in (19) by letting $a=\boldsymbol{a}^2\boldsymbol{u}_0/(\boldsymbol{a}^2\boldsymbol{u}_0-\boldsymbol{u}_1)$, $\boldsymbol{w}_0=\pm\sqrt{\boldsymbol{u}_1-\boldsymbol{a}^2\boldsymbol{u}_0}/\boldsymbol{a}$, and $\boldsymbol{w}_1=\boldsymbol{a}\boldsymbol{w}_0$. Note that the additional conditions on a, $\boldsymbol{w}_0$, $\boldsymbol{w}_1$ above and on $\boldsymbol{u}_0$, $\boldsymbol{u}_1$, $\boldsymbol{a}$ in the statement imply each other. □

Corollary 5. *A properly parameterized quartic curve $\mathbf{r}(t)$ is an iPH curve of class I.1 with $\mathbf{r}'(0)\neq 0$ and $\mathbf{r}'(1)\neq 0$, if and only if its hodograph can be written in complex form as in* (5) *with $\boldsymbol{u}_0\neq 0$, $\boldsymbol{u}_1\neq 0$, $\operatorname{Im}(\boldsymbol{a})=0$, $\boldsymbol{a}\notin\{0,1\}$, and $\operatorname{Im}(\boldsymbol{a}^2\boldsymbol{u}_0/\boldsymbol{u}_1)\neq 0$.*

Proof. If $\boldsymbol{r}'(t)$ is given as in (3) with $a\notin\{0,1\}$ and $b=0$, then we can rewrite it as in (5) by letting $\boldsymbol{u}_0=a^2\boldsymbol{w}$, $\boldsymbol{u}_1=(1-a)^2(1+\boldsymbol{k})\boldsymbol{w}$, and $\boldsymbol{a}=(a-1)/a$. Vice versa, we can convert a hodograph from the form in (5) to the form in (3) by letting $a=1/(1-\boldsymbol{a})$, $b=0$, $\boldsymbol{k}=(\boldsymbol{u}_1-\boldsymbol{a}^2\boldsymbol{u}_0)/(\boldsymbol{a}^2\boldsymbol{u}_0)$, and $\boldsymbol{w}=(1-\boldsymbol{a})^2\boldsymbol{u}_0$. As in the previous proofs, the additional conditions imply each other. □

As before, these corollaries pave the way for an algebraic characterization of these kind of quartic curves in terms of the Bézier control edges.

Theorem 5. *A properly parameterized quartic curve $\mathbf{r}(t)$ is a PH curve with $\mathbf{r}'(0)\neq 0$ and $\mathbf{r}'(1)\neq 0$, if and only if its Bézier control polygon is non-degenerate and its control edges satisfy* (9)*,* $\boldsymbol{e}_0\boldsymbol{e}_3\neq\boldsymbol{e}_1\boldsymbol{e}_2$*, and*

$$\operatorname{Im}\left(2\frac{\boldsymbol{e}_1\boldsymbol{e}_3-\boldsymbol{e}_2^2}{\boldsymbol{e}_0\boldsymbol{e}_3-\boldsymbol{e}_1\boldsymbol{e}_2}\right)\neq 0 \qquad \textit{and} \qquad 4\left(\frac{\boldsymbol{e}_1\boldsymbol{e}_3-\boldsymbol{e}_2^2}{\boldsymbol{e}_0\boldsymbol{e}_3-\boldsymbol{e}_1\boldsymbol{e}_2}\right)^2\frac{\boldsymbol{e}_0}{\boldsymbol{e}_3}\in\mathbb{R}\setminus\{1\}. \tag{20}$$

Proof. Everything can be shown as in the proof of Theorem 3, except for the necessity of the conditions in (20). The first of these conditions follows exactly as in the proof of Theorem 2, and the second is a direct consequence from Corollary 4, which guarantees that $\boldsymbol{a}^2\boldsymbol{u}_0/\boldsymbol{u}_1 \in \mathbb{R}\setminus\{1\}$. □

Theorem 6. *A properly parameterized quartic curve $\boldsymbol{r}(t)$ is an iPH curve of class I.1 with $\boldsymbol{r}'(0)\neq 0$ and $\boldsymbol{r}'(1)\neq 0$, if and only if its Bézier control polygon is non-degenerate and its control edges satisfy* (9)*, $\boldsymbol{e}_0\boldsymbol{e}_3 \neq \boldsymbol{e}_1\boldsymbol{e}_2$, and*

$$2\frac{\boldsymbol{e}_1\boldsymbol{e}_3-\boldsymbol{e}_2^2}{\boldsymbol{e}_0\boldsymbol{e}_3-\boldsymbol{e}_1\boldsymbol{e}_2} \in \mathbb{R}\setminus\{1\} \qquad \textit{and} \qquad \operatorname{Im}\left(4\left(\frac{\boldsymbol{e}_1\boldsymbol{e}_3-\boldsymbol{e}_2^2}{\boldsymbol{e}_0\boldsymbol{e}_3-\boldsymbol{e}_1\boldsymbol{e}_2}\right)^2\frac{\boldsymbol{e}_0}{\boldsymbol{e}_3}\right)\neq 0. \tag{21}$$

Proof. Again, the proof is similar to the ones above and hinges on the conditions related to $\boldsymbol{a}$ in Corollary 5. In particular, the control polygon is non-degenerate, because $\operatorname{Im}(\boldsymbol{u}_0/\boldsymbol{u}_1) = \operatorname{Im}(\boldsymbol{a}^2\boldsymbol{u}_0/\boldsymbol{u}_1)/\boldsymbol{a}^2 \neq 0$, as in the proof of Theorem 4, and the case $\boldsymbol{e}_0\boldsymbol{e}_3 = \boldsymbol{e}_1\boldsymbol{e}_2$, which is equivalent to $\boldsymbol{a}^3\boldsymbol{e}_0/\boldsymbol{e}_3 = \boldsymbol{a}^3\boldsymbol{u}_0/\boldsymbol{u}_1 = 1$ is ruled out by this property, too. □

3.4 C^1 Hermite interpolation

An important consequence of Theorems 2–6 is that the C^1 Hermite interpolation problem can always be solved by a quartic Bézier curve that admits rational offsets. Given two points $\boldsymbol{q}_0, \boldsymbol{q}_1$ and two non-vanishing tangent vectors $\boldsymbol{t}_0, \boldsymbol{t}_1$ in the plane, the C^1 *Hermite interpolation problem* consists of finding a curve $\boldsymbol{r}(t)$ that interpolates this data at its endpoints. For a quartic Bézier curve, these constraints are met, if and only if

$$\boldsymbol{p}_0 = \boldsymbol{q}_0, \qquad \boldsymbol{p}_1 = \boldsymbol{q}_0 + \boldsymbol{t}_0/4, \qquad \boldsymbol{p}_3 = \boldsymbol{q}_1 - \boldsymbol{t}_1/4, \qquad \boldsymbol{p}_4 = \boldsymbol{q}_1, \tag{22}$$

and the remaining control point $\boldsymbol{p}_2$ may be used to guarantee that the curve has rational offsets. Without loss of generality, let us assume that the four control points in (22) are not collinear, since otherwise the problem is solved by a straight line, which clearly admits rational offsets. It then remains to substitute

$$\boldsymbol{e}_0 = \boldsymbol{t}_0/4, \qquad \boldsymbol{e}_1 = \boldsymbol{p}_2 - \boldsymbol{p}_1, \qquad \boldsymbol{e}_2 = \boldsymbol{p}_3 - \boldsymbol{p}_2, \qquad \boldsymbol{e}_3 = \boldsymbol{t}_1/4,$$

in (9), using the values of $\boldsymbol{p}_1$ and $\boldsymbol{p}_3$ from (22), yielding a quartic equation in the unknown $\boldsymbol{p}_2$. This equation can be solved algebraically, with at least one and up to four solutions. Note that the existence of these solutions was also derived by Lü [14], albeit by resorting to a different quartic equation in an auxiliary variable $\boldsymbol{x}$, and the advantage of our approach is that we compute $\boldsymbol{p}_2$ directly.

Due to the assumption above, each solution corresponds to a quartic Bézier curve $\boldsymbol{r}(t)$ with a non-degenerate control polygon, and Theorems 2–6 allow us to distinguish the following cases:

- If $\boldsymbol{e}_0\boldsymbol{e}_3 = \boldsymbol{e}_1\boldsymbol{e}_2$, then $\boldsymbol{r}(t)$ is a class II quartic iPH curve with control edges in geometric progression.
- If $\boldsymbol{e}_0\boldsymbol{e}_3 \neq \boldsymbol{e}_1\boldsymbol{e}_2$, then $\boldsymbol{a} = 2(\boldsymbol{e}_1\boldsymbol{e}_3 - \boldsymbol{e}_2^2)/(\boldsymbol{e}_0\boldsymbol{e}_3 - \boldsymbol{e}_1\boldsymbol{e}_2)$ is well-defined and nonzero, and we have five sub-cases:
  - If $\operatorname{Im}(\boldsymbol{e}_0\boldsymbol{a}^2/\boldsymbol{e}_3) \neq 0$ and $\operatorname{Im}(\boldsymbol{a}) \neq 0$, then $\boldsymbol{r}(t)$ is a class II quartic iPH curve.
  - If $\operatorname{Im}(\boldsymbol{e}_0\boldsymbol{a}^2/\boldsymbol{e}_3) \neq 0$ and $\boldsymbol{a} \in \mathbb{R}\setminus\{1\}$, then $\boldsymbol{r}(t)$ is a class I.1 quartic iPH curve with a tangent continuous singularity at $t = 1/(1-\boldsymbol{a})$.
  - If $\operatorname{Im}(\boldsymbol{e}_0\boldsymbol{a}^2/\boldsymbol{e}_3) \neq 0$ and $\boldsymbol{a} = 1$, then $\boldsymbol{r}(t)$ is a quadratic iPH curve.
  - If $\boldsymbol{e}_0\boldsymbol{a}^2/\boldsymbol{e}_3 \in \mathbb{R}\setminus\{0,1\}$ and $\operatorname{Im}(\boldsymbol{a}) \neq 0$, then $\boldsymbol{r}(t)$ is a quartic PH curve with a cusp at $t = \boldsymbol{e}_0\boldsymbol{a}^2/(\boldsymbol{e}_0\boldsymbol{a}^2 - \boldsymbol{e}_3) \notin \{0,1\}$.
  - If $\boldsymbol{e}_0\boldsymbol{a}^2/\boldsymbol{e}_3 = 1$ and $\operatorname{Im}(\boldsymbol{a}) \neq 0$, then $\boldsymbol{r}(t)$ is a cubic PH curve.

Note that the missing case with $\boldsymbol{e}_0\boldsymbol{e}_3 \neq \boldsymbol{e}_1\boldsymbol{e}_2$, $\operatorname{Im}(\boldsymbol{e}_0\boldsymbol{a}^2/\boldsymbol{e}_3) = 0$, and $\operatorname{Im}(\boldsymbol{a}) = 0$, with $\boldsymbol{a} \neq 0$ defined as in the second set of cases, cannot occur. Indeed, if $\operatorname{Im}(\boldsymbol{a}) = 0$, then it follows from $\operatorname{Im}(\boldsymbol{e}_0\boldsymbol{a}^2/\boldsymbol{e}_3) = 0$ that $\boldsymbol{e}_0$ and $\boldsymbol{e}_3$ are parallel. We further conclude as in the proof of Theorem 2 that $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ can be expressed as in (8), that is, as linear combinations of $\boldsymbol{e}_0$ and $\boldsymbol{e}_3$, and therefore they are also parallel to $\boldsymbol{e}_0$, which contradicts our assumption that $\boldsymbol{p}_0$, $\boldsymbol{p}_1$, $\boldsymbol{p}_3$, and $\boldsymbol{p}_4$ are not collinear.

Figure 3 shows several examples of quartic Bézier curves with rational offsets that interpolate the Hermite data given in Table 1 and compares them to the corresponding quintic PH interpolants. The examples confirm that all the cases listed above can occur in practice as special cases and that there are often two quartic curves that are quite similar to the visually most pleasing quintic curve (see also Figure 1). An

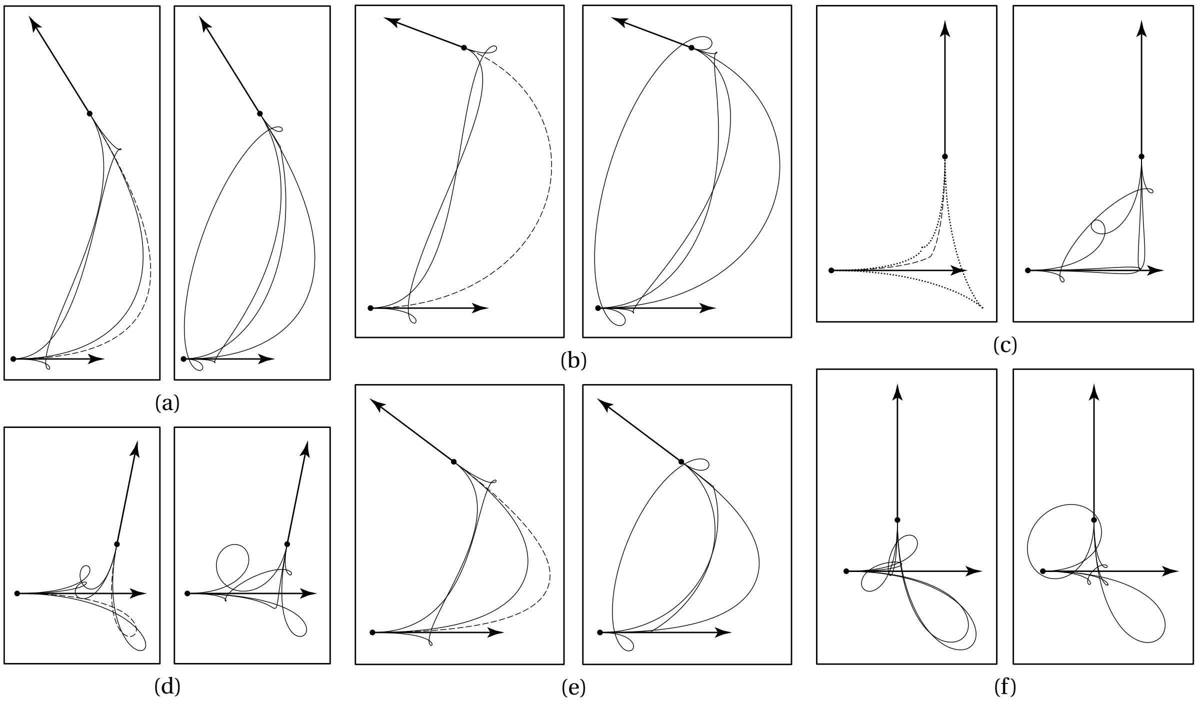

Figure 3: Examples of C^1 Hermite interpolation with quartic iPH (left) and quintic PH curves (right), including some special cases (dashed and dotted curves): (a) quadratic iPH curve, (b) class II quartic iPH curve with control edges in geometric progression, (c) irregular class I.1 quartic iPH curve with singularity at $t = 1/2$ (dashed) and irregular quartic PH curves with cusps at $t = 1/2$ (dotted), (d) cubic PH curve, (e) regular class I.1 quartic iPH curve. Note that the prescribed tangents are scaled by a factor of 1/5 to better fit the images.

	$\boldsymbol{q}_0$	$\boldsymbol{q}_1$	$\boldsymbol{t}_0$	$\boldsymbol{t}_1$
(a)	$(0,0)$	$\left(\frac{1}{6},\frac{8}{15}\right)$	$(1,0)$	$\left(-\frac{2}{3},\frac{16}{15}\right)$
(b)	$(0,0)$	$\left(\frac{78857}{500000},\frac{54819}{125000}\right)$	$(1,0)$	$\left(-\frac{85293}{125000},\frac{8019}{31250}\right)$
(c)	$(0,0)$	$\left(\frac{1}{6},\frac{1}{6}\right)$	$(1,0)$	$(0,1)$
(d)	$(0,0)$	$\left(\frac{29}{75}-\frac{\sqrt{2\sqrt{26}+2}}{15},\frac{4}{15}-\frac{\sqrt{2\sqrt{26}-2}}{15}\right)$	$(1,0)$	$\left(\frac{4}{25},\frac{4}{5}\right)$
(e)	$(0,0)$	$\left(\frac{11969}{97200},\frac{523}{2025}\right)$	$(1,0)$	$\left(-\frac{16}{25},\frac{12}{25}\right)$
(f)	$(0,0)$	$\left(\frac{3}{40},\frac{3}{40}\right)$	$(1,0)$	$(0,1)$

Table 1: Data for the C^1 Hermite interpolants in Figure 3.

exception is the example in Figure 3 (c), where all quartic curves have vanishing first derivative at $t = 1/2$. Two of these curves (dotted) are quartic PH curves with cusps at $t = 1/2$, and the third curve (dashed) is a class I.1 quartic iPH curve. The latter actually corresponds to a double root of the quartic equation induced by condition (9), just like the class II quartic iPH curve with control edges in geometric progression in Figure 3 (b), which explains why there are only three different quartic Bézier curves that solve the Hermite interpolation problem in both examples.

4 Geometric considerations

In Section 3, we learned that there are several kinds of polynomial curves with degree at most four, which admit rational offsets and are characterized by the common property that the control edges from the representation as a quartic Bézier curve satisfy condition (9). We shall now turn to two different geometric characterizations of these curves, but let us first introduce a common name for uniting them.

Definition 2. We say that a properly parameterized polynomial curve $\boldsymbol{r}(t)$ is a *quartic (i)PH curve*, if it is either a class II or class I.1 quartic iPH curve, a quartic or cubic PH curve, or a quadratic iPH curve with $\boldsymbol{r}'(0) \neq 0$ and $\boldsymbol{r}'(1) \neq 0$.

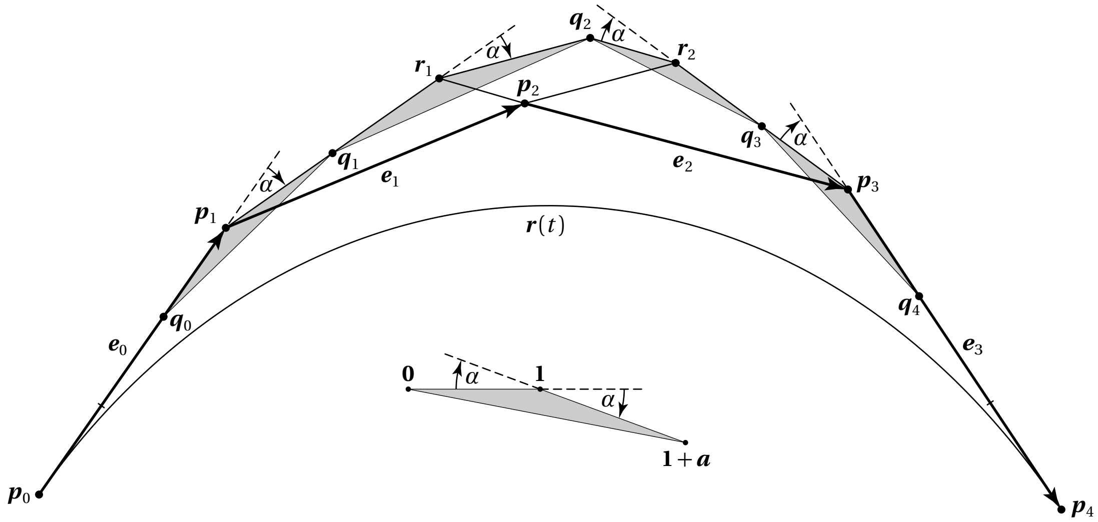

Figure 4: Notation for points, edges, and angles related to the Bézier control polygon of a quartic (i)PH curve used in Theorem 7.

Note that the additional conditions on the regularity of $\boldsymbol{r}(t)$ at $t = 0$ and $t = 1$ avoid that the edges $\boldsymbol{e}_0$ and $\boldsymbol{e}_3$ of the quartic Bézier control polygon vanish and that they apply only to class I.1 quartic iPH and quartic PH curves, since the other curves are regular for all $t \in \mathbb{R}$.

4.1 First characterization

Theorem 7. *A properly parameterized quartic Bézier curve $\boldsymbol{r}(t)$ with non-degenerate control polygon is a quartic (i)PH curve, if and only if there exist two points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ such that the four triangles*

$$[\boldsymbol{q}_0,\boldsymbol{p}_1,\boldsymbol{q}_1],\qquad [\boldsymbol{q}_1,\boldsymbol{r}_1,\boldsymbol{q}_2],\qquad [\boldsymbol{q}_2,\boldsymbol{r}_2,\boldsymbol{q}_3],\qquad [\boldsymbol{q}_3,\boldsymbol{p}_3,\boldsymbol{q}_4] \tag{23}$$

are similar[1]*, where*

$$\boldsymbol{q}_0=\frac{\boldsymbol{p}_0+2\boldsymbol{p}_1}{3},\qquad \boldsymbol{q}_1=\frac{\boldsymbol{p}_1+\boldsymbol{r}_1}{2},\qquad \boldsymbol{q}_2=\boldsymbol{r}_1+\boldsymbol{r}_2-\boldsymbol{p}_2,\qquad \boldsymbol{q}_3=\frac{\boldsymbol{r}_2+\boldsymbol{p}_3}{2},\qquad \boldsymbol{q}_4=\frac{2\boldsymbol{p}_3+\boldsymbol{p}_4}{3}, \tag{24}$$

as shown in Figure 4.

Proof. We first prove the necessity and assume, by Corollaries 1–5 and (7), that the hodograph of $\boldsymbol{r}(t)$ can be written as in (5) for some $\boldsymbol{a} \neq 0$ and with $\boldsymbol{u}_0 = 4\boldsymbol{e}_0 \neq 0$ and $\boldsymbol{u}_1 = 4\boldsymbol{e}_3 \neq 0$. Letting $\boldsymbol{r}_1 = \boldsymbol{p}_1 + \frac{2}{3}\boldsymbol{a}\,\boldsymbol{e}_0$ and $\boldsymbol{r}_2 = \boldsymbol{p}_3 - \frac{2}{3}\boldsymbol{a}^{-1}\boldsymbol{e}_3$, we then use (8) and the definition of the $\boldsymbol{q}_i$ in (24) to get

$$\frac{\boldsymbol{q}_1-\boldsymbol{p}_1}{\boldsymbol{p}_1-\boldsymbol{q}_0}=\frac{\frac{1}{3}\boldsymbol{a}\,\boldsymbol{e}_0}{\frac{1}{3}(\boldsymbol{p}_1-\boldsymbol{p}_0)}=\boldsymbol{a},\qquad \frac{\boldsymbol{q}_2-\boldsymbol{r}_1}{\boldsymbol{r}_1-\boldsymbol{q}_1}=\frac{\boldsymbol{e}_2-\frac{2}{3}\boldsymbol{a}^{-1}\boldsymbol{e}_3}{\frac{1}{3}\boldsymbol{a}\,\boldsymbol{e}_0}=\frac{\frac{1}{3}\boldsymbol{a}^2\boldsymbol{e}_0}{\frac{1}{3}\boldsymbol{a}\,\boldsymbol{e}_0}=\boldsymbol{a},$$

which, according to the triangle similarity test with two sides and included angle (SAS), implies that the triangles $[\boldsymbol{q}_0,\boldsymbol{p}_1,\boldsymbol{q}_1]$ and $[\boldsymbol{q}_1,\boldsymbol{r}_1,\boldsymbol{q}_2]$ are both similar to the triangle $[\boldsymbol{0},\boldsymbol{1},\boldsymbol{1}+\boldsymbol{a}]$, where $\boldsymbol{0}=(0,0)$ and $\boldsymbol{1}=(1,0)$. The similarity of the triangles $[\boldsymbol{q}_2,\boldsymbol{r}_2,\boldsymbol{q}_3]$ and $[\boldsymbol{q}_3,\boldsymbol{p}_3,\boldsymbol{q}_4]$ to $[\boldsymbol{0},\boldsymbol{1},\boldsymbol{1}+\boldsymbol{a}]$ can be shown analogously.

To prove the sufficiency, we first conclude from the similarity of the triangles in (23) that the four ratios

$$\boldsymbol{a}_1=\frac{\boldsymbol{q}_1-\boldsymbol{p}_1}{\boldsymbol{p}_1-\boldsymbol{q}_0},\qquad \boldsymbol{a}_2=\frac{\boldsymbol{q}_2-\boldsymbol{r}_1}{\boldsymbol{r}_1-\boldsymbol{q}_1},\qquad \boldsymbol{a}_3=\frac{\boldsymbol{q}_3-\boldsymbol{r}_2}{\boldsymbol{r}_2-\boldsymbol{q}_2},\qquad \boldsymbol{a}_4=\frac{\boldsymbol{q}_4-\boldsymbol{p}_3}{\boldsymbol{p}_3-\boldsymbol{q}_3} \tag{25}$$

are all equal to a common value $\boldsymbol{a}=\boldsymbol{a}_1=\boldsymbol{a}_2=\boldsymbol{a}_3=\boldsymbol{a}_4$. By the definition of the $\boldsymbol{q}_i$ in (24), we then have

$$\boldsymbol{r}_1-\boldsymbol{p}_1=2(\boldsymbol{q}_1-\boldsymbol{p}_1)=2\boldsymbol{a}_1(\boldsymbol{p}_1-\boldsymbol{q}_0)=\frac{2}{3}\boldsymbol{a}\,\boldsymbol{e}_0$$

[1] Note that we call two triangles *similar*, if one can be obtained from the other by translation, rotation, and uniform scaling, but not by reflection. That is, both triangles have the same shape and orientation.

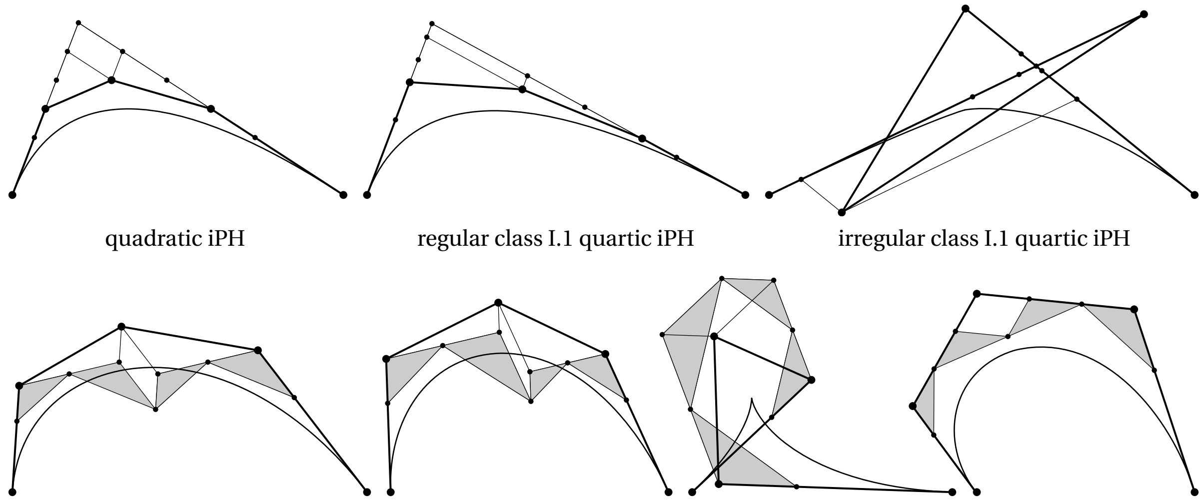

Figure 5: Special cases of quartic (i)PH curves, corresponding to the characterization in Theorem 7.

and

$$\boldsymbol{p}_2-\boldsymbol{r}_1=\boldsymbol{r}_2-\boldsymbol{q}_2=\boldsymbol{a}_3^{-1}(\boldsymbol{q}_3-\boldsymbol{r}_2)=\boldsymbol{a}_3^{-1}(\boldsymbol{p}_3-\boldsymbol{q}_3)=\boldsymbol{a}_3^{-1}\boldsymbol{a}_4^{-1}(\boldsymbol{q}_4-\boldsymbol{p}_3)=\frac{1}{3}\boldsymbol{a}^{-2}\boldsymbol{e}_3.$$

Therefore,

$$\boldsymbol{e}_1=\boldsymbol{p}_2-\boldsymbol{p}_1=\frac{2}{3}\boldsymbol{a}\,\boldsymbol{e}_0+\frac{1}{3}\boldsymbol{a}^{-2}\boldsymbol{e}_3,$$

which is the first condition in (8), and the second condition can be shown similarly. □

If $\boldsymbol{r}(t)$ is a quartic Bézier curve that satisfies the geometric conditions in Theorem 7, then we can further identify a number of special cases (see Figure 5). On the one hand, if the common value $\boldsymbol{a}=\boldsymbol{a}_1=\boldsymbol{a}_2=\boldsymbol{a}_3=\boldsymbol{a}_4$ of the ratios in (25) has a vanishing imaginary part, then it follows from Corollaries 3 and 5 that $\boldsymbol{r}(t)$ is either a quadratic iPH curve, if $\boldsymbol{a}=1$, or a class I.1 quartic iPH curve, if $\boldsymbol{a}\neq 1$. In both cases, the points $\boldsymbol{p}_0$, $\boldsymbol{p}_1$, $\boldsymbol{q}_1$, $\boldsymbol{r}_1$, $\boldsymbol{q}_2$ are collinear and partition the line segment $[\boldsymbol{p}_0,\boldsymbol{q}_2]$ in the ratios $3:\boldsymbol{a}:\boldsymbol{a}:\boldsymbol{a}^2$, and the points $\boldsymbol{q}_2$, $\boldsymbol{r}_2$, $\boldsymbol{q}_3$, $\boldsymbol{p}_3$, $\boldsymbol{p}_4$ are collinear and partition $[\boldsymbol{q}_2,\boldsymbol{p}_4]$ in the ratios $1:\boldsymbol{a}:\boldsymbol{a}:3\boldsymbol{a}^2$. Moreover, the curve is regular for $t\in[0,1]$, if and only if $\boldsymbol{a}>0$.

On the other hand, if the vectors $\boldsymbol{r}_1-\boldsymbol{p}_1=\frac{2}{3}\boldsymbol{a}\,\boldsymbol{e}_0$ and $\boldsymbol{p}_3-\boldsymbol{r}_2=\frac{2}{3}\boldsymbol{a}^{-1}\boldsymbol{e}_3$ are parallel, then $\boldsymbol{a}\,\boldsymbol{e}_0=\lambda\boldsymbol{a}^{-1}\boldsymbol{e}_3$ for some $\lambda\in\mathbb{R}\setminus\{0\}$, hence $\boldsymbol{a}^2\boldsymbol{u}_0=\lambda\boldsymbol{u}_1$, and it follows from Corollaries 2 and 4 that $\boldsymbol{r}(t)$ is either a cubic PH curve, if $\lambda=1$, or a quartic PH curve, if $\lambda\neq 1$. The curve is regular for $t\in[0,1]$, if and only if $\lambda>0$, that is, whenever $\boldsymbol{r}_1-\boldsymbol{p}_1$ and $\boldsymbol{p}_3-\boldsymbol{r}_2$ do not point in opposite directions.

Finally, if $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, and $\boldsymbol{r}_1$ are collinear, with $\boldsymbol{r}_1$ splitting the segment $[\boldsymbol{p}_1,\boldsymbol{p}_2]$ in the ratio $2\mu:1$ for some $\mu>0$, then $\boldsymbol{p}_2$, $\boldsymbol{p}_3$, and $\boldsymbol{r}_2$ are also collinear, with $\boldsymbol{r}_2$ splitting the segment $[\boldsymbol{p}_2,\boldsymbol{p}_3]$ in the ratio $\mu:2$, and vice versa. This indicates the case of a curve, for which all exterior angles of the control polygon are equal to the argument of $\boldsymbol{a}$, and the special case of a class II quartic iPH curve with control edges in geometric progression occurs if and only if $\mu=1$.

4.1.1 Construction of quartic (i)PH curves

Moreover, we would like to point out that the geometric characterization in Theorem 7 can also be used to construct the control polygons of quartic (i)PH curves. To this end, we may start with arbitrary control points $\boldsymbol{p}_0$, $\boldsymbol{p}_1\neq\boldsymbol{p}_0$, $\boldsymbol{p}_2$, and choose some $\boldsymbol{a}\neq 0$. We then let $\boldsymbol{q}_0$ be the point that splits $[\boldsymbol{p}_0,\boldsymbol{p}_1]$ in the ratio $2:1$ and construct $\boldsymbol{q}_1$ such that $[\boldsymbol{q}_0,\boldsymbol{p}_1,\boldsymbol{q}_1]$ is similar to $T=[\mathbf{0},\mathbf{1},\mathbf{1}+\boldsymbol{a}]$. Adding $\boldsymbol{q}_1-\boldsymbol{p}_1$ to $\boldsymbol{q}_1$ gives $\boldsymbol{r}_1$ and $\boldsymbol{q}_2$ is determined by the condition that also $[\boldsymbol{q}_1,\boldsymbol{r}_1,\boldsymbol{q}_2]$ must be similar to T. We further construct $\boldsymbol{r}_2$ such that $[\boldsymbol{r}_1,\boldsymbol{p}_2,\boldsymbol{r}_2,\boldsymbol{q}_2]$ is a parallelogram, $\boldsymbol{q}_3$ according to the similarity of $[\boldsymbol{q}_2,\boldsymbol{r}_2,\boldsymbol{q}_3]$ to T, and $\boldsymbol{p}_3$ by adding $\boldsymbol{q}_3-\boldsymbol{r}_2$ to $\boldsymbol{q}_3$. The condition that $[\boldsymbol{q}_3,\boldsymbol{p}_3,\boldsymbol{q}_4]$ must be similar to T specifies $\boldsymbol{q}_4$, and we finally get $\boldsymbol{p}_4$ by adding $2(\boldsymbol{q}_4-\boldsymbol{p}_3)$ to $\boldsymbol{q}_4$.

Alternatively, we may choose some $\boldsymbol{r}_1\neq\boldsymbol{p}_1$ instead of $\boldsymbol{a}$. In this case, we first construct $\boldsymbol{q}_0$ as before and $\boldsymbol{q}_1$ as the midpoint between $\boldsymbol{p}_1$ and $\boldsymbol{r}_1$. The rest of the construction remains the same, except that we use $[\boldsymbol{q}_0,\boldsymbol{p}_1,\boldsymbol{q}_1]$ as the triangle T to which the other three triangles must be similar. Note that we do not need $\boldsymbol{a}$ in this approach, but that it can be constructed or computed easily. In fact, the signed angle $\alpha = \sphericalangle(\boldsymbol{p}_1 - \boldsymbol{q}_0, \boldsymbol{q}_1 - \boldsymbol{p}_1)$ gives the argument of $\boldsymbol{a}$ (see Figure 4) and the inverse of the ratio between the lengths of these vectors gives the modulus of $\boldsymbol{a}$, and any of the ratios in (25) can be used to determine $\boldsymbol{a}$ in terms of complex arithmetic.

In both cases, the considerations above can be taken into account for constructing the quartic Bézier control polygons of quadratic and class I.1 quartic iPH curves, cubic and quartic PH curves, and class II quartic iPH curves with control edges in geometric progression by appropriately constraining the choice of $\boldsymbol{a}$ or the position of $\boldsymbol{r}_1$.

4.2 Second characterization

Let us now turn to the second geometric characterization and start with an auxiliary result.

Corollary 6. *A properly parameterized quartic Bézier curve $\boldsymbol{r}(t)$ with non-degenerate control polygon is a quartic (i)PH curve, if and only if its hodograph can be written in complex form as*

$$\boldsymbol{r}'(t) = ((1-t) + \boldsymbol{b}^2 t)\left(\boldsymbol{v}_0(1-t) + \frac{\boldsymbol{v}_1}{\boldsymbol{b}} t\right)^2 \tag{26}$$

with $\boldsymbol{v}_0 \neq 0$, $\boldsymbol{v}_1 \neq 0$, $\boldsymbol{b} \neq 0$, and $\arg(\boldsymbol{b}) \in (-\frac{\pi}{2}, \frac{\pi}{2}]$.

Proof. We recall from Corollaries 1–5, that $\boldsymbol{r}(t)$ is a quartic (i)PH curve, if and only if its hodograph can be expressed as in (5) for some $\boldsymbol{u}_0 \neq 0$, $\boldsymbol{u}_1 \neq 0$, and $\boldsymbol{a} \neq 0$. Given this form, we can rewrite $\boldsymbol{r}'(t)$ as in (26) by letting $\boldsymbol{v}_0 = \pm\sqrt{\boldsymbol{u}_0}$, $\boldsymbol{v}_1 = \pm\sqrt{\boldsymbol{u}_1}$, $\boldsymbol{b} = \boldsymbol{a}^{-1}\boldsymbol{v}_1/\boldsymbol{v}_0$ and choosing the signs of $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$ appropriately such that $\arg(\boldsymbol{b}) \in (-\frac{\pi}{2}, \frac{\pi}{2}]$. Vice versa, if $\boldsymbol{r}'(t)$ is given as in (26), then we get back to the form in (5) by letting $\boldsymbol{u}_0 = \boldsymbol{v}_0^2$, $\boldsymbol{u}_1 = \boldsymbol{v}_1^2$, and $\boldsymbol{a} = \boldsymbol{b}^{-1}\boldsymbol{v}_1/\boldsymbol{v}_0$. □

To proceed, let $\boldsymbol{h} = \sqrt{\boldsymbol{e}_0\boldsymbol{e}_3}$. In terms of vectors, $\boldsymbol{h}$ is the *halfway vector*[2] between $\boldsymbol{e}_0$ and $\boldsymbol{e}_3$, either in clockwise or counterclockwise direction, with length $\|\boldsymbol{h}\| = \sqrt{\|\boldsymbol{e}_0\|\|\boldsymbol{e}_3\|}$. We further define the lines

$$L_1 = \{\boldsymbol{p}_1 + \lambda\boldsymbol{e}_0 : \lambda \in \mathbb{R}\}, \qquad L_2 = \{\boldsymbol{p}_2 + \lambda\boldsymbol{h} : \lambda \in \mathbb{R}\}, \qquad L_3 = \{\boldsymbol{p}_3 + \lambda\boldsymbol{e}_3 : \lambda \in \mathbb{R}\},$$

as shown in Figure 6. For any $\beta \in \mathbb{R}$, we denote by $L_2(\beta)$ the line that we get after rotating L_2 by β around $\boldsymbol{p}_2$, that is $L_2(\beta) = \{\boldsymbol{p}_2 + \lambda\boldsymbol{h}\exp(\mathbf{i}\beta) : \lambda \in \mathbb{R}\}$ and consider the rays

$$R_1(\beta) = \{\boldsymbol{p}_1 + \lambda\boldsymbol{e}_0\exp(2\mathbf{i}\beta) : \lambda \in \mathbb{R}, \lambda > 0\}, \qquad R_3(\beta) = \{\boldsymbol{p}_3 - \lambda\boldsymbol{e}_3\exp(-2\mathbf{i}\beta) : \lambda \in \mathbb{R}, \lambda > 0\}.$$

Moreover, we let $\boldsymbol{s}_1(\beta)$ and $\boldsymbol{s}_2(\beta)$ be the intersections of $R_1(\beta)$ and $R_3(\beta)$ with $L_2(-\beta)$ and $L_2(\beta)$, respectively. Note that the point $\boldsymbol{s}_1(\beta)$ may not exist for certain values of β and that it is not unique, if $\boldsymbol{p}_1 \in L_2(-\beta)$ and $\boldsymbol{p}_2 \in R_1(\beta)$. In that case, $\boldsymbol{s}_1(\beta)$ may be any point of $R_1(\beta)$, and likewise for $\boldsymbol{s}_2(\beta)$.

Theorem 8. *A properly parameterized quartic Bézier curve $\boldsymbol{r}(t)$ with non-degenerate control polygon is a quartic (i)PH curve, if and only if there exists some $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\boldsymbol{s}_1(\beta)$ and $\boldsymbol{s}_2(\beta)$ exist and lie on the same side of L_2, and the lengths $E_0 = \|\boldsymbol{e}_0\|$, $E_3 = \|\boldsymbol{e}_3\|$, and $F_i = \|\boldsymbol{f}_i\|$, where*

$$\boldsymbol{f}_0 = \boldsymbol{s}_1(\beta) - \boldsymbol{p}_1, \qquad \boldsymbol{f}_1 = \boldsymbol{p}_2 - \boldsymbol{s}_1(\beta), \qquad \boldsymbol{f}_2 = \boldsymbol{s}_2(\beta) - \boldsymbol{p}_2, \qquad \boldsymbol{f}_3 = \boldsymbol{p}_3 - \boldsymbol{s}_2(\beta), \tag{27}$$

satisfy the conditions

$$E_0F_2 = 3F_0F_1, \qquad E_3F_1 = 3F_2F_3, \qquad F_1F_2 = 4F_0F_3. \tag{28}$$

Proof. To prove the necessity, we recall from Corollary 6 that the hodograph of $\boldsymbol{r}(t)$ can be expressed as in (26). Comparing the hodograph in (26) with the one in (6), we observe that

$$\boldsymbol{e}_0 = \frac{1}{4}\boldsymbol{v}_0^2, \qquad \boldsymbol{e}_3 = \frac{1}{4}\boldsymbol{v}_1^2,$$

and

$$\boldsymbol{e}_1 = \frac{1}{3}\boldsymbol{b}^2\boldsymbol{e}_0 + \frac{2}{3}\boldsymbol{b}^{-1}\boldsymbol{h}, \qquad \boldsymbol{e}_2 = \frac{2}{3}\boldsymbol{b}\boldsymbol{h} + \frac{1}{3}\boldsymbol{b}^{-2}\boldsymbol{e}_3, \tag{29}$$

[2] That is, the signed angles $\sphericalangle(\boldsymbol{e}_0, \boldsymbol{h})$ and $\sphericalangle(\boldsymbol{h}, \boldsymbol{e}_3)$ are equal.

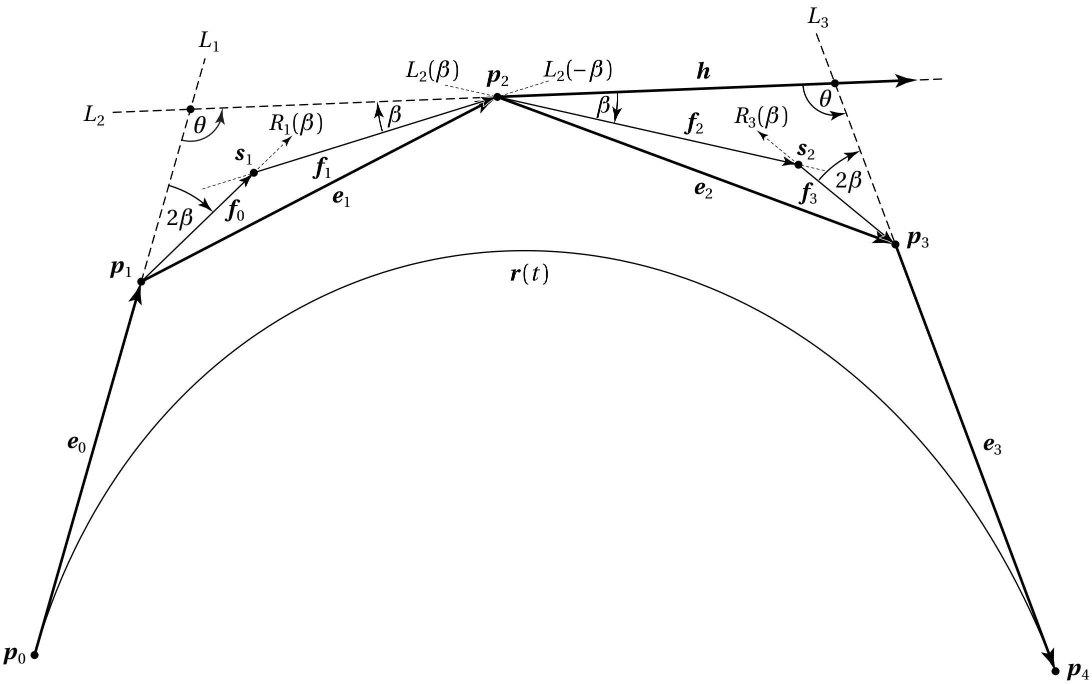

Figure 6: Notation for points, edges, angles, and lines related to the Bézier control polygon of a quartic (i)PH curve used in Theorem 8.

where $\boldsymbol{h} = \boldsymbol{v}_0\boldsymbol{v}_1/4 = \sqrt{\boldsymbol{e}_0\boldsymbol{e}_3}$. Now let $\beta = \arg(\boldsymbol{b})$, $\boldsymbol{s}_1 = \boldsymbol{p}_2 - \frac{2}{3}\boldsymbol{b}^{-1}\boldsymbol{h}$, and $\boldsymbol{s}_2 = \boldsymbol{p}_2 + \frac{2}{3}\boldsymbol{b}\,\boldsymbol{h}$. Since

$$\sphericalangle(\boldsymbol{h}, \boldsymbol{p}_2 - \boldsymbol{s}_1) = \arg\left(\frac{\boldsymbol{p}_2 - \boldsymbol{s}_1}{\boldsymbol{h}}\right) = \arg(\boldsymbol{b}^{-1}) = -\beta = -\arg(\boldsymbol{b}) = -\arg\left(\frac{\boldsymbol{s}_2 - \boldsymbol{p}_2}{\boldsymbol{h}}\right) = -\sphericalangle(\boldsymbol{h}, \boldsymbol{s}_2 - \boldsymbol{p}_2),$$

it is not only clear that $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are on the same side of L_2, but also that they lie on $L_2(-\beta)$ and $L_2(\beta)$, respectively. To see that $\boldsymbol{s}_1$ lies on $R_1(\beta)$, we let $b = \|\boldsymbol{b}\|$ and use (29) to rewrite $\boldsymbol{s}_1$ as

$$\boldsymbol{s}_1 = \boldsymbol{p}_1 + \boldsymbol{e}_1 - \frac{2}{3}\boldsymbol{b}^{-1}\boldsymbol{h} = \boldsymbol{p}_1 + \frac{1}{3}\boldsymbol{b}^2\boldsymbol{e}_0 = \boldsymbol{p}_1 + \frac{b^2}{3}\exp(2\mathbf{i}\beta)\boldsymbol{e}_0.$$

A similar argument shows that $\boldsymbol{s}_2$ lies on $R_3(\beta)$, and therefore $\boldsymbol{s}_1 = \boldsymbol{s}_1(\beta)$ and $\boldsymbol{s}_2 = \boldsymbol{s}_2(\beta)$. By (27), we then have

$$\boldsymbol{f}_0 = \frac{1}{3}\boldsymbol{b}^2\boldsymbol{e}_0, \qquad \boldsymbol{f}_1 = \frac{2}{3}\boldsymbol{b}^{-1}\boldsymbol{h}, \qquad \boldsymbol{f}_2 = \frac{2}{3}\boldsymbol{b}\,\boldsymbol{h}, \qquad \boldsymbol{f}_3 = \frac{1}{3}\boldsymbol{b}^{-2}\boldsymbol{e}_3$$

and further, since $\|\boldsymbol{h}\| = \|\sqrt{\boldsymbol{e}_0\boldsymbol{e}_3}\| = \sqrt{E_0E_3}$,

$$F_0 = \frac{b^2}{3}E_0, \qquad F_1 = \frac{2}{3b}\sqrt{E_0E_3}, \qquad F_2 = \frac{2b}{3}\sqrt{E_0E_3}, \qquad F_3 = \frac{1}{3b^2}E_3,$$

which implies (28).

For proving the sufficiency, let $b = \sqrt{3F_0/E_0}$ and $\boldsymbol{b} = b\exp(\mathbf{i}\beta)$. As $\boldsymbol{s}_1(\beta)$ lies on $R_1(\beta)$, we have $\boldsymbol{f}_0 = \lambda_0\boldsymbol{e}_0\exp(2\mathbf{i}\beta)$ for $\lambda_0 = F_0/E_0 > 0$, hence $\boldsymbol{f}_0 = \frac{1}{3}\boldsymbol{b}^2\boldsymbol{e}_0$. Similarly, we find that $\boldsymbol{f}_3 = \frac{1}{3}\boldsymbol{b}^{-2}\boldsymbol{e}_3$ after noticing that the first two conditions in (28) give $F_3 = \frac{1}{9}E_0E_3/F_0 = \frac{1}{3}E_3/b^2$. Since $\boldsymbol{s}_1(\beta)$ and $\boldsymbol{s}_2(\beta)$ lie on $L_2(-\beta)$ and $L_2(\beta)$, respectively, and are on the same side of L_2, we can assume without loss of generality that $\boldsymbol{h}$ is oriented such that $\sphericalangle(\boldsymbol{f}_1, \boldsymbol{h}) = \sphericalangle(\boldsymbol{h}, \boldsymbol{f}_2) = \beta$. Otherwise, we simply replace $\boldsymbol{h}$ by $-\boldsymbol{h}$. Therefore, $\boldsymbol{f}_1 = \lambda_1\boldsymbol{h}\exp(-\mathbf{i}\beta)$ for $\lambda_1 = F_1/h > 0$ and $\boldsymbol{f}_2 = \lambda_2\boldsymbol{h}\exp(\mathbf{i}\beta)$ for $\lambda_2 = F_2/h > 0$, where $h = \|\boldsymbol{h}\| = \|\sqrt{\boldsymbol{e}_0\boldsymbol{e}_3}\|$. We now observe that the first condition in (28) implies $F_2 = b^2F_1$, and substituting this, as well as the previous expression for F_3, into the last condition in (28) gives $b^2F_1^2 = \frac{4}{9}E_0E_3$ and further $F_1 = \frac{2}{3}h/b$. Therefore, $\boldsymbol{f}_1 = \frac{2}{3}\boldsymbol{b}^{-1}\boldsymbol{h}$ and similarly $\boldsymbol{f}_2 = \frac{2}{3}\boldsymbol{b}\,\boldsymbol{h}$, which yields the conditions in (29), because $\boldsymbol{f}_0 + \boldsymbol{f}_1 = \boldsymbol{e}_1$ and $\boldsymbol{f}_2 + \boldsymbol{f}_3 = \boldsymbol{e}_2$. Setting $\boldsymbol{v}_0 = \pm 2\sqrt{\boldsymbol{e}_0}$ and $\boldsymbol{v}_1 = \pm 2\sqrt{\boldsymbol{e}_3}$, with the signs chosen such that $\boldsymbol{v}_0\boldsymbol{v}_1 = 4\boldsymbol{h}$, we can then write the hodograph of the curve as in (26). $\square$

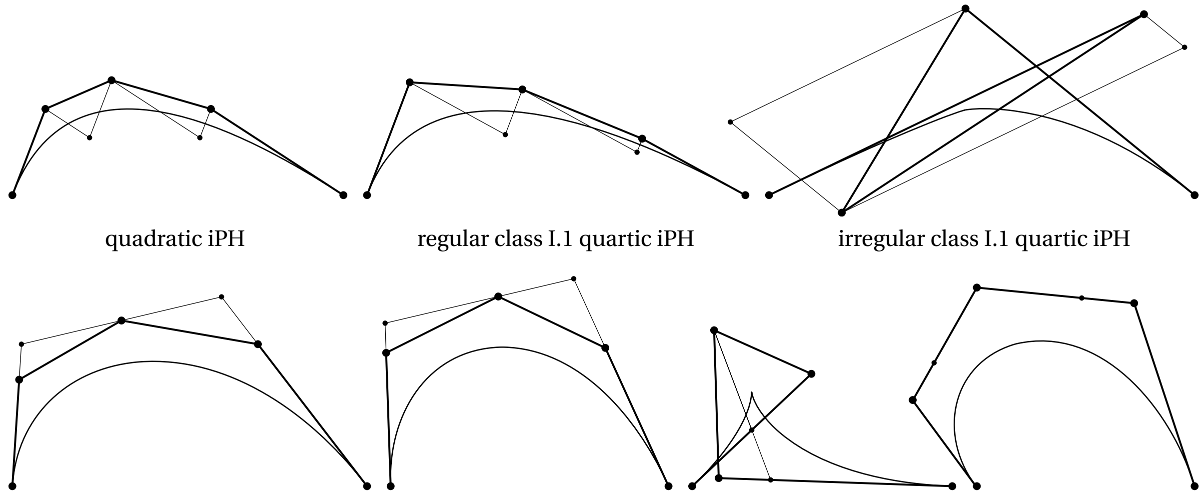

Figure 7: Special cases of quartic (i)PH curves, corresponding to the characterization in Theorem 8.

Let us now take a closer look at the special cases of quartic Bézier curves $\boldsymbol{r}(t)$ that satisfy the geometric conditions in Theorem 8 (see Figure 7). The first case occurs when $\beta = 0$, so that $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are the intersection points of L_1 and L_3 with L_2. The length conditions in (28) are then identical to the conditions in [23, Theorem 5] and reveal that $\boldsymbol{r}(t)$ is a PH curve (cf. Figure 2). More precisely, $\boldsymbol{r}(t)$ is a cubic PH curve, if $\boldsymbol{p}_2$ is the midpoint of the segment $[\boldsymbol{s}_1, \boldsymbol{s}_2]$ and, equivalently, if $\boldsymbol{p}_1$ and $\boldsymbol{p}_3$ split the segments $[\boldsymbol{p}_0, \boldsymbol{s}_1]$ and $[\boldsymbol{p}_4, \boldsymbol{s}_2]$ in the ratio $3:1$. Otherwise, $\boldsymbol{r}(t)$ is a quartic PH curve and regular for $t \in [0,1]$. The second case occurs when $\beta = \pi/2$, so that $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are the intersection points of L_1 and L_3 with the line $L_2^{\perp}$ through $\boldsymbol{p}_2$ and orthogonal to L_2. In this case, as shown in [23], $\boldsymbol{r}(t)$ is a quartic PH curve with a cusp at some $t \in (0,1)$. Note that in both cases we have $\operatorname{Im}(\boldsymbol{b}^2) = 0$, so that the previous statements also follow from Corollaries 2, 4, and 6 by observing that $\boldsymbol{b}^{-2} = \boldsymbol{a}^2 \boldsymbol{u}_0 / \boldsymbol{u}_1$. Moreover, these corollaries reveal that $\boldsymbol{r}(t)$ is a cubic PH curve, if $\boldsymbol{b} = 1$, and a quartic PH curve with a cusp at $t = 1/(1-\boldsymbol{b}^2)$, otherwise.

If $\boldsymbol{f}_1$ is parallel to $\boldsymbol{e}_0$, say $\boldsymbol{f}_1 = \frac{2}{3}\lambda \boldsymbol{e}_0$ for some $\lambda \in \mathbb{R} \setminus \{0\}$, that is, $\lambda = \boldsymbol{b}^{-1}\boldsymbol{h}/\boldsymbol{e}_0 = \boldsymbol{b}^{-1}\boldsymbol{v}_1/\boldsymbol{v}_0$, then it follows from Corollaries 3, 5, and 6 that $\boldsymbol{r}(t)$ is either a quadratic iPH curve, if $\lambda = 1$, or a class I.1 quartic iPH curve, if $\lambda \neq 1$. Moreover, the curve is regular for $t \in [0,1]$, if and only if $\lambda > 0$, that is, whenever $\boldsymbol{f}_1$ and $\boldsymbol{e}_0$ do not point in opposite directions. The initial condition on $\boldsymbol{f}_1$ is actually equivalent to the condition that $\boldsymbol{f}_2$ is parallel to $\boldsymbol{e}_3$ with $\boldsymbol{f}_2 = \frac{2}{3}\lambda^{-1}\boldsymbol{e}_3$, and both imply that $\boldsymbol{f}_0 = \frac{1}{3}\lambda^{-2}\boldsymbol{e}_3$ and $\boldsymbol{f}_3 = \frac{1}{3}\lambda^2\boldsymbol{e}_0$. Moreover, $\boldsymbol{r}(t)$ is a quadratic iPH curve, if and only if $\boldsymbol{e}_0 = \boldsymbol{f}_1 + \boldsymbol{f}_3$ and $\boldsymbol{e}_3 = \boldsymbol{f}_0 + \boldsymbol{f}_2$.

The last special case happens when $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, and $\boldsymbol{s}_1$ are collinear, with $\boldsymbol{s}_1$ splitting the segment $[\boldsymbol{p}_1, \boldsymbol{p}_2]$ in the ratio $1:2\mu$ for some $\mu > 0$. This condition turns out to be equivalent to the condition that $\boldsymbol{p}_2$, $\boldsymbol{p}_3$, and $\boldsymbol{s}_2$ are collinear, with $\boldsymbol{s}_2$ splitting the segment $[\boldsymbol{p}_2, \boldsymbol{p}_3]$ in the ratio $2:\mu$, and this case is further characterized by the condition that all exterior angles of the control polygon are equal, with $\sphericalangle(\boldsymbol{e}_{i-1}, \boldsymbol{e}_i) = 2\beta$ for $i = 1,2,3$. Moreover, $\boldsymbol{r}(t)$ is a class II quartic iPH curve with control edges in geometric progression, if and only if $\mu = 1$.

4.2.1 Construction of quartic (i)PH curves

As in Section 4.1.1, we can use the geometric characterization in Theorem 8 to construct the control polygons of quartic (i)PH curves. Like before, we start with arbitrary control points $\boldsymbol{p}_0$, $\boldsymbol{p}_1 \neq \boldsymbol{p}_0$, $\boldsymbol{p}_2$, and choose some $\boldsymbol{s}_1$ with $\boldsymbol{s}_1 \neq \boldsymbol{p}_1$ and $\boldsymbol{s}_1 \neq \boldsymbol{p}_2$. The choice of $\boldsymbol{s}_1$ uniquely determines the angle $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2}]$, as well as the lines $L_2(-\beta)$, L_2, and $L_2(\beta)$ (see Figure 6). The location of $\boldsymbol{s}_2$ is then given by the first condition in (28) and the constraint that $\boldsymbol{s}_2$ must lie on the same side of L_2 as $\boldsymbol{s}_1$, and $\boldsymbol{p}_3$ is determined by the third condition in (28) and the observation that $\sphericalangle(\boldsymbol{f}_2, \boldsymbol{f}_3)$ must be equal to $\sphericalangle(\boldsymbol{f}_0, \boldsymbol{f}_1)$. Using β, we finally construct L_3 and $\boldsymbol{p}_4$ by using the second condition in (28).

The considerations above further indicate how the special cases of quartic Bézier control polygons for cubic and quartic PH curves, quadratic and class I.1 quartic iPH curves, and for class II quartic iPH curves with control edges in geometric progression can be constructed by suitably constraining the position of $\boldsymbol{s}_1$.

5 Conclusions

In this paper, we set out to find algebraic and geometric characterizations of quartic iPH curves. While deriving a set of algebraic conditions for the subset of class II quartic iPH curves, we noticed that these conditions naturally extend to other quartic Bézier curves with rational offsets, namely quartic and cubic PH, as well as class I.1 quartic and quadratic iPH curves. After all, this is not too surprising, as these curves can be seen as "limit cases" of class II quartic iPH curves. Going back to Theorem 1 and the representation of the hodograph in (4), it is clear that $\boldsymbol{r}'(t)$ becomes the hodograph of a PH curve as $\mathrm{Im}(\boldsymbol{k}) \to 0$, and that the degree of this curve is cubic if and only if $\boldsymbol{k} \to 0$ and quartic otherwise. Similarly, $\boldsymbol{r}'(t)$ turns into the hodograph of a class I.1 quartic or quadratic iPH curve as $\mathrm{Im}(\boldsymbol{w}_1/\boldsymbol{w}_0) \to 0$, with the quadratic case occurring if and only if $\boldsymbol{w}_1/\boldsymbol{w}_0 \to 1$. This motivated us to introduce a new term and to refer to all these curves with rational offsets as *quartic (i)PH curves*. We further derived two different sets of geometric conditions that can be used for identifying and constructing quartic (i)PH curves. Note that we did not include the case when $\mathrm{Im}(\boldsymbol{k}) \to 0$ *and* $\mathrm{Im}(\boldsymbol{w}_0/\boldsymbol{w}_1) \to 0$, so that $\boldsymbol{r}'(t)$ is the product of a real polynomial $p(t)$ of degree at most 3 with a complex constant $\boldsymbol{w} \neq 0$, since it corresponds to the trivial case where the curve $\boldsymbol{r}(t)$ is a line segment. Our algebraic and geometric characterizations also do not cover class I.0 and class I.2 quartic iPH curves, since they are not "limit cases" of class II quartic iPH curves, and it remains future work to analyse them.

Quartic (i)PH curves constitute an interesting family of polynomial curves with rational offsets, as they can be used for solving the general C^1 Hermite interpolation problem and thus offer a viable alternative to quintic PH curves, sharing with the latter the fact that there can be up to four solutions (see Figure 1). This allows a curve designer to select from eight instead of only four curves and to find the "best" Hermite interpolant with rational offsets, for example, by considering the *absolute rotation index* [9] or the *elastic bending energy* [7]. It remains future work to investigate under which conditions quartic (i)PH curves are "better" than quintic PH curves and vice versa. More analysis is also needed for identifying upfront, if one or more of the quartic (i)PH Hermite interpolants might be irregular, which can happen for quartic PH or iPH curves of class I.1. We should further stress that the rational offsets of quartic (i)PH curves are generally of degree 14, compared to the degree 9 rational offsets of quintic PH curves, which may be considered a disadvantage of quartic (i)PH curves.

Last but not least we would like to return to our observation that a special case of quartic (i)PH curves are the quartic Bézier curves with control edges in geometric progression, a property which also happens to characterize cubic PH curves. This is actually not coincidental and carries over to Bézier curves of arbitrary degree. In fact, if $\boldsymbol{r}(t)$ is a non-degenerate Bézier curve with control edges $\boldsymbol{e}_i = \boldsymbol{a}^i \boldsymbol{e}_0$ for $i = 1, \ldots, n$ and some constant $\boldsymbol{a}$ with $\mathrm{Im}(\boldsymbol{a}) \neq 0$, then the hodograph of the curve is

$$\boldsymbol{r}'(t) = n \sum_{i=0}^{n-1} \boldsymbol{a}^i \boldsymbol{e}_0 B_i^{n-1}(t) = n((1-t) + \boldsymbol{a}\, t)^{n-1} \boldsymbol{e}_0,$$

which shows, by Theorem 1, that $\boldsymbol{r}(t)$ is a PH curve for n odd and an iPH curve for n even.

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