# Fitting Free Form Surfaces

Kai Hormann[*]

*University of Erlangen*

The problem of reconstructing smooth surfaces from discrete scattered data arises in many fields of science and engineering and has now been studied thoroughly for nearly 40 years. The data sources include measured values (meteorology, oceanography, optics, geodetics, geology, laser range scanning, etc.) as well as experimental results (from physical, chemical or engineering experiments) and computational values (evaluation of mathematical functions, finite element solutions of partial differential equations or results of other numerical simulations). Due to the vast variety of data sources many different methods have been developed, each of them more or less suited to a specific problem.

In the field of geology, meteorology, cartography, a. o., the problem can typically be stated as follows: given data points $(x_i, y_i, z_i) \in \mathbb{R}^3$, find a scalar function $F : \mathbb{R}^2 \to \mathbb{R}$ that approximates or interpolates the value $z_i$ at $(x_i, y_i)$, i.e. $F(x_i, y_i) \approx z_i$. This problem is generally known as *Scattered Data Interpolation* (cf. Figure 1) and there exist many solutions to that problem which include Shepard's methods [42], radial basis functions [26] and finite element methods. Good surveys of these methods and further references can be found in [1, 17, 35, 40].

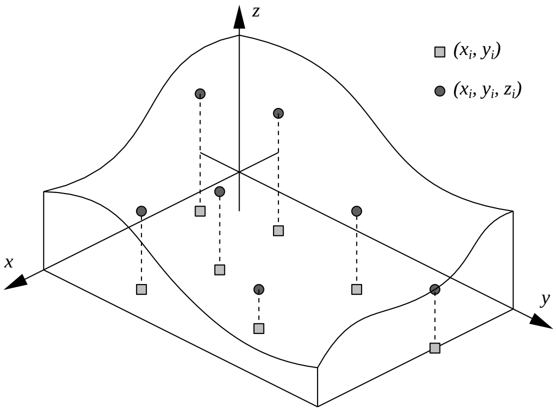

Figure 1: Scattered Data Interpolation.

[*] hormann@informatik.uni-erlangen.de

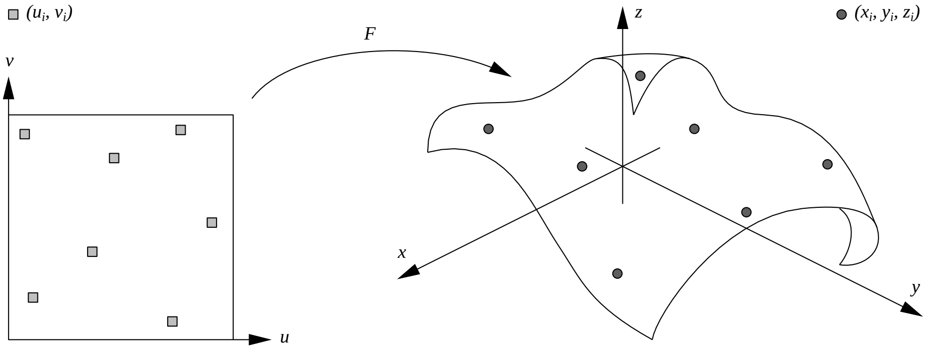

Figure 2: The parametric surface fitting problem.

In contrast to this *scalar problem* there is the *parametric problem*, where the task is to find a parameterized surface $F : \mathbb{R}^2 \to \mathbb{R}^3$ that approximates or interpolates the data points. This is usually done by specifying additional parameter values $(u_i, v_i) \in \mathbb{R}^2$ and by determining $F$ such that $F(u_i, v_i) \approx (x_i, y_i, z_i)$ (cf. Figure 2). While the theory of parametric surfaces was well understood in differential geometry, dating back to the time of Gauß [9], their potential for the representation of surfaces in the field of engineering remained unknown for a long time. “The exploration of the use of parametric curves and surfaces can be viewed as the origin of *Computer Aided Geometric Design (CAGD)*” ([11], p. xv).

The fundamental ideas in CAGD have been developed at the French and US-American car industry in the 1960ies. While de Casteljau at Citroën and Bézier at Renault independently developed the theory of Bézier curves and surfaces, Coons at Ford and Gordon at General Motors worked on *transfinite interpolation methods*, which can be used to ‘fill in’ a network of curves. At the same time *mathematical splines*, introduced by Schoenberg in the 1940ies, were used by Ferguson at Boeing and Sabin at the British Aircraft Corporation.

Combining the ideas of Bézier curves and splines naturally leads to an efficient and numerically stable representation of B-splines and further on to non-uniform rational B-splines (NURBS), which is the de-facto standard of today’s CAD systems. Detailed information about the theoretical and practical aspects of B-splines and NURBS can be found in [4, 8, 11, 41, 43].

# 1 Parameterization of Scattered 3D Data

The problem of parameterizing 3D data points is fundamental to many applications in CAGD, especially for the surface fitting problem. In general, a set of data points $P_i \in \mathbb{R}^3$ and a parameter domain $\Omega \subset \mathbb{R}^2$, over which the points are to be parameterized, are given. In addition, the topology of the point set has to be specified as different topologies lead to different approximation (or interpolation) problems (cf. Figure 3). This topological information is usually

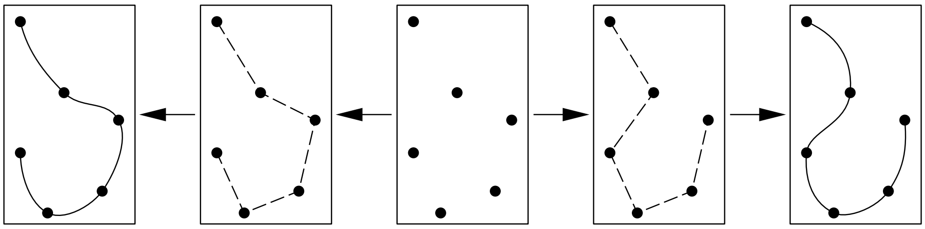

Figure 3: A point set (middle) with two different topologies (left and right).

given in terms of a triangulation of the data points, i.e. there exists a list of triangles $T_j = \Delta(P_{j_0}, P_{j_1}, P_{j_2})$.

The task now is to find parameter values $p_i \in \Omega$, one for each data point $P_i$, so that the topology of the point set is being preserved, i.e. the triangles in the parameter domain $t_j = \Delta(p_{j_0}, p_{j_1}, p_{j_2})$ must not overlap. Note, that the parameterization is implicitly given in the case of the *scalar problem*: for each point $P_i = (x_i, y_i, z_i)$, the first two components define the parameter value $p_i = (x_i, y_i)$.

While it is quite clear how to solve the *local* problem, i.e. parameterizing a set of points surrounding a reference point $R$, which can be done e.g. by an exponential mapping or by projection into an adequate tangent plane at $R$, the *global* problem is more complicated and has been discussed in several papers before.

Bennis et al. [2] propose a method that is based on differential geometry: they map isoparametric curves of the surface onto curves in the parameter domain such that the geodesic curvature at each point is preserved. The parameterization is then extended to both sides of that initial curve until some distortion threshold is reached. But this method as well as the one presented in [33] by Maillot et al. need the surface to be split into several independent regions and can therefore not be seen as solutions to the *global* problem.

Ma and Kruth [32] project the data points $P_i$ onto a parametric base surface $S : \Omega \to \mathrm{I\!R}^3$ and the parameter values of the projected points are taken as $p_i$. But since this method works only if the shape of the base surface is close to that of the triangulated data, it is not suitable for arbitrary data sets.

The approaches in [10, 13, 22, 27] have the following strategy in common:

1. find a parameterization for the *boundary points*, and
2. minimize an edge-based energy function

$$E = \frac{1}{2} \sum_{\{i,j\} \in \text{Edges}} c_{ij} \, \|p_i - p_j\|^2 \qquad (1)$$

to determine the parameterization for the *inner points*.

The edge coefficients $c_{ij}$ can be chosen in different ways. While Floater chooses them so that the geometric shape of the surface is preserved [13], Greiner

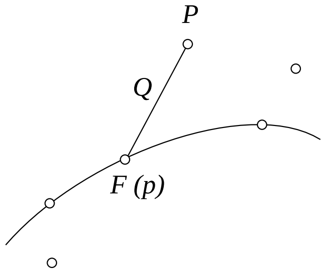

Figure 4: $p$ should be corrected.

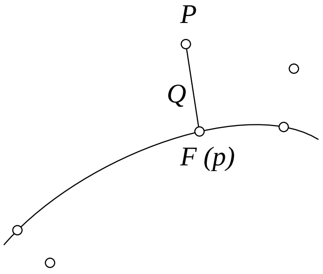

Figure 5: $p$ is well chosen.

and Hormann set $c_{ij} = \frac{1}{\|P_i - P_j\|^r}$ for some $r \geq 0$, as they want to minimize the energy of a network of springs [22, 27]. Both methods are generalizations of well-known results for the parameterization of curves, namely the *chord length* and *centripetal* parameterization [11, 14, 30].

A different method is introduced by Pinkall and Polthier in [37] and Eck et al. in [10]. They consider the discrete harmonic piecewise linear function between the surface triangles $T_j$ and the corresponding parameter triangles $t_j$ which leads to minimizing Equation 1 with $c_{ij} = \frac{1}{4}(\cot\alpha + \cot\beta)$, where $\alpha$ and $\beta$ are the angles opposite to the edge $\overline{P_i P_j}$ in the two adjacent surface triangles.

In all cases, minimizing Equation 1 is equivalent to solving a non-singular sparse matrix system, that is (apart from Floater's method) even symmetric positive definite. Though this is a comparatively fast way to find a parameterization, it suffers from the fact that it is not clear how to choose the initial parameterization of the *boundary points*. Floater maps them to the boundary of the unit square using chord length parameterization, Greiner and Hormann project them into the plane that fits all *boundary points* best in the least square sense, and Eck et al. use parameter values lying on a circle.

## 2 Parameter Correction

Once the parameter values are determined and a surface $F : \Omega \to \mathbb{R}^3$ approximating the data points is found, one can try to improve the parameterization, thereby exploiting the additional information given by the approximating surface itself [28, 39]. The standard method, introduced by Hoschek in [28], is derived from the observation that the error vector $Q = P - F(p)$ will not be orthogonal with respect to the approximating surface in general and will thus not represent the minimal distance between the surface and the data point (cf. Figures 4 and 5). If that happens, the parameter value $p$ should be corrected.

This is done by linearly approximating the surface at $F(p)$ and projecting the data point $P$ orthographically onto this tangent plane ($L$ in Figure 6). The correction term $\Delta p = (\Delta u, \Delta v)$ can be determined by solving $L = F(p) +$

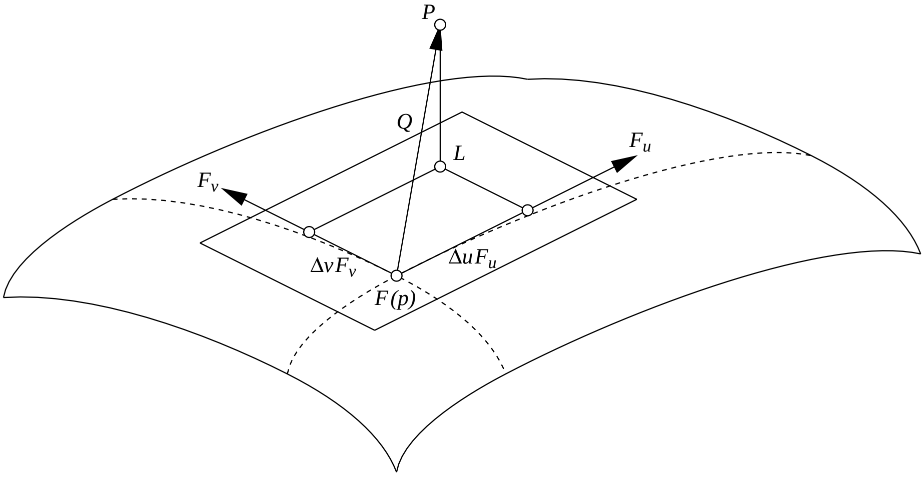

Figure 6: Parameter correction.

$\Delta u\, F_u + \Delta v\, F_v$, which can be rewritten as

$$\Delta p = \begin{pmatrix} F_u^2 & F_u F_v \\ F_u F_v & F_v^2 \end{pmatrix}^{-1} \begin{pmatrix} \langle Q|F_u\rangle \\ \langle Q|F_v\rangle \end{pmatrix} .$$

The approximation process will then be repeated with the improved parameter values $\tilde{p} = p + \Delta p$. The underlying idea of this method is to split the non-linear approximation problem, in which the surface parameters and the parameter values are unknowns, into two steps:

1. an *approximation step*, which finds the optimal surface parameters for given parameter values, and

2. a *parameter correction step*, which determines the optimal parameter values while the surface parameters are fixed.

Solving these problems alternately leads to good results after a few iterations only [27].

# 3 Interpolation and Approximation

The process of reconstructing smooth surfaces from discrete data offers two possibilities: interpolation and approximation. Since the solution of the interpolation problem coincides with the data points, one might assume it to be the more accurate reconstruction. But if the samples carry some noise (which is e.g. due to measurement errors) an approximating surface will be more adequate. Therefore, the choice of the appropriate method will depend on the specific structure of the problem (cf. Figure 7).

Figure 7: The configuration to the left is suitable for interpolation while approximation should be used for the noisy data points to the right.

The interpolation problem can be stated as follows: given a set of $n$ data points $P_i = (x_i, y_i, z_i) \in \mathrm{I\!R}^3$ with corresponding parameter values $p_i = (u_i, v_i) \in \mathrm{I\!R}^2$ and some class $\mathcal{S}$ of parameterized surfaces $F : \mathrm{I\!R}^2 \to \mathrm{I\!R}^3$, find $F \in \mathcal{S}$ such that $F(p_i) = (P_i)$ for all $i$. $\mathcal{S}$ may consist of bivariate polynomials, tensor product B-splines or NURBS, piecewise Bézier patches or triangular splines. If $\mathcal{S}$ is spanned by basis functions $B_1, \ldots, B_k$, the interpolation function $F = \sum_{j=1}^{k} c_j B_j$ satisfies $F(p_i) = \sum_j c_j B_j(p_i) = P_i$ for all $i$, hence the unknowns $c_1, \ldots, c_k$ can be found by solving the linear system

$$\boldsymbol{Bc} = \boldsymbol{p}$$

with

$$\boldsymbol{B} = \begin{pmatrix} B_1(p_1) & \cdots & B_k(p_1) \\ \vdots & \ddots & \vdots \\ B_1(p_n) & \cdots & B_k(p_n) \end{pmatrix}, \quad \boldsymbol{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}, \quad \boldsymbol{p} = \begin{pmatrix} P_1 \\ \vdots \\ P_n \end{pmatrix}. \tag{2}$$

Note, that this problem will not be solvable in general if $k < n$ and that the existence of a unique solution requires $k = n$.

In case of curve interpolation with polynomials, $\boldsymbol{B}$ will be a *Vandermonde-Matrix*, that is nonsingular for mutually different parameter values $p_i$. An interpolating B-spline curve can be found if and only if the *Schoenberg-Whitney* conditions are fulfilled [4, 8, 11]. These results can be extended to surface interpolation with polynomials or tensor product B-splines if the parameter values $p_i$ are distributed on a grid. The problem then decouples into a sequence of curve interpolation problems [16, 32].

However, if the parameter values are not gridded, the problem becomes more complicated. E.g., a biquadratic polynomial ($k = 6$) that interpolates $n = 6$ given data points cannot be found if the corresponding 6 parameter values lie on an algebraic curve of degree 2 (e.g. circles, ellipses and hyperbolas), although $k = n$. Another problem is that the interpolating surfaces might have unwanted oscillations.

To overcome these difficulties, a variational approach can be used for solving the interpolation problem. The idea is to start with a class of surfaces having more degrees of freedom which are strictly necessary to fulfill the interpolation conditions and to use the remaining degrees of freedom to smooth the surface [20, 21]. This is achieved by minimizing a functional that somehow measures the smoothness (see Section 4).

If $\mathcal{J} : \mathcal{S} \rightarrow \mathrm{I\!R}$ is such a fairness functional, the task is now to find $F \in \mathcal{S}$ such that

1. $F(p_i) = (P_i)$ for all $i$, and
2. $\mathcal{J}(F) \leq \mathcal{J}(G)$ for any $G \in \mathcal{S}$ satisfying 1.

Many reasonable measures of smoothness can be expressed by a quadratic and positive semidefinite functional (see Section 4), i.e. $\mathcal{J}(F) = \langle F|F\rangle_{\mathcal{J}}$ for a suitable positive semidefinite, symmetric, bilinear form $\langle\cdot|\cdot\rangle_{\mathcal{J}}$. By introducing *Lagrangian multipliers* $\lambda_i$, the smooth interpolation function $F = \sum_{i=1}^{k} c_i B_i$ can be found by solving the linear system

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B}^t \\ \boldsymbol{B} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} 0 \\ \boldsymbol{p} \end{pmatrix} \tag{3}$$

with

$$\boldsymbol{A} = \begin{pmatrix} \langle B_1|B_1\rangle_{\mathcal{J}} & \cdots & \langle B_1|B_k\rangle_{\mathcal{J}} \\ \vdots & \ddots & \vdots \\ \langle B_k|B_1\rangle_{\mathcal{J}} & \cdots & \langle B_k|B_k\rangle_{\mathcal{J}} \end{pmatrix}, \quad \boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix},$$

and $\boldsymbol{B}, \boldsymbol{c}, \boldsymbol{p}$ as defined in Equation 2.

However, interpolation is not always the best method for surface fitting. If the data points are not exact it is advisable to give up interpolation and better look for an approximating surface. The most common approach to this problem is the classical method of *least-squares*, whereby the approximation error

$$\mathcal{E}(F) = \sum_{i=1}^{n} (F(p_i) - P_i)^2 \tag{4}$$

is to be minimized. With the definitions from above this approximation error can be rewritten as

$$\mathcal{E}(F) = (\boldsymbol{Bc} - \boldsymbol{p})^t(\boldsymbol{Bc} - \boldsymbol{p}) = \boldsymbol{c}^t\boldsymbol{B}^t\boldsymbol{Bc} - 2\boldsymbol{c}^t\boldsymbol{B}^t\boldsymbol{p} + \boldsymbol{p}^t\boldsymbol{p}$$

and the minimum can be found by solving

$$\boldsymbol{B}^t\boldsymbol{Bc} = \boldsymbol{B}^t\boldsymbol{p}. \tag{5}$$

Equation 5 is called the *normal equation*. Note, that the matrix $\boldsymbol{M} = \boldsymbol{B}^t\boldsymbol{B}$ is symmetric, positive semidefinite. In case of approximation with B-spline curves, $\boldsymbol{M}$ is also sparse and has band structure. Furthermore, if the *Schoenberg-Whitney* conditions are fullfilled, the matrix is actually invertible and the normal equation can be solved by the method of Cholesky, QR decomposition or the conjugate gradient method [19]. However, in the bivariate case such simple conditions do not exist unless the given data is gridded and the uniqueness of the solution cannot be guaranteed.

Although straightforward least-squares fitting is often appropriate, the same problem as with interpolating surfaces may occur: it produces a surface that is not sufficiently smooth. In such cases it may be better to minimize a combination of the error functional $\mathcal{E}$ and the fairing functional $\mathcal{J}$:

$$\mathcal{K}_\omega(F) = \omega\mathcal{J}(F) + \mathcal{E}(F)$$

with a *smoothing parameter* $\omega \geq 0$ that controls the tradeoff between the smoothness and the approximation quality of the surface. Using the definitions from above, the minimum of this combined functional $\mathcal{K}_\omega$ can be found by solving

$$(\omega\boldsymbol{A} + \boldsymbol{B}^t\boldsymbol{B})\boldsymbol{c} = \boldsymbol{B}^t\boldsymbol{p}. \tag{6}$$

Assuming $\mathcal{J}$ to be a symmetric positive definite smoothing functional, Equation 6 always has a unique solution, call it $F_\omega$, for which some interesting properties hold [27]:

1. the function $\omega \to \mathcal{E}(F_\omega)$ is monotone increasing,
2. the function $\omega \to \mathcal{J}(F_\omega)$ is monotone decreasing,
3. $F_0 := \lim_{\omega\to 0} F_\omega$ exists,
4. $F_0$ minimizes $\mathcal{E}$ and is among all minima the one that additionally minimizes $\mathcal{J}$.

The first and second property support the statements on the effect of the smoothing parameter $\omega$: the smaller $\omega$ is chosen, the smaller the approximation error is, while the smoothness of the surface increases by increasing $\omega$. An immediate consequence of the last property is, that if there exist interpolating surfaces in $\mathcal{S}$, then $F_0$ is the interpolant with optimal fairness. Thus, it will be the solution to Equation 3, too. A detailed discussion of penalized least square methods can be found in [18].

# 4 Fairness Functionals

The construction of 'fair' or 'visually pleasing' surfaces is of vital importance in many areas of geometric modeling, especially in industrial design and styling [38]. While the human eye can easily rate the quality of a surface, the translation of this rating strategy to mathematical formulas is a crucial step. A lot of work has been dedicated to this problem and lots of different approaches how to measure the quality of a surface have been proposed by different authors.

In principal there are two approaches: one can either construct a functional by physical analogy (e.g. minimizing the energy of thin plates) or by geometric reasoning (e.g. minimizing area, curvature or the variation of curvature). But since every reasonable physical quantity has to be a generic property of the surface, it does not depend on the special parameterization that is used to describe the surface. This is, by definition, a geometric property, and therefore every physical quantity is a geometric one.

When selecting a fairness functional one has to take the following points into consideration.

1. Does the functional yield surfaces with a pleasant shape?
2. Can the functional be minimized in a reasonable amount of time?

It turns out that the functionals involving surface curvature yield surfaces of high quality, but an enormous amount of time is necessary to compute these solutions. The *thin plate energy functional*

$$\int a(\kappa_1^2 + \kappa_2^2) + 2(1-b)\kappa_1\kappa_2 \, d\omega$$

that describes the energy of a thin plate [7] is of that type. Here, $\kappa_1$ and $\kappa_2$ are the principle curvatures of the surface and $a$ and $b$ are constants describing properties of the material of the thin plate (resistance to bending and sheering). In [24] a special case ($a = b = 1$) of this functional,

$$\int \kappa_1^2 + \kappa_2^2 \, d\omega, \tag{7}$$

is used to determine smooth tensor product B-spline surfaces. To overcome the difficulty that this functional is highly nonlinear and thus the numerical solution is very time consuming, the integrand of Equation 7 is evaluated only at the corners of the rectangular control grid. In [34] the functional

$$\int \left(\frac{\partial \kappa_1}{\partial \epsilon_1}\right)^2 + \left(\frac{\partial \kappa_2}{\partial \epsilon_2}\right)^2 d\omega$$

with $\epsilon_1$ and $\epsilon_2$ denoting the directions of principle curvature is considered. Although this MVC functional that minimizes the variation of the curvature has proven to yield surfaces of perfect shape, the complexity and hence the numerical treatment is even worse than for Equation 7.

In contrast to these highly nonlinar functionals there are simpler, quadratic functionals that can be minimized by solving a linear system and are thus suitable for interactive use. The most commonly used is the *simplified* version of the thin plate energy functional in Equation 7:

$$\int F_{uu}^2 + 2F_{uv}^2 + F_{vv}^2 \, du \, dv,$$

where $F_{uu}$, $F_{uv}$ and $F_{vv}$ denote the second order partial derivatives of the surface $F$. As this functional is quadratic and much easier to minimize than the exact version, it is widely used [12, 23, 25, 36, 45]. However, it is a good approximation to Equation 7 only in the functional case, whereas the surfaces obtained in the parametric case may fail to have a pleasant shape. The more general quadratic, second order functional

$$\int \alpha_{11}F_u^2 + \alpha_{12}F_uF_v + \alpha_{22}F_v^2 + \beta_{11}F_{uu}^2 + \beta_{12}F_{uv}^2 + \beta_{22}F_{vv}^2 \, du \, dv$$

has been used in [6], where the coefficients $\alpha_{ij}, \beta_{ij}$ are chosen by physical reasoning, and in [44], where the choice is based on geometric properties.

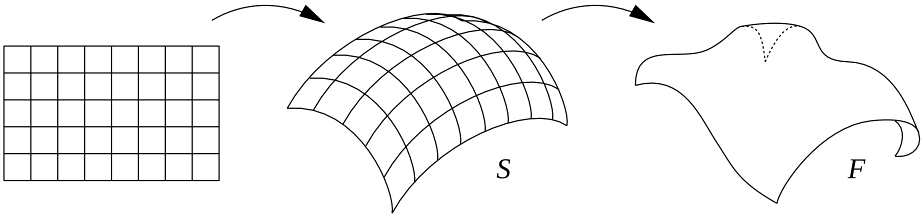

Figure 8: Parameter space, reference surface $S$ and surface $F$.

In contrast to the functionals based on the surface curvature, these simplified versions depend on the parameterization of the surface, i.e. different parameterizations of the same surface will lead to different values of the functional. This statement also holds for energy functionals based on the third order derivatives, like

$$\int F_{uuu}^2 + F_{vvv}^2 \, du \, dv,$$

as proposed in [3] and

$$\int (F_{uuu} + F_{uvv})^2 + (F_{uuv} + F_{vvv})^2 \, du \, dv,$$

introduced in [21]. A good compromise between the simplicity of the quadratic functionals and the quality of the exact ones are data dependent functionals, introduced in [20, 21, 23].

The basic idea is the following: since the simple quadratic functionals approximate the exact energies well if the shape of the surface is nearly planar, i.e. being similar to the shape of the parameter space, one can take the inverse approach and adapt the parameter space in such a way that its shape is close to that of the surface (cf. Figure 8). The concepts needed to do so are well-known in differential geometry [5, 29]. Considering a *reference surface* $S : \mathbb{R}^2 \to \mathbb{R}^3$ as parameter space and *gradient* $\mathrm{grad}_S$, *divergence* $\mathrm{div}_S$ and *Laplacian* $\Delta_S$ with respect to that reference surface, we can introduce the data dependent energy functionals

$$\begin{gathered} \int \mathrm{grad}_S(F)^2 \, d\omega_S \\ \int \Delta_S(F)^2 \, d\omega_S \\ \int \mathrm{grad}_S(\mathrm{div}_S(\mathrm{grad}_S(F))^2 \, d\omega_S \end{gathered} \tag{8}$$

with $d\omega_S$ denoting the surface element $\|S_u \times S_v\| \, du \, dv$. These functionals are still quadratic and can be minimized by solving a linear system.

If the reference surface $S$ is close to $F$, the data dependent functionals will be good approximations to the exact energies. Note that the functionals in Equation 8 do not depend on the specific parameterization of the reference surface $S$.

# 5 Hierarchical B-Splines

A major problem that arises by using the surface fitting methods discussed in Section 3 is the size of the linear systems that have to be solved. One way to overcome this drawback is the use of hierarchical surface models like the *Hierarchical B-Splines* introduced by Forsey and Bartels in [15]. Although these surfaces were originally designed for the purpose of modeling, the ideas have successfully been transfered to the surface fitting problem [16, 22, 31].

The basic idea is to subdivide the *global* approximation problem adaptively into several *local* problems where only a comparatively small part of the data has to be taken into account. These local problems will lead to linear systems of small size which can be solved efficiently. The first step of this method is to start with a class of surfaces $\mathcal{S}_0$ defined on a coarse control lattice, thus having only a small number of control points, and determine the surface $F_0 \in \mathcal{S}_0$ that solves the problem

$$\min_{F \in \mathcal{S}_0} \mathcal{K}_\omega(F),$$

where $\mathcal{K}_\omega$ is the combined fairness and error functional defined in Section 3.

If the quality of $F_0$ suits our needs, i.e. the approximation error is within a specified tolerance and the surface is sufficiently 'fair', nothing has to be done. Otherwise we will either have to adapt the smoothing parameter $\omega$ or increase the number of degrees of freedom. Halving the grid size of the control lattice yields a new class of surfaces $\mathcal{S}_1 \supset \mathcal{S}_0$ whose dimension is roughly four times the dimension of $\mathcal{S}_0$. However, this *global refinement* step increases the size of the linear system that has to be solved by factor 16. The strategy of *local refinement* offers a better alternative: only at those regions where the approximation error exceeds the tolerance, local overlay patches $\mathcal{T}_k$ with finer control lattices are added to the surface (cf. Figure 9). Now, instead of solving the approximation problem in the globally refined space $\mathcal{S}_1$ we only look for the optimal surface contained in $\tilde{\mathcal{S}}_1 = \mathcal{S}_0 + \mathcal{T} \subset \mathcal{S}_1$ with $\mathcal{T} = \sum_k \mathcal{T}_k$.

If the overlay patches are chosen such that they are orthogonal, i.e. having disjunct support, this problem decomposes into several local problems. Indeed, if $F_1$ solves the optimization problem

$$\min_{F \in \tilde{\mathcal{S}}_1} \mathcal{K}_\omega(F) = \min_{G \in \mathcal{T}} \mathcal{K}_\omega(F_0 + G),$$

then $F_1$ can be written as $F_1 = F_0 + \sum_k G_k$ with overlay patches $G_k \in \mathcal{T}_k$ determined by solving the local problems

$$\min_{G \in \mathcal{T}_k} \mathcal{K}_\omega(F_0 + G),$$

which can be solved efficiently due to their small size. Obviously, this strategy can be applied recursively.

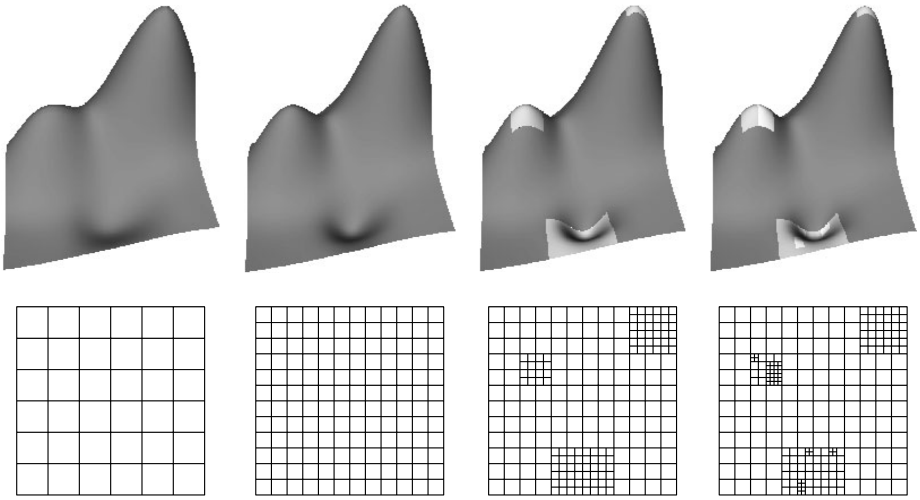

Figure 9: Approximating surface with one global and two local refinement steps.

# References

[1] R. E. Barnhill. Representation and approximation of surfaces. In J. D. Rice, editor, *Mathematical Software III*, pages 69–120. Academic Press, 1977.

[2] C. Bennis, J.-M. Vézien, and G. Iglésias. Piecewise surface flattening for non-distorted texture mapping. In *ACM Computer Graphics (SIGGRAPH '92 Proceedings)*, pages 237–246, 1991.

[3] M. I. Bloor, M. J. Wilson, and H. Hagen. The smoothing properties of variational schemes for surface design. *Computer Aided Geometric Design*, 12:381–394, 1995.

[4] C. de Boor. *A Practical Guide to Splines.* Springer, New York, 1987.

[5] M. P. do Carmo. *Differential Geometry of Curves and Surfaces.* Prentice-Hall Inc., 1976.

[6] G. Celniker and D. Gossard. Deformable curve and surface finite-element for free-form shape design. In *ACM Computer Graphics (SIGGRAPH '91 Proceedings)*, pages 257–266, 1991.

[7] R. Courant and D. Hilbert. *Methods of Mathematical Physics*, volume 1. Wiley, New York, 1953.

[8] P. Dierckx. *Curve and Surface Fitting with Splines.* Oxford Clarendon Press, 1995.

[9] P. Dombrowski and C. F. Gauß. *150 Years after Gauss' "Disquisitiones Generales Circa Superficies Curvas".* Paris Societé Mathématique de France, 1979.

[10] M. Eck, T. DeRose, T. Duchamp, H. Hoppe, M. Lounsbery, and W. Stuetzle. Multiresolution analysis of arbitrary meshes. In R. Cook, editor, *SIGGRAPH '95 Conference Proceedings*, Annual Conference Series, pages 173–182. Addison Wesley, August 1995.

[11] G. Farin. *Curves and Surfaces for Computer Aided Geometric Design.* Acadamic Press, Boston, 1993.

[12] G. E. Fasshauer and L. L. Schumaker. Minimal energy surfaces using parametric splines. *Computer Aided Geometric Design*, 13:45–79, 1996.

[13] M. S. Floater. Parameterization and smooth approximation of surface triangulations. *Computer Aided Geometric Design*, 14:231–250, 1997.

[14] T. A. Foley and G. M. Nielson. Knot selection for parametric spline interpolation. In T. Lyche and L. L. Schumaker, editors, *Mathematical Methods in Computer Aided Geometric Design*, pages 261–271. Academic Press, Boston, 1989.

[15] D. R. Forsey and R. H. Bartels. Hierarchical B-spline refinement. In *ACM Computer Graphics (SIGGRAPH '88 Proceedings)*, pages 205–212, 1988.

[16] D. R. Forsey and R. H. Bartels. Surface fitting with hierarchical splines. *ACM Transactions on Graphics*, 14:134–161, 1995.

[17] R. Franke and G. M. Nielson. Scattered data interpolation and applications: A tutorial and survey. In H. Hagen and D. Roller, editors, *Geometric Modeling*, pages 131–161. Springer, Berlin, 1991.

[18] M. von Golitschek and L. L. Schumaker. Data fitting by penalized least squares. In J. C. Mason, editor, *Algorithms for approximation II*, pages 210–227, Shrivenham, 1988.

[19] G. H. Golub and C. F. van Loan. *Matrix Computations.* The John Hopkins University Press, Baltimore, Maryland, 1989.

[20] G. Greiner. Surface construction based on variational principles. In P. J. Laurent, A. LeMéhauté, and L. L. Schumaker, editors, *Wavelets, Images, and Surface Fitting*, pages 277–286. AK Peters, Wellesley, 1994.

[21] G. Greiner. Variational design and fairing of spline surfaces. In *Computer Graphics Forum (EUROGRAPHICS '94 Proceedings)*, volume 13, pages 143–154, 1994.

[22] G. Greiner and K. Hormann. Interpolating and approximating scattered 3D data with hierarchical tensor product B-splines. In A. Méhauté, C. Rabut, and L. L. Schumaker, editors, *Surface Fitting and Multiresolution Methods*, pages 163–172. Vanderbilt University Press, 1997.

[23] G. Greiner, J. Loos, and W. Wesselink. Data dependent thin plate energy and its use in interactive surface modeling. In *Computer Graphics Forum (EUROGRAPHICS '96 Proceedings)*, volume 15, pages 175–185, 1996.

[24] H. Hagen, G.-P. Bonneau, and S. Hahmann. Variational design and surface interrogation. *Computer Graphics Forum (EUROGRAPHICS '93 Proceedings)*, 13:447–459, 1993.

[25] M. Halstead, M. Kass, and T. DeRose. Efficient, fair interpolation using Catmull-Clark surfaces. In *ACM Computer Graphics (SIGGRAPH '93 Proceedings)*, pages 35–44, 1993.

[26] R. L. Hardy. Multiquadric equation of topography and other irregular surfaces. *Journal Geophysical Research*, 76:1905–1915, 1971.

[27] K. Hormann. Glatte Approximation mit hierarchischen Splineflächen. Master's thesis, IMMD IX, University of Erlangen-Nürnberg, 1997.

[28] J. Hoschek. Intrinsic parametrization for approximation. *Computer Aided Geometric Design*, 5:27–31, 1988.

[29] W. Klingenberg. *A Course in Differential Geometry.* Springer, Berlin/Heidelberg, 1978.

[30] E. T. Lee. Choosing nodes in parametric curve interpolation. *Computer Aided Design*, 21(6):363–370, 1989.

[31] S. Lee, G. Wolberg, and S. Y. Shin. Scattered data interpolation with multilevel B-splines. *IEEE Transactions on Visualization and Computer Graphics*, 3(3):1–17, 1997.

[32] W. Ma and J. P. Kruth. Parameterization of randomly measured points for least squares fitting of B-spline curves and surfaces. *Computer Aided Design*, 27:663–675, 1995.

[33] J. Maillot, H. Yahia, and A. Verroust. Interactive texture mapping. In *ACM Computer Graphics (SIGGRAPH '93 Proceedings)*, pages 27–34, 1993.

[34] H. P. Moreton and C. H. Séquin. Functional optimization for fair surface design. In *ACM Computer Graphics (SIGGRAPH '92 Proceedings)*, pages 167–176, 1992.

[35] G. M. Nielson. Scattered data modeling. *IEEE Computer Graphics & Applications*, pages 60–70, January 1993.

[36] R. Pfeifle. *Approximation und Interpolation mit quadratischen Dreiecks-B-Splines.* PhD thesis, University of Erlangen-Nürnberg, 1995.

[37] U. Pinkall and K. Polthier. Computing discrete minimal surfaces and their conjugates. *Experimental Mathematics*, 2(1):15–36, 1993.

[38] N. S. Sapidis. *Designing Fair Curves and Surfaces.* SIAM, Philadelphia, 1994.

[39] B. Sarkar and C.-H. Menq. Parameter optimization in approximating curves and surfaces to measurement data. *Computer Aided Geometric Design*, 8:267–290, 1991.

[40] L. L. Schumaker. Fitting surfaces to scattered data. In G. G. Lorentz, C. K. Chui, and L. L. Schumaker, editors, *Approximation Theory II*, pages 203–268. Academic Press, Boston, 1976.

[41] H.-P. Seidel. Geometrische Grundlagen des Computer Aided Geometric Design. In *Geometrie und ihre Anwendungen*, pages 201–246. Hanser, 1994.

[42] D. Shepard. A two dimensional interpolation function for irregularly spaced data. In *Proceedings of ACM 23rd National Conference*, pages 517–524, 1968.

[43] W. Straßer and H.-P. Seidel. *Theory and Practice of Geometric Modeling.* Springer, 1989.

[44] W. Welch and A. Witkin. Variational surface modeling. In *ACM Computer Graphics (SIGGRAPH '92 Proceedings)*, pages 157–166, 1992.

[45] W. Wesselink. *Variational Modeling of Curves and Surfaces.* PhD thesis, University of Technology, Eindhoven, 1996.