Four-point curve subdivision based on iterated chordal and centripetal parameterizations

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Abstract

Dubuc's interpolatory four-point scheme inserts a new point by fitting a cubic polynomial to neighbouring points over uniformly spaced parameter values. In this paper we replace uniform parameter values by chordal and centripetal ones. Since we update the parameterization at each refinement level, both schemes are non-linear. Because of this data-dependent parameterization, the schemes are only invariant under solid body and isotropic scaling transformations, but not under general affine transformations. We prove convergence of the two schemes and bound the distance between the limit curve and the initial control polygon. Numerical examples indicate that the limit curves are smooth and that the centripetal one is tighter, as suggested by the distance bounds. Similar to cubic spline interpolation, the use of centripetal parameter values for highly non-uniform initial data yields better results than the use of uniform or chordal ones.

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1 Introduction

Dubuc's four-point subdivision scheme [3] is a method for generating a smooth curve passing through a sequence of points in $\mathbb{R}^d$. The algorithm is based on fitting cubic polynomials to local data, parameterized uniformly. This scheme was generalized by Daubechies, Guskov, and Sweldens [2] to allow non-uniform parameter values. Yet their scheme is linear in the data. Here we generalize further by determining the parameterization at each refinement level according to the geometry of the points at that level. We focus on the chordal and centripetal parameterizations [1, 5, 7]. The resulting two schemes are non-linear and of a new type. To the best of our knowledge, these schemes cannot be analyzed by existing techniques, which are applicable only to other types of non-linear schemes, e.g., the technique of proximity to a linear scheme [8].

Specifically, let $P_0 = \{\boldsymbol{p}_{0,k} : k \in \mathbb{Z}\}$ with $\boldsymbol{p}_{0,k} \in \mathbb{R}^d$ and $\boldsymbol{p}_{0,k+1} \neq \boldsymbol{p}_{0,k}$, be the initial set of control points, and let $P_j = \{\boldsymbol{p}_{j,k} : k \in \mathbb{Z}\}$ with $\boldsymbol{p}_{j,k} \in \mathbb{R}^d$ be the refined set of control points at level j. These points determine the set of parameter values $\{t_{j,k} : k \in \mathbb{Z}\}$ with $t_{j,0} = 0$ and $t_{j,k+1} - t_{j,k} = \|\boldsymbol{p}_{j,k+1} - \boldsymbol{p}_{j,k}\|^\alpha$ for $k \in \mathbb{Z}$, where $\alpha = 1$ gives chordal parameter values and $\alpha = 1/2$ gives centripetal ones. Note that $\alpha = 0$ corresponds to uniform parameterization. The refinement rule is then

$$\begin{aligned} \boldsymbol{p}_{j+1,2k} &= \boldsymbol{p}_{j,k}, \\ \boldsymbol{p}_{j+1,2k+1} &= \pi_{j,k}(t_*), \end{aligned} \tag{1}$$

where $\pi_{j,k}$ is the parametric cubic polynomial that interpolates $\boldsymbol{p}_{j,k-1}, \boldsymbol{p}_{j,k}, \boldsymbol{p}_{j,k+1}, \boldsymbol{p}_{j,k+2}$ at the values $t_{j,k-1}$, $t_{j,k}$, $t_{j,k+1}$, $t_{j,k+2}$ and $t_* = (t_{j,k} + t_{j,k+1})/2$; see Figure 1. We note that the four values $t_{j,k-1}$, $t_{j,k}$, $t_{j,k+1}$, $t_{j,k+2}$ must be distinct for the Lagrange interpolation to be well-defined. This in turn requires that each pair of consecutive points $\boldsymbol{p}_{j,k}$ and $\boldsymbol{p}_{j,k+1}$ be distinct. We assume this property holds for $j = 0$ and we prove that it holds for $j \geq 1$ for the chordal and centripetal schemes ($\alpha = 1$ and $\alpha = 1/2$, respectively). Although one could consider other choices of the evaluation parameter t_*, this particular choice simplifies the analysis of the limit curves.

We prove convergence of these two schemes and derive upper bounds on the distance between the limit curve and the initial control polygon. These schemes are very easy to implement and our numerical examples suggest that the limit curves are C^1, like those of Dubuc's scheme, but we have not so far been able to prove this. It is clear, however, that they cannot be C^2 in general, because the subdivision rule (1) as well as the limit curve are independent of the parameter α in the particular case that P_0 is a regular n-gon[1]. This example also shows that we cannot expect to get C^2 limit curves by using other choices of the evaluation point in (1), as

[1]After this article had gone to press, we realized that this is not entirely correct. If P_0 is a regular n-gon, then all edges of P_1 have the same length and so P_2 does not depend on α. But the edges of P_2 have different lengths, because P_1 is *not* a regular $2n$-gon. Therefore, P_j depends on α for $j \geq 3$ and so does the limit curve.

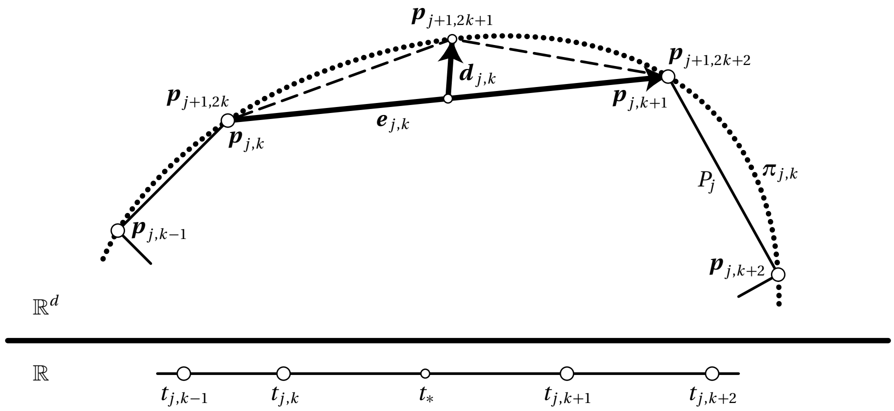

Figure 1: Insertion of a new point.

long as the rules for choosing it are symmetric, since any symmetric rule leads to the choice t_* in the regular setting.

Moreover, the numerical examples and our upper bounds indicate that the centripetal limit curve is tighter than the chordal and Dubuc's curves and gives better shaped curves whenever the edge lengths of the initial control polygon vary significantly.

2 Cubic Lagrange interpolation

In order to analyze the schemes we need to establish some properties of cubic Lagrange interpolation. Consider functional data $f_0, f_1, f_2, f_3 \in \mathbb{R}$ given at the points t_0, t_1, t_2, t_3 and let f be the cubic polynomial satisfying $f(t_i) = f_i$ for $i = 0, 1, 2, 3$. Further let g be the linear polynomial that interpolates f_1 and f_2 at t_1 and t_2. Let $[s_0, s_1, \ldots, s_k]f$ denote the divided difference of f of order k at the points $s_0, s_1, \ldots, s_k$.

Lemma 1. *For $t \in \mathbb{R}$,*

$$f(t) - g(t) = \frac{(t-t_1)(t-t_2)}{t_3 - t_0}\big((t_3 - t)[t_0, t_1, t_2]f + (t - t_0)[t_1, t_2, t_3]f\big).$$

Proof. By inserting the recurrence formula

$$[t_0, t_1, t_2, t_3]f = \big([t_1, t_2, t_3]f - [t_0, t_1, t_2]f\big)/(t_3 - t_0)$$

into the Newton form

$$f(t) = g(t) + (t-t_1)(t-t_2)[t_0, t_1, t_2]f + (t-t_0)(t-t_1)(t-t_2)[t_0, t_1, t_2, t_3]f,$$

the result follows. □

At the midpoint $t_* = (t_1 + t_2)/2$ of the interval $[t_1, t_2]$, Lemma 1 yields

$$f(t_*) - \frac{f_1 + f_2}{2} = -\frac{1}{4}\frac{(t_2 - t_1)^2}{t_3 - t_0}\big((t_3 - t_*)[t_0, t_1, t_2]f + (t_* - t_0)[t_1, t_2, t_3]f\big). \tag{2}$$

Consider now the subdivision scheme (1) and let $\boldsymbol{d}_{j,k}$ be the vector

$$\boldsymbol{d}_{j,k} = \boldsymbol{p}_{j+1,2k+1} - (\boldsymbol{p}_{j,k} + \boldsymbol{p}_{j,k+1})/2$$

depicted in Figure 1. Let $\boldsymbol{e}_{j,k} = \boldsymbol{p}_{j,k+1} - \boldsymbol{p}_{j,k}$ and consider divided differences at level j,

$$\boldsymbol{p}^{[1]}_{j,k} = \frac{\boldsymbol{p}_{j,k+1} - \boldsymbol{p}_{j,k}}{t_{j,k+1} - t_{j,k}} = \frac{\boldsymbol{e}_{j,k}}{\|\boldsymbol{e}_{j,k}\|^\alpha}$$

$$\boldsymbol{p}^{[2]}_{j,k} = \frac{\boldsymbol{p}^{[1]}_{j,k+1} - \boldsymbol{p}^{[1]}_{j,k}}{t_{j,k+2} - t_{j,k}} = \left(\frac{\boldsymbol{e}_{j,k+1}}{\|\boldsymbol{e}_{j,k+1}\|^\alpha} - \frac{\boldsymbol{e}_{j,k}}{\|\boldsymbol{e}_{j,k}\|^\alpha}\right)\frac{1}{\|\boldsymbol{e}_{j,k+1}\|^\alpha + \|\boldsymbol{e}_{j,k}\|^\alpha}.$$

Combining Equation (2) with the subdivision rule in (1), we get

Lemma 2. *For all* $\alpha \in [0,1]$,

$$\boldsymbol{d}_{j,k} = -\frac{1}{4}\frac{(t_{j,k+1}-t_{j,k})^2}{a+b+1}\big((a+1/2)\boldsymbol{p}^{[2]}_{j,k} + (b+1/2)\boldsymbol{p}^{[2]}_{j,k-1}\big) \tag{3}$$

with $a = (t_{j,k}-t_{j,k-1})/(t_{j,k+1}-t_{j,k})$ *and* $b = (t_{j,k+2}-t_{j,k+1})/(t_{j,k+1}-t_{j,k})$.

Lemma 3. *For* $\alpha = 0$ *(uniform parameterization),*

$$\|\boldsymbol{d}_{j,k}\| \le \frac{1}{8}\max\{\|\boldsymbol{e}_{j,k-1}\|, \|\boldsymbol{e}_{j,k+1}\|\}, \tag{4}$$

for $\alpha = 1/2$ *(centripetal parameterization),*

$$\|\boldsymbol{d}_{j,k}\| \le \frac{1}{4}\|\boldsymbol{e}_{j,k}\|, \tag{5}$$

and for $\alpha = 1$ *(chordal parameterization),*

$$\|\boldsymbol{d}_{j,k}\| \le \frac{3}{8}\max\{\|\boldsymbol{e}_{j,k-1}\|, \|\boldsymbol{e}_{j,k}\|, \|\boldsymbol{e}_{j,k+1}\|\}. \tag{6}$$

Proof. Consider first the case $\alpha = 0$. Then

$$\boldsymbol{p}^{[2]}_{j,k} = (\boldsymbol{e}_{j,k+1} - \boldsymbol{e}_{j,k})/2,$$

and since $a = b = 1$, Equation (3) reduces to

$$\boldsymbol{d}_{j,k} = -\frac{1}{16}(\boldsymbol{e}_{j,k+1} - \boldsymbol{e}_{j,k-1}),$$

so that the estimate (4) follows immediately.

In the case $\alpha = 1/2$, since $\|\boldsymbol{p}^{[1]}_{j,k}\| = \|\boldsymbol{e}_{j,k}\|^{1/2}$, we have

$$\|\boldsymbol{p}^{[2]}_{j,k}\| \le \frac{\|\boldsymbol{p}^{[1]}_{j,k+1}\| + \|\boldsymbol{p}^{[1]}_{j,k}\|}{\|\boldsymbol{e}_{j,k+1}\|^{1/2} + \|\boldsymbol{e}_{j,k}\|^{1/2}} = 1,$$

and using this inequality in (3) gives (5).

To prove (6) we write (3) as

$$\boldsymbol{d}_{j,k} = -\frac{1}{4}\frac{t_{j,k+1}-t_{j,k}}{a+b+1}\big(A(\boldsymbol{p}^{[1]}_{j,k+1} - \boldsymbol{p}^{[1]}_{j,k}) + B(\boldsymbol{p}^{[1]}_{j,k} - \boldsymbol{p}^{[1]}_{j,k-1})\big),$$

where

$$A = \frac{a+1/2}{b+1} \qquad \text{and} \qquad B = \frac{b+1/2}{a+1}.$$

Then, since $\|\boldsymbol{p}^{[1]}_{j,k}\| = 1$, we get

$$\|\boldsymbol{d}_{j,k}\| \le \frac{1}{4}\frac{\|\boldsymbol{e}_{j,k}\|}{a+b+1}(A + |A-B| + B).$$

Now suppose that $a \ge b$. Then $A \ge B$ and

$$\|\boldsymbol{d}_{j,k}\| \le \frac{1}{4}\frac{\|\boldsymbol{e}_{j,k}\|}{a+b+1}\frac{2a+1}{b+1} \le \frac{2a+1}{4(a+1)}\|\boldsymbol{e}_{j,k}\|. \tag{7}$$

For $a \le 1$, this immediately gives

$$\|\boldsymbol{d}_{j,k}\| \le \frac{3}{8}\|\boldsymbol{e}_{j,k}\|,$$

and for $a \ge 1$, since $a = \|\boldsymbol{e}_{j,k-1}\|/\|\boldsymbol{e}_{j,k}\|$, we have

$$\|\boldsymbol{d}_{j,k}\| \le \frac{2a+1}{4a(a+1)}\|\boldsymbol{e}_{j,k-1}\| \le \frac{3}{8}\|\boldsymbol{e}_{j,k-1}\|.$$

Since the opposite case $a \le b$ is similar with $\|\boldsymbol{e}_{j,k+1}\|$ replacing $\|\boldsymbol{e}_{j,k-1}\|$, Equation (6) follows. □

We are now able to show that the centripetal and chordal subdivision schemes are well-defined.

Theorem 1. *For* $\alpha = 1/2$ *and* $\alpha = 1$ *any two consecutive points* $\boldsymbol{p}_{j,k}$ *and* $\boldsymbol{p}_{j,k+1}$ *are distinct.*

Proof. It is sufficient to show that

$$\|\boldsymbol{d}_{j,k}\| < \frac{1}{2}\|\boldsymbol{e}_{j,k}\|.$$

In the centripetal case, $\alpha = 1/2$, this follows immediately from (5). In the chordal case, $\alpha = 1$, it follows from (7) if $a \ge b$ and similarly for $a \le b$. □

3 Convergence

In this section we prove the convergence of the centripetal and the chordal schemes. A key ingredient of the proof is the fact that the edge lengths $\|\boldsymbol{e}_{j,k}\|$ converge to zero as j increases, which follows directly from Lemma 3.

Lemma 4. *For* $\alpha = 0$,

$$\max\{\|\boldsymbol{e}_{j+1,2k}\|, \|\boldsymbol{e}_{j+1,2k+1}\|\} \le \frac{5}{8} \max\{\|\boldsymbol{e}_{j,k-1}\|, \|\boldsymbol{e}_{j,k}\|, \|\boldsymbol{e}_{j,k+1}\|\},$$

for $\alpha = 1/2$,

$$\max\{\|\boldsymbol{e}_{j+1,2k}\|, \|\boldsymbol{e}_{j+1,2k+1}\|\} \le \frac{3}{4} \|\boldsymbol{e}_{j,k}\|, \tag{8}$$

and for $\alpha = 1$,

$$\max\{\|\boldsymbol{e}_{j+1,2k}\|, \|\boldsymbol{e}_{j+1,2k+1}\|\} \le \frac{7}{8} \max\{\|\boldsymbol{e}_{j,k-1}\|, \|\boldsymbol{e}_{j,k}\|, \|\boldsymbol{e}_{j,k+1}\|\}. \tag{9}$$

Proof. By the definition of $\boldsymbol{e}_{j,k}$ and $\boldsymbol{d}_{j,k}$ we have $\boldsymbol{e}_{j+1,2k} = \boldsymbol{e}_{j,k}/2 + \boldsymbol{d}_{j,k}$ and $\boldsymbol{e}_{j+1,2k+1} = \boldsymbol{e}_{j,k}/2 - \boldsymbol{d}_{j,k}$. The statement then follows by using the triangle inequality and the bounds on $\|\boldsymbol{d}_{j,k}\|$ from Lemma 3. □

Next, we represent each polygon P_j parametrically as the continuous piecewise linear function $\boldsymbol{f}_j : \mathbb{R} \to \mathbb{R}^d$ that interpolates the data $(2^{-j}k, \boldsymbol{p}_{j,k})$ and show that the sequence $\boldsymbol{f}_0, \boldsymbol{f}_1, \ldots$ is a Cauchy sequence in the sup norm.

Theorem 2. *The centripetal and chordal subdivision schemes converge.*

Proof. Since

$$\|\boldsymbol{f}_{j+1} - \boldsymbol{f}_j\|_\infty = \sup_{t\in\mathbb{R}} \|\boldsymbol{f}_{j+1}(t) - \boldsymbol{f}_j(t)\| = \sup_{k\in\mathbb{Z}} \|\boldsymbol{d}_{j,k}\|,$$

it follows from Lemma 3 that

$$\|\boldsymbol{f}_{j+1} - \boldsymbol{f}_j\|_\infty \le \frac{3}{8} \sup_{k\in\mathbb{Z}} \|\boldsymbol{e}_{j,k}\|.$$

Since by Lemma 4,

$$\sup_{k\in\mathbb{Z}} \|\boldsymbol{e}_{j,k}\| \le \mu \sup_{k\in\mathbb{Z}} \|\boldsymbol{e}_{j-1,k}\| \le \cdots \le \mu^j \sup_{k\in\mathbb{Z}} \|\boldsymbol{e}_{0,k}\|, \tag{10}$$

with $\mu < 1$, the sequence $\{\boldsymbol{f}_j : j \in \mathbb{N}_0\}$ is a Cauchy sequence in the sup norm and therefore converges to a continuous limit

$$\boldsymbol{f} = \lim_{j\to\infty} \boldsymbol{f}_j.$$

□

We note that the estimates (6) in Lemma 3 and (9) in Lemma 4 actually hold for all $\alpha \in [0,1]$ if the scheme is well-defined, so that the above proof implies convergence in that case. However, we found examples for $\alpha \in (0,1/2)$ where the scheme fails because it generates identical consecutive points.

4 Distance bounds

In a similar way that Lemma 3 led to the convergence proof in the previous section, the same lemma can also be used to derive upper bounds on the Hausdorff distance d_{H} between the piece of the limit curve $\{\boldsymbol{f}(s) : s \in [k, k+1]\}$ and the line segment $[\boldsymbol{p}_{0,k}, \boldsymbol{p}_{0,k+1}]$. In order to prove these bounds, let us first establish a local variant of the estimate in Equation (10).

Lemma 5. *For* $\alpha = 0$,

$$\max_{2^j k - 2 \le i \le 2^j (k+1)+1} \|\boldsymbol{e}_{j,i}\| \le \left(\frac{5}{8}\right)^j \max_{k-2 \le \ell \le k+2} \|\boldsymbol{e}_{0,\ell}\|,$$

for $\alpha = 1/2$,

$$\max_{2^j k \le i \le 2^j (k+1)-1} \|\boldsymbol{e}_{j,i}\| \le \left(\frac{3}{4}\right)^j \|\boldsymbol{e}_{0,k}\|,$$

and for $\alpha = 1$,

$$\max_{2^j k - 2 \le i \le 2^j (k+1)+1} \|\boldsymbol{e}_{j,i}\| \le \left(\frac{7}{8}\right)^j \max_{k-2 \le \ell \le k+2} \|\boldsymbol{e}_{0,\ell}\|.$$

Proof. Since all the control points at level j between $\boldsymbol{p}_{0,k} = \boldsymbol{p}_{j,2^j k}$ and $\boldsymbol{p}_{0,k+1} = \boldsymbol{p}_{j,2^j(k+1)}$ depend only on the six initial points $\boldsymbol{p}_{0,k-2}, \boldsymbol{p}_{0,k-1}, \dots, \boldsymbol{p}_{0,k+3}$, the first and third inequalities follow from Lemma 4 by induction on j. The second inequality also follows by induction on j from (8) in Lemma 4. □

The upper bound on the Hausdorff distance now follows from this lemma and Lemma 3.

Theorem 3. *For* $\alpha = 0$,

$$d_{\mathrm{H}}\big(\boldsymbol{f}([k,k+1]), [\boldsymbol{p}_{0,k}, \boldsymbol{p}_{0,k+1}]\big) \le \frac{3}{13} \max_{k-2\le\ell\le k+2} \|\boldsymbol{e}_{0,\ell}\|,$$

for $\alpha = 1/2$,

$$d_{\mathrm{H}}\big(\boldsymbol{f}([k,k+1]), [\boldsymbol{p}_{0,k}, \boldsymbol{p}_{0,k+1}]\big) \le \frac{5}{7} \|\boldsymbol{e}_{0,k}\|,$$

and for $\alpha = 1$,

$$d_{\mathrm{H}}\big(\boldsymbol{f}([k,k+1]), [\boldsymbol{p}_{0,k}, \boldsymbol{p}_{0,k+1}]\big) \le \frac{11}{5} \max_{k-2\le\ell\le k+2} \|\boldsymbol{e}_{0,\ell}\|.$$

Proof. Let $s_{j,i} = 2^{-j} i$ and consider the difference between $\boldsymbol{f}_{j+2}$ and $\boldsymbol{f}_j$. Since

$$\boldsymbol{f}_{j+2}(s) - \boldsymbol{f}_j(s) = \begin{cases} \boldsymbol{0}, & s = s_{j+2,4i}, \\ \boldsymbol{d}_{j,i}/2 + \boldsymbol{d}_{j+1,2i}, & s = s_{j+2,4i+1}, \\ \boldsymbol{d}_{j,i}, & s = s_{j+2,4i+2}, \\ \boldsymbol{d}_{j,i}/2 + \boldsymbol{d}_{j+1,2i+1}, & s = s_{j+2,4i+3}, \end{cases}$$

we have

$$\sup_{s_{j,i}\le s\le s_{j,i+1}} \|\boldsymbol{f}_{j+2}(s) - \boldsymbol{f}_j(s)\| \le \max\{\|\boldsymbol{d}_{j,i}\|/2 + \|\boldsymbol{d}_{j+1,2i}\|, \|\boldsymbol{d}_{j,i}\|, \|\boldsymbol{d}_{j,i}\|/2 + \|\boldsymbol{d}_{j+1,2i+1}\|\}.$$

Using Lemmas 3 and 4 we get the estimates

$$\max\{\|\boldsymbol{d}_{j+1,2i}\|, \|\boldsymbol{d}_{j+1,2i+1}\|\} \le \begin{cases} \frac{5}{64} \max\limits_{i-2\le\ell\le i+2} \|\boldsymbol{e}_{j,\ell}\|, & \alpha = 0, \\ \frac{3}{16} \|\boldsymbol{e}_{j,i}\|, & \alpha = 1/2, \\ \frac{21}{64} \max\limits_{i-2\le\ell\le i+2} \|\boldsymbol{e}_{j,\ell}\|, & \alpha = 1, \end{cases}$$

and conclude that

$$\sup_{s_{j,i}\le s\le s_{j,i+1}} \|\boldsymbol{f}_{j+2}(s) - \boldsymbol{f}_j(s)\| \le \begin{cases} \frac{9}{64} \max\limits_{i-2\le\ell\le i+2} \|\boldsymbol{e}_{j,\ell}\|, & \alpha = 0, \\ \frac{5}{16} \|\boldsymbol{e}_{j,i}\|, & \alpha = 1/2, \\ \frac{33}{64} \max\limits_{i-2\le\ell\le i+2} \|\boldsymbol{e}_{j,\ell}\|, & \alpha = 1. \end{cases}$$

Considering now all intervals $[s_{j,i}, s_{j,i+1}]$ between k and $k+1$, i.e., $2^j k \le i \le 2^j(k+1) - 1$, and taking Lemma 5 into account, we have

$$\sup_{k\le s\le k+1} \|\boldsymbol{f}_{j+2}(s) - \boldsymbol{f}_j(s)\| \le \begin{cases} \frac{9}{64} \left(\frac{5}{8}\right)^j \max\limits_{k-2\le\ell\le k+2} \|\boldsymbol{e}_{0,\ell}\|, & \alpha = 0, \\ \frac{5}{16} \left(\frac{3}{4}\right)^j \|\boldsymbol{e}_{0,k}\|, & \alpha = 1/2, \\ \frac{33}{64} \left(\frac{7}{8}\right)^j \max\limits_{k-2\le\ell\le k+2} \|\boldsymbol{e}_{0,\ell}\|, & \alpha = 1. \end{cases}$$

The statement then follows because the Hausdorff distance is clearly bounded from above by the parametric distance between $\boldsymbol{f}$ and $\boldsymbol{f}_0$,

$$d_{\mathrm{H}}\big(\boldsymbol{f}([k,k+1]), [\boldsymbol{p}_{0,k}, \boldsymbol{p}_{0,k+1}]\big) \le \sup_{k\le s\le k+1} \|\boldsymbol{f}(s) - \boldsymbol{f}_0(s)\| \le \sum_{j=0}^{\infty} \sup_{k\le s\le k+1} \|\boldsymbol{f}_{2j+2}(s) - \boldsymbol{f}_{2j}(s)\|,$$

and by noticing that $\frac{9}{64} \sum_{j=0}^{\infty} \left(\frac{5}{8}\right)^{2j} = \frac{3}{13}$, $\frac{5}{16} \sum_{j=0}^{\infty} \left(\frac{3}{4}\right)^{2j} = \frac{5}{7}$, and $\frac{33}{64} \sum_{j=0}^{\infty} \left(\frac{7}{8}\right)^{2j} = \frac{11}{5}$. □

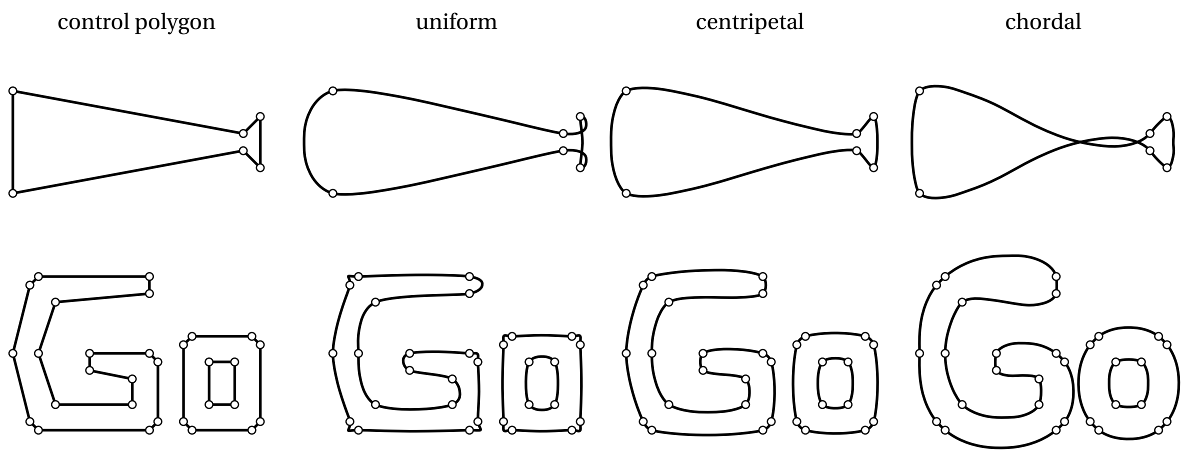

Figure 2: Examples of the four-point schemes.

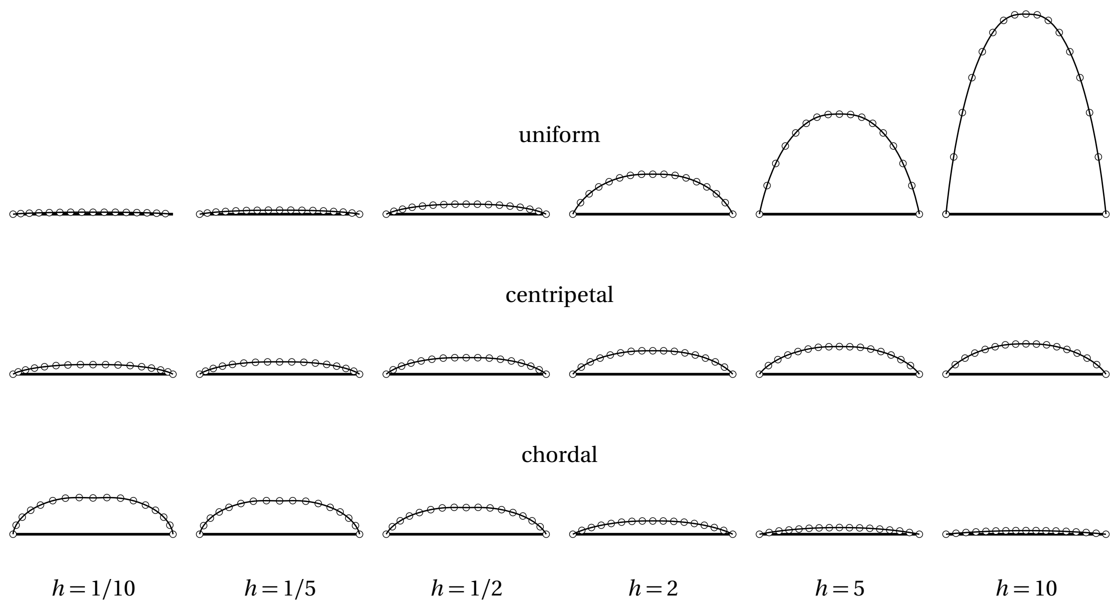

Figure 3: Shape effect over a rectangular control polygon.

5 Numerical examples

We have implemented Dubuc's scheme and its non-linear siblings corresponding to $\alpha = 1/2$ and $\alpha = 1$ in *C++*; see [6] for details. Figure 2 shows the different limit curves for several initial control polygons. The plots confirm the well-known effect that Dubuc's scheme tends to give curves that are very tight to long edges and overshoot at short ones, often leading to unwanted cusps and loops. On the other hand, the non-linear chordal scheme leads to very roundish shapes that closely follow the short edges and have relatively large distance to the long ones. The limit curves of the centripetal scheme nicely mediate between these two extremes: they are relatively close to all initial edges and still have a pleasing shape. Similar effects are known for cubic spline interpolation with uniform, centripetal, and chordal parameterization [4].

Another example that illustrates these shape effects is given in Figure 3 which shows the local behaviour of the limit curve over the top edge of a rectangle with fixed width 1 and varying height h. The dots mark the vertices of the refined polygon after four subdivision steps.

The examples above strongly suggest that the limit curves generated by the centripetal and chordal schemes are C^1, just like the limit curves of Dubuc's scheme, and indeed we conjecture that this is true in general. In the absence of a proof, numerical tests of smoothness are of great interest in order to support our conjecture. Figures 4, 5, and 6 visualize the results of such tests concerning the smoothness of the curves in the top row of Figure 2.

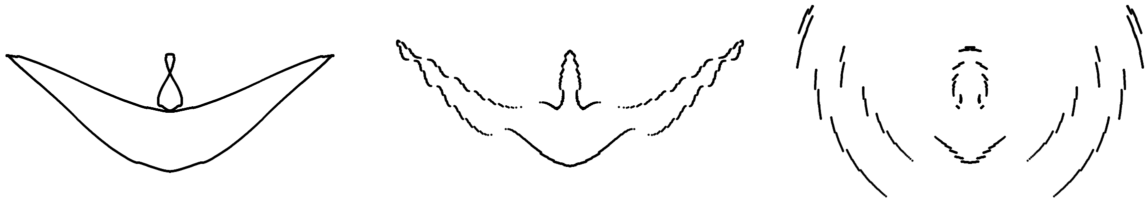

Figure 4: First divided differences $\{\hat{\boldsymbol{p}}_{j,k}^{[1]}\}$ with respect to uniform parameterization after $j = 10$ subdivision steps for the example in the top row of Figure 2.

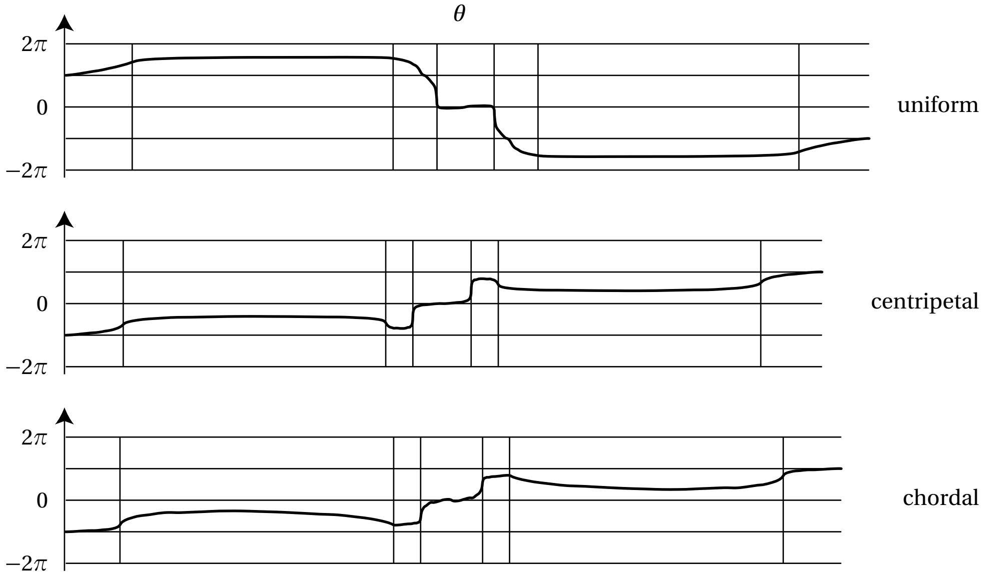

Figure 5: Normal angle over arc length for the example in the top row of Figure 2.

One difficulty in analyzing C^1-smoothness is that it is not clear with respect to which parameterization to work. Let us start with the uniform parameter values $\{\hat{t}_{j,k} = 2^{-j}k\}$ that we also use for the limit curve $\boldsymbol{f}$ in Theorem 2 and consider the first divided differences with respect to this parameterization,

$$\hat{\boldsymbol{p}}_{j,k}^{[1]} = \frac{\boldsymbol{p}_{j,k+1} - \boldsymbol{p}_{j,k}}{\hat{t}_{j,k+1} - \hat{t}_{j,k}} = 2^j(\boldsymbol{p}_{j,k+1} - \boldsymbol{p}_{j,k}),$$

regardless of the particular parameterization used to actually compute the points $\boldsymbol{p}_{j,k}$. It is then straightforward to prove that $\boldsymbol{f}$ is differentiable if the sequence of piecewise linear functions $\hat{\boldsymbol{f}}_j^{[1]}$ which interpolate $\hat{\boldsymbol{p}}_{j,k}^{[1]}$ at $\hat{t}_{j,k}$ converges uniformly, but Figure 4 indicates that the latter is not necessarily the case for the centripetal and chordal schemes.

Instead of the uniform parameterization, it turns out to be more promising to work with the chordal parameter values $\{\tilde{t}_{j,k}\}$. In our first numerical test we let $\theta_{j,k}$ be the angle of the local, outward-pointing unit normal vector

$$\boldsymbol{n}_{j,k} = \begin{pmatrix} \cos\theta_{j,k} \\ \sin\theta_{j,k} \end{pmatrix} \perp (\boldsymbol{p}_{j,k+1} - \boldsymbol{p}_{j,k})$$

and let $\theta_j \colon \mathbb{R} \to \mathbb{R}$ be the piecewise linear function that interpolates the data $(\tilde{t}_{j,k}, \theta_{j,k})$. In all our examples, the sequence $\theta_0, \theta_1, \dots$ converges to a continuous limit θ, as shown in Figure 5: here, each plot starts at the leftmost point of the curve where the normal is $(-1, 0)$, follows the curve in clockwise orientation, and the vertical lines refer to the positions of the initial control points. From this we conclude that also the piecewise linear functions $\tilde{\boldsymbol{f}}_j^{[1]}$ which interpolate the data $(\tilde{t}_{j,k}, \tilde{\boldsymbol{p}}_{j,k}^{[1]})$, with

$$\tilde{\boldsymbol{p}}_{j,k}^{[1]} = \frac{\boldsymbol{p}_{j,k+1} - \boldsymbol{p}_{j,k}}{\tilde{t}_{j,k+1} - \tilde{t}_{j,k}} = \frac{\boldsymbol{p}_{j,k+1} - \boldsymbol{p}_{j,k}}{\|\boldsymbol{p}_{j,k+1} - \boldsymbol{p}_{j,k}\|} = \begin{pmatrix} -\sin\theta_{j,k} \\ \cos\theta_{j,k} \end{pmatrix},$$

converge to a continuous limit, namely $\tilde{\boldsymbol{f}}^{[1]} = (-\sin\theta, \cos\theta)$.

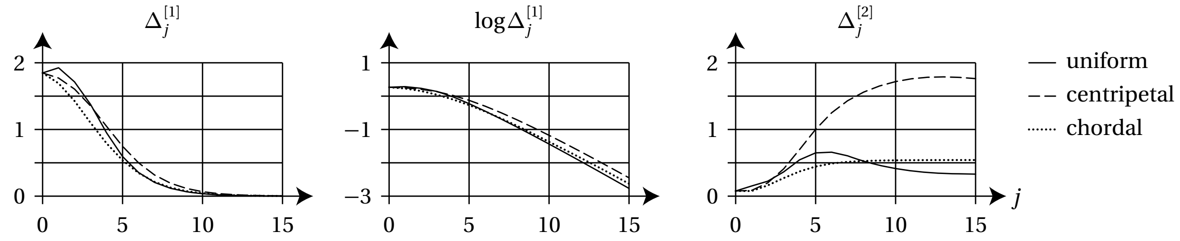

Figure 6: Maximum difference of the first and second divided differences with respect to chordal parameterization over the number of iterations for the example in the top row of Figure 2.

A second numerical test supports this conjecture: we check whether or not the lengths of the difference vectors $\boldsymbol{\delta}_{j,k}^{[1]} = \tilde{\boldsymbol{p}}_{j,k+1}^{[1]} - \tilde{\boldsymbol{p}}_{j,k}^{[1]}$ converge to zero as j increases, which is a necessary condition for the uniform convergence of the functions $\tilde{\boldsymbol{f}}_j^{[1]}$. In all our tests, this is indeed the case and we further find that the maximum length

$$\Delta_j^{[1]} = \sup_{k \in \mathbb{Z}} \|\boldsymbol{\delta}_{j,k}^{[1]}\|$$

decreases by a constant factor in the limit; compare the linear behaviour for large j in the log-plots of $\Delta_j^{[1]}$ in the middle of Figure 6. However, we can construct examples where $\Delta_j^{[1]}$ increases in the first two or even three subdivision steps before finally going to zero, which suggests that a general proof might be hard to establish. Figure 6 also shows the maximum difference $\Delta_j^{[2]}$ of the second divided differences $\tilde{\boldsymbol{p}}_{j,k}^{[2]}$, again with respect to chordal parameterization. It seems to always converge to a strictly positive value, indicating that the limit curves are not C^2, as expected.

We finally note that showing the C^1-continuity of the limit curve $\boldsymbol{f}$ not only requires to prove the uniform convergence of the $\tilde{\boldsymbol{f}}_j^{[1]}$, but also the convergence of the chordal parameterizations. Although the latter seems obvious, since the numerical results in Figures 2 and 3 indicate that all limit curves have finite length, we did not succeed in formally proving this statement.

References

[1] J. H. Ahlberg, E. N. Nilson, and J. L. Walsh. *The Theory of Splines and Their Applications*, volume 38 of *Mathematics in Science and Engineering*. Academic Press, New York, 1967.

[2] I. Daubechies, I. Guskov, and W. Sweldens. Regularity of irregular subdivision. *Constructive Approximation*, 15(3):381–426, Oct. 1999.

[3] S. Dubuc. Interpolation through an iterative scheme. *Journal of Mathematical Analysis and Applications*, 114(1):185–204, Feb. 1986.

[4] M. S. Floater. On the deviation of a parametric cubic spline interpolant from its data polygon. *Computer Aided Geometric Design*, 25(3):148–156, Mar. 2008.

[5] M. S. Floater and T. Surazhsky. Parameterization for curve interpolation. In K. Jetter, M. D. Buhmann, W. Haussmann, R. Schaback, and J. Stöckler, editors, *Topics in Multivariate Approximation and Interpolation*, volume 12 of *Studies in Computational Mathematics*, pages 39–54. Elsevier, Amsterdam, 2006.

[6] K. Hormann. Efficient evaluation of interpolating cubic polynomials. Technical Report IfI-08-04, Department of Informatics, Clausthal University of Technology, Aug. 2008.

[7] E. T. Y. Lee. Choosing nodes in parametric curve interpolation. *Computer-Aided Design*, 21(6):363–370, July–Aug. 1989.

[8] J. Wallner and N. Dyn. Convergence and C^1 analysis of subdivision schemes on manifolds by proximity. *Computer Aided Geometric Design*, 22(7):593–622, Oct. 2005.