Reconstruction of a function from Hermite–Birkhoff data

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Abstract

Birkhoff (or lacunary) interpolation is an extension of polynomial interpolation that appears when observation gives irregular information about some function and its derivatives. A Birkhoff interpolation problem is not always solvable even in the appropriate polynomial or rational space. In this paper we split up the initial problem in subproblems having a unique polynomial solution and use multinode rational basis functions in order to obtain a global interpolant.

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1 Birkhoff interpolation

In 1906, G. D. Birkhoff [4] studied the problem related to lacunary interpolation, that is, interpolation which appears whenever observation gives irregular information about a function and its derivatives. Few years later, in 1931, G. Pólya [22] gave a notable contribution by introducing certain algebraic inequalities that must be satisfied by the interpolation scheme to be regular, that is, solvable for each choice of pairwise distinct nodes and associated interpolation data. These papers received little attention until I. J. Schoenberg revived interest on the subject in 1966 [24], when he provided a generalization of the Pólya's theorem which gives a necessary condition to the existence of the solution. Lacunary or Birkhoff interpolation, in polynomial space, radically differs from Lagrange or Hermite interpolation in both its problems and its methods [20].

More precisely, let $X = \{x_1, x_2, \dots, x_n\}$ be a set of pairwise distinct real numbers, for which we assume that $x_1 < x_2 < \dots < x_n$. In the problem of interpolation of given data $f_{i,j} = f^{(j)}(x_i)$, $i = 1, \dots, n$, $j \in \mathcal{J}_i \subset \mathbb{N}$, by a polynomial p of appropriate degree,

$$p^{(j)}(x_i) = f_{i,j}$$

we mainly distinguish between *Hermite interpolation* and *Birkhoff interpolation*. We have a Hermite interpolation problem if, for each i, the indices j in the set $\mathcal{J}_i$ form an unbroken sequence, that is, $\mathcal{J}_i = \{0, 1, \dots, j_i\}$, a Birkhoff interpolation problem otherwise. In contrast to Hermite interpolation, a Birkhoff interpolation problem does not always have a unique solution or, even worse, does not have a solution. It is, however, convenient to consider Hermite interpolation to be a special case of lacunary interpolation and to deal with Hermite–Birkhoff interpolation.

For instance, there is no quadratic polynomial $p(x) = ax^2 + bx + c$ such that

$$p(-1) = p(1) = 0, \qquad p'(0) = 1. \tag{1}$$

In this case we can try to enlarge the space of possible solutions by considering rational functions

$$\left(\frac{p}{q}\right)^{(j)}(x_i) = f_{i,j},$$

instead of polynomials, hoping that the problem is solvable in the larger space. In [19], the univariate Birkhoff rational interpolation problem is investigated. First, The Birkhoff rational interpolation problem is converted into a polynomial system solving problem. Then the polynomial system is solved by means of Gröbner basis and thus the solution of the Birkhoff rational interpolation is obtained. However, by easy calculations, we can see that the problem

$$\left(\frac{p}{q}\right)(-1) = \left(\frac{p}{q}\right)(1) = 0, \qquad \left(\frac{p}{q}\right)'(0) = 1 \tag{2}$$

does not have a solution in the space of rational functions of the form

$$\left(\frac{p}{q}\right)(x) = \frac{ax + b}{x + c},$$

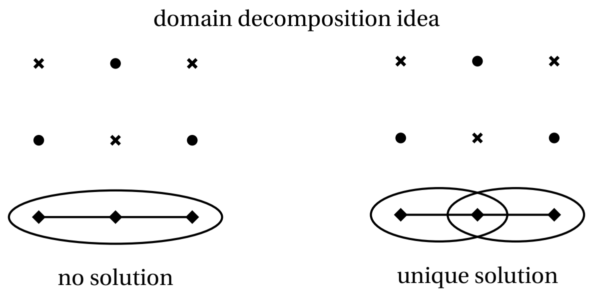

Figure 1: The unsolvable Hermite–Birkhoff problem (on the left) is split up into two solvable subproblems (on the right). We denote by a ball the available data and by a cross the not-available data. The first line relates to $f(x_i)$, the second one to $f'(x_i)$.

neither in the space of rational functions of the form

$$\left(\frac{p}{q}\right)(x) = \frac{a}{x^2 + bx + c},$$

which are appropriate spaces to consider if we are looking for a unique solution of (2). The interest in this kind of interpolation lies in the fact that, in recent years, many scholars applied the Birkhoff interpolation in numerically solving boundary value problems or initial-value problems and rational functions sometimes are superior to polynomials with the same interpolation data because they can achieve more accurate approximations with the same amount of computation (see [26] and the references therein).

In this paper we propose to split up the unsolvable problems in two or more solvable subproblems, as shown in Figure 1 for the particular case in (1), whose solutions can be blended together. Here we consider the case of *multinode basis functions* [17] as blending functions. To this end we consider a covering $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ of X by subsets $F_k \subset X$, such that, for each $k = 1, \ldots, m$, the corresponding Hermite–Birkhoff interpolation subproblem $p^{(j)}(x_i) = f_{i,j}$, $x_i \in F_k$, $j \in \mathcal{J}_i$ has a unique solution, and we associate to each F_k, $k = 1, \ldots, m$, a multinode basis function (see Section 2). The latter are then used in combination with the local Hermite–Birkhoff polynomials that interpolate the data associated to F_k (see Sections 3 and 4) and the approximation order of the combination is studied (see Section 6.1). Finally, we provide numerical experiments which confirm the theoretical results on the approximation order and show a good accuracy of approximation (see Section 6).

2 Multinode basis functions

Let us consider a covering $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ of X by its not empty subsets $F_k \subset X$, that is,

$$\bigcup_{k=1}^{m} F_k = X, \qquad F_k \neq \emptyset, \qquad \text{for each } k = 1, \ldots, m. \tag{3}$$

The multinode basis functions with respect to the covering $\mathcal{F}$ are defined by

$$B_{\mu,k}(x) = \frac{\prod_{x_i \in F_k} |x - x_i|^{-\mu}}{\sum_{l=1}^{m} \prod_{x_i \in F_l} |x - x_i|^{-\mu}}, \qquad k = 1, \ldots, m, \tag{4}$$

where $\mu > 0$ is a parameter that determines the differentiability class of the basis functions and controls the range of influence of the data values, in a sense that we specify later in Section 6 (see Figure 2). The multinode basis functions (4) are non-negative and form a partition of unity, that is,

$$\sum_{k=1}^{m} B_{\mu,k}(x) = 1, \tag{5}$$

but instead of being cardinal they satisfy the following properties.

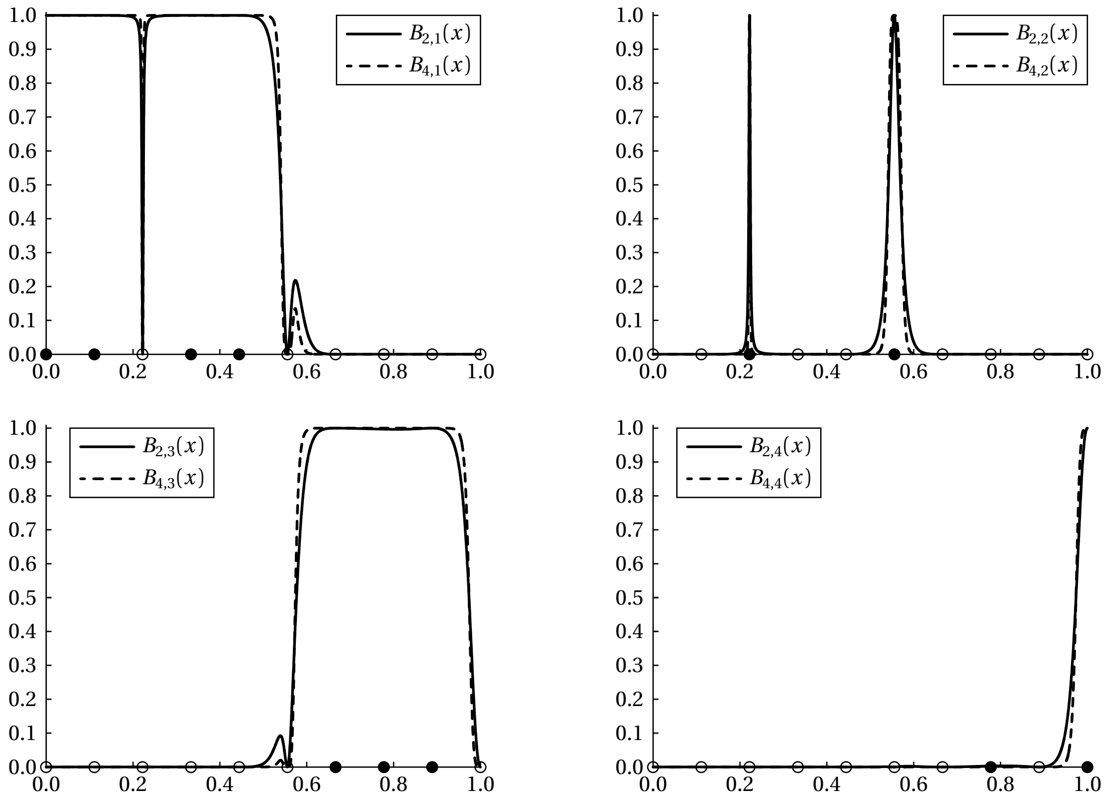

Figure 2: Multinode basis functions $B_{2,j}(x)$ and $B_{4,j}(x)$ for the subsets F_k displayed with black balls.

Proposition 1. *The multinode basis functions in* (4) *vanish at all nodes x_j that are not in F_k,*

$$B_{\mu,k}(x_j) = 0, \qquad \mu > 0, \tag{6}$$

for any $k = 1, \ldots, m$ and $x_j \notin F_k$, and

$$\sum_{k \in K_i} B_{\mu,k}(x_i) = 1, \qquad \mu > 0, \tag{7}$$

where

$$K_i = \{l \in \{1, \ldots, m\} : x_i \in F_l\} \neq \emptyset \tag{8}$$

is the set of indices of all subsets of $\mathcal{F}$ that contain x_i.

Proof. If we multiply both the numerator and the denominator of (4) with $|x - x_j|^\mu$, then

$$B_{\mu,k}(x) = \frac{C_k(x)}{\sum_{l=1}^{m} C_l(x)},$$

where

$$C_l(x) = |x - x_j|^\mu \prod_{x_i \in F_l} \frac{1}{|x - x_i|^\mu}, \qquad l = 1, \ldots, m.$$

Then, $C_l(x_j) = 0$ if and only if $l \notin K_j$, and (6) follows because $k \notin K_j$. Equality (7) follows by (6) and the partition of unity property (5). □

Proposition 2. *For $\mu > 0$ even integer, the multinode basis functions* (4) *are rational and have no real poles, otherwise their class of differentiability is $\mu - 1$ for μ odd integer and $[\mu]$, the largest integer not greater than μ, in all remaining cases. Moreover, all derivatives of order $\ell > 0$ vanish at all nodes x_j that are not in F_k,*

$$B^{(\ell)}_{\mu,k}(x_j) = 0, \tag{9}$$

for any $k = 1, \dots, m$ *and* $x_j \notin F_k$ *and*

$$\sum_{k\in K_i} B^{(\ell)}_{\mu,k}(x_i) = 0, \qquad \mu > 1. \tag{10}$$

Proof. If $\mu > 1$, then $C_l(x)$ is differentiable at x_j, and (9) follows because

$$B'_{\mu,k}(x) = \frac{C'_k(x)\sum_{l=1}^{m} C_l(x) + C_k(x)\sum_{k=1}^{m} C'_l(x)}{\left(\sum_{l=1}^{m} C_l(x)\right)^2}$$

and $C'_k(x_j) = 0$ for $x_j \notin F_k$. The procedure can be iterated until $C^{(\ell)}_l(x)$ exists at x_j. Equation (10) follows by differentiating both sides of (5) and by using Equation (9). □

Proposition 3. *The multinode basis functions* (4) *may be written in the following form*

$$B_{\mu,k}(x) = \frac{\prod_{x_i\notin F_k} |x - x_i|^{\mu}}{\sum_{l=1}^{m}\prod_{x_i\notin F_l} |x - x_i|^{\mu}}. \tag{11}$$

Proof. Expression (11) is obtainable from (4) by elementary calculations. □

3 Multinode local Hermite–Birkhoff interpolation

Let us consider the Hermite–Birkhoff interpolation problem

$$p^{(j)}[f](x_i) = f^{(j)}(x_i), \qquad i = 1, \dots, n, \quad j \in \mathcal{J}_i, \tag{12}$$

and let us assume that for each $k = 1, \dots, m$, the Hermite–Birkhoff interpolation subproblems

$$P^{(j)}_k[f](x_i) = f^{(j)}(x_i), \qquad x_i \in F_k, \quad j \in \mathcal{J}_i \tag{13}$$

have a unique solution in their appropriate polynomial spaces $\mathcal{P}^{q_k}_x$, where $q_k = \sum_{x_i\in F_k} \#(\mathcal{J}_i) - 1$. In order to test the solvability of the problems (12) and (13), one can use the necessary Pólya's condition [20] or the sufficient condition given by the Atkinson–Sharma theorem [5] and, in case of existence, the solution can be computed by solving a linear system of q_k linear equations in q_k unknowns. In fact, there are no explicit expressions for the polynomials $P_k(x)$, even though many algorithms for calculating them are known [18, 21, 23].

The use of determinants allows also to give an explicit expression for the remainder term

$$R_k[f](x) = f(x) - P_k[f](x),$$

which is obtained in [4] by using the Peano's kernel theorem [15] and from which a bound can be computed. Nonetheless, this formulation, for its extreme generality is not useful for our purpose. Therefore we state the following

Theorem 1. *Let us suppose, for each* $k = 1, \dots, m$, *that* $F_k = \{x_{k_1}, x_{k_2}, \dots, x_{k_l}\}$ *with* $x_{k_1} < x_{k_2} < \cdots < x_{k_l}$. *Let us set* $q_{\max} = \max_k q_k$ *and let* Ω *be a closed interval containing* X. *If* $f \in C^{q_{\max}+1}(\Omega)$, *then*

$$|f(x) - P_k[f](x)| \le (1 + \|P_k\|_\infty)\frac{\prod_{x_i\in F_k} |x - x_i|^{\#(\mathcal{J}_i)}}{(q_k + 1)!}|f^{(q_k+1)}(\xi_k)|, \tag{14}$$

where $\min(x, x_{k_1}) < \xi_k < \max(x, x_{k_l})$.

Proof. Since we require the uniqueness of the solution of the local Hermite–Birkhoff interpolation problem (13), the operator P_k is a projector [8] on the polynomial space $\mathcal{P}^{q_k}_x$, that is, it reproduces polynomials up to the degree q_k. Let denote by $H_k[f](x)$ the Hermite interpolation polynomial on the nodes of F_k with $\#(\mathcal{J}_i)$ interpolation conditions on each node $x_i \in F_k$. Then we have

$$\begin{aligned}|f(x)-P_k[f](x)| &\leq |f(x)-H_k[f](x)|+|H_k[f](x)-P_k[f](x)| \\ &\leq |f(x)-H_k[f](x)|+|P_k H_k[f](x)-P_k[f](x)| \\ &\leq (1+\|P_k\|)\,|f(x)-H_k[f](x)|.\end{aligned} \tag{15}$$

On the other hand, it is well known a Cauchy representation for the remainder term in Hermite interpolation [15],

$$f(x)-H_k[f](x)=\frac{\prod_{x_i\in F_k}(x-x_i)^{\#(\mathcal{J}_i)}}{(q_k+1)!}f^{(q_k+1)}(\xi_k), \tag{16}$$

where $\min(x, x_{k_1}) < \xi_k < \max(x, x_{k_l})$. Inequality (14) follows by taking the modulus of both sides of (16) and by substituting it in (15). □

4 Multinode global interpolation operator

As soon as we have provided a solution for all local Hermite–Birkhoff interpolation problems, we define the multinode global interpolation operator by

$$M_\mu[f,\mathcal{F}](x)=\sum_{k=1}^{m}B_{\mu,k}(x)P_k[f](x), \tag{17}$$

where $P_k[f](x)$ is the polynomial solution of the Hermite–Birkhoff interpolation problem on F_k. The operator $M_\mu[f,\mathcal{F}]$ has remarkable properties. Firstly, it reproduces polynomials up to a certain degree as specified in the following proposition.

Proposition 4. *The operator $M_\mu[f,\mathcal{F}]$ reproduces polynomials up to the degree $q_{\min}=\min_k q_k$.*

Proof. Since the Hermite–Birkhoff interpolation subproblems (13) have a unique solution in the polynomial spaces $\mathcal{P}_x^{q_k}$, $P_k[f](x)$ reproduces polynomials up to degree q_k for $k=1,\ldots,m$. Moreover, the basis functions $B_{\mu,k}(x)$ are a partition of unity and therefore, for each $p\in\mathcal{P}_x^{q_{\min}}$,

$$M_\mu[p,\mathcal{F}](x)=\sum_{k=1}^{m}B_{\mu,k}(x)P_k[p](x)=\sum_{k=1}^{m}B_{\mu,k}(x)p(x)=p(x).$$

□

Remark 1. If the initial problem has a unique polynomial solution, by setting $\mathcal{F}=\{X\}$, $M_\mu[f,\mathcal{F}](x)$ coincides with that polynomial solution.

Secondly, the operator $M_\mu[f,\mathcal{F}]$ is an interpolation operator as specified in the following theorem.

Theorem 2. *The operator $M_\mu[f,\mathcal{F}]$ interpolates the functional data*

$$M_\mu[f,\mathcal{F}](x_i)=f(x_i), \qquad \text{for each } i, \text{ such that } 0\in\mathcal{J}_i \tag{18}$$

and if $\mathcal{F}$ is a partition of X (i.e., $F_\alpha\cap F_\beta=\emptyset$ for each $\alpha\neq\beta$), then the operator $M_\mu[f,\mathcal{F}]$ interpolates all data used in its definition, that is,

$$M_\mu^{(j)}[f,\mathcal{F}](x_i)=f^{(j)}(x_i), \qquad \text{for each } k=1,\ldots,m, \quad x_i\in F_k, \quad j\in\mathcal{J}_i.$$

Proof. Let $x_i\in X$ such that $0\in\mathcal{J}_i$. Then we have

$$M_\mu[f,\mathcal{F}](x_i)=\sum_{k=1}^{m}B_{\mu,k}(x_i)P_k[f](x_i)=\sum_{k=1}^{m}B_{\mu,k}(x_i)f(x_i)=f(x_i).$$

Let $\ell\in\{1,\ldots,q_{\min}\}$ and $\mu>q_{\min}+1$. By differentiating ℓ times (17) we get, by the Leibniz rule for differentiation,

$$M_\mu^{(\ell)}[f,\mathcal{F}](x)=\sum_{k=1}^{m}\sum_{\iota=0}^{\ell}\binom{\ell}{\iota}B_{\mu,k}^{(\ell-\iota)}(x)P_k^{(\iota)}[f](x)=\sum_{k=1}^{m}\sum_{\iota=0}^{\ell-1}\binom{\ell}{\iota}B_{\mu,k}^{(\ell-\iota)}(x)P_k^{(\iota)}[f](x)+\sum_{k=1}^{m}B_{\mu,k}(x)P_k^{(\ell)}[f](x).$$

Let $x_i \in X$ and $\ell \in \mathcal{J}_i$. Since $\mathcal{F}$ is a partition of X then $K_i = \{\kappa\}$. By properties (6) and (7) we have

$$\sum_{k=1}^{m} B_{\mu,k}(x_i) P_k^{(\ell)}[f](x_i) = \sum_{\substack{k=1\\k\neq\kappa}}^{m} B_{\mu,k}(x_i) P_k^{(\ell)}[f](x_i) + B_{\mu,\kappa}(x_i) P_\kappa^{(\ell)}[f](x_i) = f^{(\ell)}(x_i).$$

On the other hand, by properties (9) and (10) we have

$$\sum_{k=1}^{m} B_{\mu,k}^{(\ell-\iota)}(x_i) P_k^{(\iota)}[f](x_i) = \sum_{\substack{k=1\\k\neq\kappa}}^{m} B_{\mu,k}^{(\ell-\iota)}(x_i) P_k^{(\iota)}[f](x_i) + B_{\mu,\kappa}^{(\ell-\iota)}(x_i) P_\kappa^{(\iota)}[f](x_i) = 0$$

for each $\iota = 0, \ldots, \ell-1$. □

Let us observe that the operator $M_\mu[f,\mathcal{F}]$ could not interpolate all derivative data at some x_κ if $\#(K_\kappa) > 1$ and the sequence of indices in $\mathcal{J}_\kappa$ is broken.

Example 1. For example, let us assume

$$\#(K_\kappa) = 2, \qquad F_\alpha \cap F_\beta = \{x_\kappa\}, \qquad \mathcal{J}_\kappa = \{0, 2, \ldots, \ell-1, \ell\} \quad, \ell \geq 2$$

and

$$B_{\mu,\alpha}^{(\ell-1)}(x_\kappa) P'_\alpha[f](x_\kappa) + B_{\mu,\beta}^{(\ell-1)}(x_\kappa) P'_\alpha[f](x_\kappa) \neq 0.$$

We note that

$$P'_\alpha[f](x_\kappa) \neq P'_\beta[f](x_\kappa),$$

by property (10). From

$$P_\alpha^{(\ell)}[f](x_\kappa) = P_\beta^{(\ell)}[f](x_\kappa) = f^{(\ell)}(x_\kappa),$$

by properties (6) and (7), it easily follows that

$$\sum_{k=1}^{m} B_{\mu,k}(x_\kappa) P_k^{(\ell)}[f](x_\kappa) = B_{\mu,\alpha}(x_\kappa) f^{(\ell)}(x_\kappa) + B_{\mu,\beta}(x_\kappa) f^{(\ell)}(x_\kappa) = f^{(\ell)}(x_\kappa).$$

On the other hand,

$$\sum_{k=1}^{m} \sum_{\iota=0}^{\ell-1} \binom{\ell}{\iota} B_{\mu,k}^{(\ell-\iota)}(x_\kappa) P_k^{(\iota)}[f](x_\kappa) = \sum_{\iota=0}^{\ell-1} \binom{\ell}{\iota} \Big(B_{\mu,\alpha}^{(\ell-\iota)}(x_\kappa) P_\alpha^{(\iota)}[f](x_\kappa) + B_{\mu,\beta}^{(\ell-\iota)}(x_\kappa) P_\alpha^{(\iota)}[f](x_\kappa) \Big)$$

by property (9). Let us fix our attention to the right hand side of the previous equation. For each $\iota \in \mathcal{J}_\kappa$, we get

$$B_{\mu,\alpha}^{(\ell-\iota)}(x_\kappa) P_\alpha^{(\iota)}[f](x_\kappa) + B_{\mu,\beta}^{(\ell-\iota)}(x_\kappa) P_\alpha^{(\iota)}[f](x_\kappa) = \Big(B_{\mu,\alpha}^{(\ell-\iota)}(x_\kappa) + B_{\mu,\beta}^{(\ell-\iota)}(x_\kappa) \Big) f^{(\iota)}(x_\kappa) = 0$$

by property (10), but

$$B_{\mu,\alpha}^{(\ell-1)}(x_\kappa) P'_\alpha[f](x_\kappa) + B_{\mu,\beta}^{(\ell-1)}(x_\kappa) P'_\alpha[f](x_\kappa) \neq 0$$

and consequently

$$M_\mu^{(\ell)}[f,\mathcal{F}](x_\kappa) \neq f^{(\ell)}(x_\kappa).$$

In order to avoid this trouble, we proceed as follows. For each $\kappa = 1, \ldots, n$ let $\nu_\kappa = \#(K_\kappa)$ and $F_{\alpha_1}, \ldots, F_{\alpha_{\nu_\kappa}}$ the subsets of X which contain x_κ. As above, let us denote by $P_{\alpha_1}[f], \ldots, P_{\alpha_{\nu_\kappa}}[f]$ the polynomial solutions of the Hermite–Birkhoff interpolation problems on $F_{\alpha_1}, \ldots, F_{\alpha_{\nu_\kappa}}$, respectively. For all $j = 0, 1, \ldots, \max(\mathcal{J}_\kappa)$, we set

$$\tilde{f}^{(j)}(x_\kappa) = \frac{1}{\nu_\kappa} \Big(P_{\alpha_1}^{(j)}[f](x_\kappa) + \cdots + P_{\alpha_{\nu_\kappa}}^{(j)}[f](x_\kappa) \Big) \tag{19}$$

and we note that

$$\tilde{f}^{(j)}(x_\kappa) = f^{(j)}(x_\kappa), \tag{20}$$

as soon as $j \in \mathcal{J}_\kappa$. For each $k = 1, \ldots, m$, we call the Hermite interpolation problem

$$\tilde{P}_k^{(j)}[f](x_i) = \tilde{f}^{(j)}(x_i), \qquad x_i \in F_k, \quad j = 0, 1, \ldots, \max(\mathcal{J}_i) \tag{21}$$

a *Hermitian completion* of the Hermite–Birkhoff interpolation problem (13). It is well known that each interpolation problem (21) has a unique solution $\tilde{P}_k[f](x)$ in the polynomial space $\mathcal{P}_x^{d_k}$, where $d_k = \#(F_k) + \sum_{x_i \in F_k} \max(\mathcal{J}_i) - 1$, for which there are explicit formulas in Lagrange or Newton form (see [3, 11, 25] and the references therein). Nevertheless, as soon as $\nu_k > 1$ and at least two among $P_{\alpha_1}^{(j)}[f](x_\kappa), \dots, P_{\alpha_{\nu_\kappa}}^{(j)}[f](x_\kappa)$ are different, we get $q_k < d_k$; in this case, if $p \in \mathcal{P}_x^{q_k}$ is a generic polynomial, then $\tilde{P}_k[p]$ is different from p, since, by (19), we have completed the lacunary data using solutions of several interpolation problems. In fact we have

$$\tilde{q}_k = \operatorname{dex}(\tilde{P}_k[\cdot]) = \min_{j=0,1,\dots,\max(\mathcal{J}_i)} \operatorname{dex}(P_{\alpha_j}[\cdot])$$

and the proof is straightforward. Consequently, $q_k \le \tilde{q}_k$. Results on the bound for the remainder term

$$\tilde{R}_k[f](x) = f(x) - \tilde{P}_k[f](x)$$

can be obtained similarly to Theorem 1, taking into account that $\operatorname{dex}(\tilde{P}_k[\cdot]) = \tilde{q}_k$.

Theorem 3. *Let us suppose, for each $k = 1, \dots, m$, that $F_k = \{x_{k_1}, x_{k_2}, \dots, x_{k_l}\}$ with $x_{k_1} < x_{k_2} < \dots < x_{k_l}$. Let us set $q_{\max} = \max_k q_k$ and let Ω be a closed interval containing X. If $f \in C^{q_{\max}+1}(\Omega)$, then*

$$|f(x) - \tilde{P}_k[f](x)| \le (1 + \|\tilde{P}_k\|_\infty) \frac{\prod_{x_i \in F_k} (x - x_i)^{\#(\mathcal{J}_i) - \#(\mathcal{S}_i)}}{(\tilde{q}_k + 1)!} |f^{(\tilde{q}_k+1)}(\xi_k)|,$$

where $\min(x, x_{k_1}) < \xi_k < \max(x, x_{k_l})$ and $\mathcal{S}_i \subset \mathcal{J}_i$, $i = 1, \dots, n$, such that $\sum_{x_i \in F_k} \#(\mathcal{J}_i) - \sum_{x_i \in F_k} \#(\mathcal{S}_i) = \tilde{q}_k + 1$.

Proof. Since $\operatorname{dex}(\tilde{P}_k[\cdot]) = \tilde{q}_k$, the operator $\tilde{P}_k$ is a projector [8] on the polynomial space $\mathcal{P}_x^{\tilde{q}_k}$, that is, it reproduces polynomials up to the degree $\tilde{q}_k$. Let us denote by $\tilde{H}_k[f](x)$ the Hermite interpolation polynomial on the nodes of F_k with $\#(\mathcal{J}_i) - \#(\mathcal{S}_i)$ interpolation conditions on each node $x_i \in F_k$. Then we have

$$\begin{aligned} |f(x) - \tilde{P}_k[f](x)| &\le |f(x) - \tilde{H}_k[f](x)| + |\tilde{H}_k[f](x) - \tilde{P}_k[f](x)| \\ &\le |f(x) - \tilde{H}_k[f](x)| + |\tilde{P}_k \tilde{H}_k[f](x) - \tilde{P}_k[f](x)| \\ &\le (1 + \|\tilde{P}_k\|) |f(x) - \tilde{H}_k[f](x)|. \end{aligned}$$

Therefore,

$$f(x) - \tilde{H}_k[f](x) = \frac{\prod_{x_i \in F_k} (x - x_i)^{\#(\mathcal{J}_i) - \#(\mathcal{S}_i)}}{(\tilde{q}_k + 1)!} f^{(\tilde{q}_k+1)}(\xi_k),$$

where $\min(x, x_{k_1}) < \xi_k < \max(x, x_{k_l})$. □

Example 2. Let $X = \{-1, 0, 1\}$ with interpolation conditions as in (1) or Figure 1. In this case, $F_1 = \{-1, 0\}$, $F_2 = \{0, 1\}$,

$$\begin{aligned} P_1[f](x) &= f(-1) + (1 + x) f'(0), \\ P_2[f](x) &= f(1) + (-1 + x) f'(0), \end{aligned}$$

and

$$\begin{aligned} \tilde{P}_1[f](x) &= \frac{f(-1) + f(1)}{2} + f'(0)x + 3\left(\frac{f(-1) - f(1)}{2} + f'(0)\right)x^2 + \big(f(-1) + f(1) + 2f'(0)\big)x^3, \\ \tilde{P}_2[f](x) &= \frac{f(-1) + f(1)}{2} + f'(0)x + 3\left(\frac{f(1) - f(-1)}{2} - f'(0)\right)x^2 + \big(f(-1) - f(1) + 2f'(0)\big)x^3. \end{aligned}$$

Let $f(x) = \sin(x)$. In Figure 3 we show the polynomials $P_1[f]$ and $P_2[f]$, and their Hermitian completions $\tilde{P}_1[f]$ and $\tilde{P}_2[f]$, respectively. In Figure 4 we show the absolute values of the errors $e[f, \mathcal{F}](x) = f(x) - M_2[f, \mathcal{F}](x)$ and $\tilde{e}[f, \mathcal{F}](x) = f(x) - \tilde{M}_2[f, \mathcal{F}](x)$ in the interval $[-1, 1]$, where

$$M_2[f, \mathcal{F}](x) = B_{2,1}(x) P_1[f](x) + B_{2,2}(x) P_2[f](x)$$

and

$$\tilde{M}_2[f, \mathcal{F}](x) = B_{2,1}(x) \tilde{P}_1[f](x) + B_{2,2}(x) \tilde{P}_2[f](x).$$

As we can see, at least in this case, the inequality $|e[f, \mathcal{F}](x)| \le |\tilde{e}[f, \mathcal{F}](x)|$ holds for all $x \in [-1, 1]$.

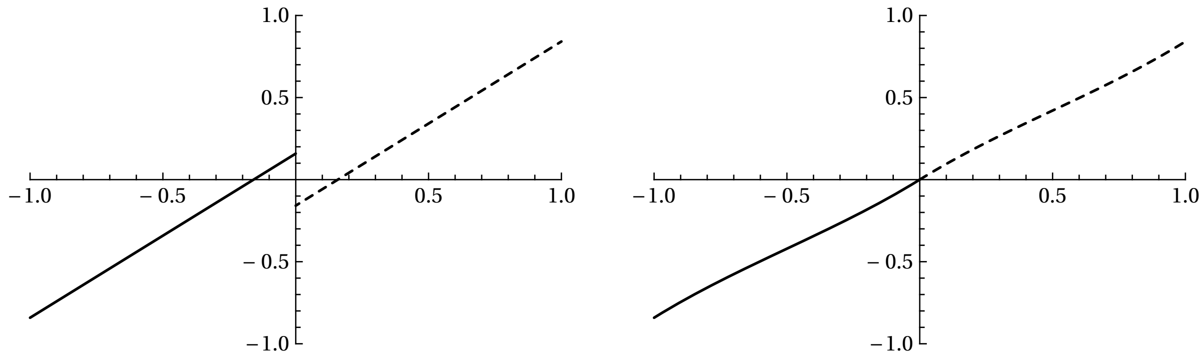

Figure 3: Polynomials $P_1[f]$ (solid) and $P_2[f]$ (dashed), on the left, and their Hermitian completions $\tilde{P}_1[f]$ and $\tilde{P}_2[f]$, respectively, on the right.

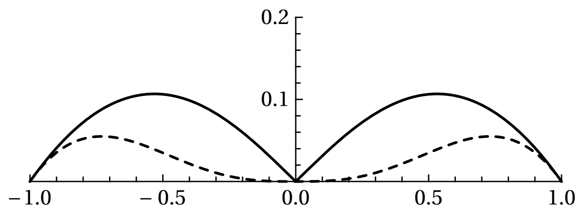

Figure 4: Comparison between the absolute values of the errors $e[f,\mathcal{F}](x)$ (solid) and $\tilde{e}[f,\mathcal{F}](x)$ (dashed) for the function $f(x)=\sin(x)$ in the interval $[-1,1]$.

We set

$$\tilde{M}_\mu[f,\mathcal{F}](x)=\sum_{k=1}^{m} B_{\mu,k}(x)\tilde{P}_k[f](x).$$

The operator $\tilde{M}_\mu[\cdot,\mathcal{F}]$ preserves the reproducing polynomial property of $M_\mu[\cdot,\mathcal{F}]$ and, in addition, interpolates all derivative data. In fact we have

Proposition 5. *The operator $\tilde{M}_\mu[f,\mathcal{F}]$*

(a) reproduces polynomials up to the degree $q_{\min}=\min_k q_k$;

(b) interpolates all data used in its definition, that is,

$$\tilde{M}_\mu^{(j)}[f,\mathcal{F}](x_i)=f^{(j)}(x_i), \qquad \text{for each } k=1,\ldots,m, \quad x_i\in F_k, \quad j\in\mathcal{J}_i.$$

Proof.

(a) Let $p\in\mathcal{P}_x^{q_{\min}}$. Since the Hermite–Birkhoff interpolation subproblems (13) have a unique solution in the polynomial spaces $\mathcal{P}_x^{q_k}$, we have $P_k[p]=p$ for each $k=1,\ldots,m$. Setting (19) now becomes

$$\tilde{p}^{(j)}(x_k)=\frac{1}{\nu_k}\left(p^{(j)}(x_k)+\cdots+p^{(j)}(x_k)\right)=p^{(j)}(x_k),$$

and consequently the Hermitian completion of the Hermite–Birkhoff interpolation problem (13),

$$\tilde{P}_k^{(j)}[p](x_i)=\tilde{p}^{(j)}(x_k), \qquad x_i\in F_k, \quad j=0,1,\ldots,\max(\mathcal{J}_i),$$

has the solution $\tilde{P}_k[p]=p$, since $q_k\le d_k$. The statement then follows, because the basis functions $B_{\mu,k}(x)$ are a partition of unity.

(b) Let $\ell\in\{0,1,\ldots,q\}$ and $\mu>q+1$. By differentiating ℓ times (17), we get, by the Leibniz rule for differentiation,

$$\begin{aligned}\tilde{M}_\mu^{(\ell)}[f,\mathcal{F}](x)&=\sum_{k=1}^{m}\sum_{\iota=0}^{\ell}\binom{\ell}{\iota}B_{\mu,k}^{(\ell-\iota)}(x)\tilde{P}_k^{(\iota)}[f](x)\\&=\sum_{k=1}^{m}\sum_{\iota=0}^{\ell-1}\binom{\ell}{\iota}B_{\mu,k}^{(\ell-\iota)}(x)\tilde{P}_k^{(\iota)}[f](x)+\sum_{k=1}^{m}B_{\mu,k}(x)\tilde{P}_k^{(\ell)}[f](x).\end{aligned}$$

Let $x_i \in X$ and $F_{\alpha_1}, \ldots, F_{\alpha_{\nu_i}}$ be the subsets of X which contain x_i and $\ell \in \mathcal{J}_i$. By properties (6) and (7), we have

$$\begin{aligned}
\sum_{k=1}^{m} B_{\mu,k}(x_i)\tilde{P}_k^{(\ell)}[f](x_i) &= \sum_{\substack{k=1 \\ k\notin\{\alpha_1,\ldots,\alpha_{\nu_i}\}}}^{m} B_{\mu,k}(x_i)\tilde{P}_k^{(\ell)}[f](x_i) + \sum_{\substack{k=1 \\ k\in\{\alpha_1,\ldots,\alpha_{\nu_i}\}}}^{m} B_{\mu,k}(x_i)\tilde{P}_k^{(\ell)}[f](x_i) \\
&= \sum_{\substack{k=1 \\ k\neq\kappa}}^{m} 0 + \tilde{f}^{(\ell)}(x_i) \sum_{\substack{k=1 \\ k\in\{\alpha_1,\ldots,\alpha_{\nu_i}\}}}^{m} B_{\mu,k}(x_i) \\
&= f^{(\ell)}(x_i),
\end{aligned}$$

since (21) and (20). On the other hand, for each $\iota = 0, \ldots, \ell-1$, we have

$$\begin{aligned}
\sum_{k=1}^{m} B_{\mu,k}^{(\ell-\iota)}(x_i)\tilde{P}_k^{(\iota)}[f](x_i) &= \sum_{\substack{k=1 \\ k\notin\{\alpha_1,\ldots,\alpha_{\nu_i}\}}}^{m} B_{\mu,k}^{(\ell-\iota)}(x_i)\tilde{P}_k^{(\iota)}[f](x_i) + \sum_{\substack{k=1 \\ k\in\{\alpha_1,\ldots,\alpha_{\nu_i}\}}}^{m} B_{\mu,k}^{(\ell-\iota)}(x_i)\tilde{P}_k^{(\iota)}[f](x_i) \\
&= \sum_{\substack{k=1 \\ k\neq\kappa}}^{m} 0 + \tilde{f}^{(\iota)}(x_i) \sum_{\substack{k=1 \\ k\in\{\alpha_1,\ldots,\alpha_{\nu_i}\}}}^{m} B_{\mu,k}^{(\ell-\iota)}(x_i) \\
&= 0,
\end{aligned}$$

since (9), (10), and (21).

□

In the next sections we discuss the approximation order of the operators $M_\mu[f,\mathcal{F}]$ and $\tilde{M}_\mu[f,\mathcal{F}]$, and we will compare their accuracies of approximation in several interesting cases.

5 The approximation order

Let us now study the approximation order of the operators $M_\mu[f,\mathcal{F}]$ and $\tilde{M}_\mu[f,\mathcal{F}]$. To this end let $\|\cdot\|$ be the maximum norm and $R_\rho(y) = \{x \in \mathbb{R} : |x-y| \le \rho\}$ be the closed interval with centre y and size 2ρ. We define

$$h = \inf\{\rho > 0 : \forall x \in \Omega\ \exists F_k \in \mathcal{F} : R_\rho(x) \cap F_k \neq \emptyset\} \tag{22}$$

and

$$L = \inf\{l > 0 : \forall F_k \in \mathcal{F}\ \exists x \in \Omega : F_k \subset R_{lh}(x)\}, \tag{23}$$

$$M = \sup_{x\in\Omega} \#\{F_k \in \mathcal{F} : R_h(x) \cap F_k \neq \emptyset\}, \tag{24}$$

$$N = \max_i \#\{F_k \in \mathcal{F} : x_i \in F_k\} = \max_i \#\{K_i\}. \tag{25}$$

A small value of h corresponds to a rather uniform distribution of nodes, but does not exclude the presence of large F_k (note that Lh is half the diameter of the largest F_k). A large value of M corresponds to clustered F_k and N is the maximum number of F_k with at least a common node ($N = 1$ means that $\mathcal{F}$ is a partition of X). Moreover, we set

$$N_k = \sum_{x_i \in F_k} \#(\mathcal{J}_i) = q_k + 1,$$

and

$$C_{F_{\max}} = \max_k \#(F_k), \qquad C_{F_{\min}} = \min_k \#(F_k),$$

$$C_{N_{\max}} = \max_k N_k, \qquad C_{N_{\min}} = \min_k N_k.$$

Theorem 4. *Let Ω be an interval that contains X, $f \in C^{q_{\max}+1}(\Omega)$, $\mu > \frac{1+C_{N_{\max}}}{C_{F_{\min}}}$, and $\#(F_k) = const$ for each k. Then,*

$$|f(x) - M_\mu[f,\mathcal{F}](x)| \le C M h^{C_{N_{\min}}} \phi_{\max}$$

for any $x \in \Omega$, with C a positive constant which depends on F_k and μ, and $\phi_{\max} = \max_{j=0,\ldots,q_{\max}} \|f^{(j)}\|_\infty$.

Proof. For $x \in \Omega$ let

$$Q_h(y) = \{x \in \mathbb{R} : y - h < x \leq y + h\}$$

be the half-open interval with centre y and size $2h$. Let $T_j = T_{j,h}(x)$ be the half-open annulus with centre x and radius $2hj$ defined by

$$T_j = Q_h(x - 2hj) \cup Q_h(x + 2hj), \qquad j = 0, \dots, N.$$

Note that $T_0 = Q_h(x)$ and that

$$\Omega \subset \bigcup_{j=0}^{N} T_j, \qquad N = \left[\frac{\mathrm{diam}(\Omega)}{2h}\right] + 1.$$

By settings (22) and (24) we have

$$1 \leq \#(X \cap T_{j,h}) \leq 2M.$$

$T_0 = Q_h(x)$ contains at least one node by setting (22); therefore, for each F_k with at least one node in T_0, we have

$$\prod_{x_i \in F_k} |x - x_i| \leq (Lh)^{\#(F_k)}, \tag{26}$$

because of 23. Let us consider now the sets F_k with at least one node in T_1 and no nodes in T_0; we have

$$h^{\#(F_k)} \leq \prod_{x_i \in F_k} |x - x_i| \leq \left[(L+3)h\right]^{\#(F_k)},$$

while for the sets F_k with at least one node in T_2 and no nodes in $T_1 \cup T_0$ we have

$$(3h)^{\#(F_k)} \leq \prod_{x_i \in F_k} |x - x_i| \leq \left[(L+5)h\right]^{\#(F_k)},$$

and in general, for the sets F_k with at least one node in T_j and no nodes in $T_{j-1} \cup \cdots \cup T_0$,

$$\left[(2j-1)h\right]^{\#(F_k)} \leq \prod_{x_i \in F_k} |x - x_i| \leq \left[(L+2j+1)h\right]^{\#(F_k)}. \tag{27}$$

Let us now turn to the approximation error

$$e(x) = |f(x) - M_\mu[f, \mathcal{F}](x)|$$

of the multinode global operator at x. By (17) and the fact that the basis functions $B_{k,\mu}(x)$ are non-negative and form a partition of unity,

$$e(x) \leq \left| \sum_{k=1}^{m} B_{k,\mu}(x) f(x) - \sum_{k=1}^{m} B_{k,\mu}(x) P_k[f](x) \right| \leq \sum_{k=1}^{m} |f(x) - P_k[f](x)| B_{k,\mu}(x).$$

Using Proposition 1 and (4), we then get

$$e(x) \leq \sum_{k=1}^{m} (1 + \|P_k\|) \frac{\prod_{x_i \in F_k} |x - x_i|^{\#(\mathcal{J}_i)}}{(q_k+1)!} |f^{(q_k+1)}(\xi_k)| \frac{\prod_{x_i \in F_k} |x - x_i|^{-\mu}}{\sum_{l=1}^{m} \prod_{x_j \in F_l} |x - x_j|^{-\mu}}. \tag{28}$$

Let $F_{k_{\min}} \in \mathcal{F}$ be the subset, such that

$$\prod_{x_i \in F_{k_{\min}}} |x - x_i| = \min_k \prod_{x_i \in F_k} |x - x_i|.$$

Then,

$$\prod_{x_i \in F_{k_{\min}}} |x - x_i| \leq (Lh)^{\#(F_{k_0})},$$

since at least one set F_{k_0} has a node in T_0. Finally, we have to bound, for each $k = 1, \dots, m$,

$$\prod_{x_i \in F_k} |x - x_i|^{\#(\mathcal{J}_i)}.$$

For each F_k with at least one node in T_0, we have

$$\prod_{x_i\in F_k}|x-x_i|^{\#(\mathcal{J}_i)} \le \prod_{x_i\in F_k}(Lh)^{\#(\mathcal{J}_i)} \le (Lh)^{N_k},$$

and for each F_k with no nodes in $T_0\cup\cdots\cup T_{j-1}$ and at least one node in T_j,

$$\prod_{x_i\in F_k}|x-x_i|^{\#(\mathcal{J}_i)} \le \prod_{x_i\in F_k}\big((L+2j+1)h\big)^{\#(\mathcal{J}_i)} \le \big((L+2j+1)h\big)^{N_k}.$$

Then we get

$$\begin{aligned}
e(x) &\le \sum_{k=1}^{m}(1+\|P_k\|)\frac{\prod_{x_i\in F_k}|x-x_i|^{\#(\mathcal{J}_i)}}{(q_k+1)!}|f^{(q_k+1)}(\xi_k)|\frac{\prod_{x_i\in F_k}|x-x_i|^{-\mu}}{\prod_{x_i\in F_{k_{\min}}}|x-x_i|^{-\mu}}\\
&\le \prod_{x_i\in F_{k_{\min}}}|x-x_i|^{\mu}\sum_{k=1}^{m}(1+\|P_k\|)\frac{\prod_{x_i\in F_k}|x-x_i|^{\#(\mathcal{J}_i)}}{(q_k+1)!}|f^{(q_k+1)}(\xi_k)|\prod_{x_i\in F_k}|x-x_i|^{-\mu}.
\end{aligned}$$

We denote by $\mathcal{I}_j$ the set of subsets F_k with at least one node in T_j and no nodes in $T_0\cup\cdots\cup T_{j-1}$. By construction, $\bigcup_{j=0}^{N}\mathcal{I}_j=\mathcal{F}$ and $\bigcap_{j=0}^{N}\mathcal{I}_j=\emptyset$. Moreover, if we let

$$\begin{aligned}
P_{\max} &= \max_k \|P_k\|,\\
\phi_{\max} &= \max_{j=0,\dots,q_{\max}}\|f^{(j)}\|_\infty,
\end{aligned}$$

then by bounding (28) we have

$$\begin{aligned}
e(x) &\le \frac{(1+P_{\max})}{(q_{\min}+1)!}\phi_{\max}\prod_{x_i\in F_{k_{\min}}}|x-x_i|^{\mu}\sum_{k=1}^{m}\prod_{x_i\in F_k}|x-x_i|^{\#(\mathcal{J}_i)}\prod_{x_i\in F_k}|x-x_i|^{-\mu}\\
&\le \frac{(1+P_{\max})}{(q_{\min}+1)!}\phi_{\max}\prod_{x_i\in F_{k_{\min}}}|x-x_i|^{\mu}\left(\sum_{j=0}^{N}\sum_{F_k\in\mathcal{I}_j}\prod_{x_i\in F_k}|x-x_i|^{\#(\mathcal{J}_i)}\prod_{x_i\in F_k}|x-x_i|^{-\mu}\right)\\
&\le \frac{(1+P_{\max})}{(q_{\min}+1)!}\phi_{\max}\left(\sum_{F_k\in\mathcal{I}_0}\prod_{x_i\in F_k}|x-x_i|^{\#(\mathcal{J}_i)}\left(\frac{\prod_{x_i\in F_{k_{\min}}}|x-x_i|}{\prod_{x_i\in F_k}|x-x_i|}\right)^{\mu}\right.\\
&\qquad\left.+\sum_{j=1}^{N}\sum_{F_k\in\mathcal{I}_j}\prod_{x_i\in F_k}|x-x_i|^{\#(\mathcal{J}_i)}\left(\frac{\prod_{x_i\in F_{k_{\min}}}|x-x_i|}{\prod_{x_i\in F_k}|x-x_i|}\right)^{\mu}\right).
\end{aligned}$$

Since

$$\frac{\prod_{x_i\in F_{k_{\min}}}|x-x_i|}{\prod_{x_i\in F_k}|x-x_i|}\le 1$$

for the subsets $F_k\in\mathcal{I}_0$ and

$$\frac{\prod_{x_i\in F_{k_{\min}}}|x-x_i|}{\prod_{x_i\in F_k}|x-x_i|}\le\frac{(Lh)^{\#(F_{k_0})}}{\big[(2j-1)h\big]^{\#(F_k)}}=\frac{L^{\#(F_{k_0})}}{(2j-1)^{\#(F_k)}}h^{\#(F_{k_0})-\#(F_k)}$$

for $F_k\in\mathcal{I}_j$, $j\ge1$, then

$$\begin{aligned}
e(x) &\le \frac{(1+P_{\max})}{(q_{\min}+1)!}\phi_{\max}\left(\sum_{F_k\in\mathcal{I}_0}\prod_{x_i\in F_k}|x-x_i|^{\#(\mathcal{J}_i)}+\sum_{j=1}^{N}\sum_{F_k\in\mathcal{I}_j}\prod_{x_i\in F_k}|x-x_i|^{\#(\mathcal{J}_i)}\left(\frac{L^{\#(F_{k_0})}}{(2j-1)^{\#(F_k)}}h^{\#(F_{k_0})-\#(F_k)}\right)^{\mu}\right)\\
&\le \frac{(1+P_{\max})}{(q_{\min}+1)!}\phi_{\max}\left(\sum_{F_k\in\mathcal{I}_0}(Lh)^{N_k}+\sum_{j=1}^{N}\sum_{F_k\in\mathcal{I}_j}\big((L+2j+1)h\big)^{N_k}\left(\frac{L^{\#(F_{k_0})}}{(2j-1)^{\#(F_k)}}h^{\#(F_{k_0})-\#(F_k)}\right)^{\mu}\right)\\
&\le \frac{(1+P_{\max})}{(q_{\min}+1)!}\phi_{\max}\left(\sum_{F_k\in\mathcal{I}_0}L^{N_k}h^{N_k}+\sum_{j=1}^{N}\sum_{F_k\in\mathcal{I}_j}(L+2j+1)^{N_k}h^{N_k}\left(\frac{L^{\#(F_{k_0})}}{(2j-1)^{\#(F_k)}}h^{\#(F_{k_0})-\#(F_k)}\right)^{\mu}\right)\\
&\le \frac{(1+P_{\max})}{(q_{\min}+1)!}\phi_{\max}h^{C_{N_{\min}}}\left(\sum_{F_k\in\mathcal{I}_0}L^{C_{N_{\max}}}+\sum_{j=1}^{N}\sum_{F_k\in\mathcal{I}_j}(L+2j+1)^{C_{N_{\max}}}\left(\frac{L^{C_{F_{\max}}}}{(2j-1)^{C_{F_{\min}}}}\right)^{\mu}h^{\left(\#(F_{k_0})-\#(F_k)\right)\mu}\right).
\end{aligned}$$

Let us assume that $\#(F_k) = const$ for each k. Then,

$$e(x) \le \frac{(1+P_{\max})}{(q_{\min}+1)!}\phi_{\max} h^{C_{N_{\min}}} \left(M L^{C_{N_{\max}}} + M \sum_{j=1}^{N} (L+2j+1)^{C_{N_{\max}}} \left(\frac{L^{C_{F_{\max}}}}{(2j-1)^{C_{F_{\min}}}} \right)^{\mu} \right)$$
$$\le \frac{(1+P_{\max})}{(q_{\min}+1)!}\phi_{\max} h^{C_{N_{\min}}} M \left(L^{C_{N_{\max}}} + \left(L^{C_{F_{\max}}}\right)^{\mu} \sum_{j=1}^{N} \frac{(L+2j+1)^{C_{N_{\max}}}}{(2j-1)^{\mu C_{F_{\min}}}} \right).$$

Let us consider the series

$$\sum_{j=1}^{\infty} \frac{(L+2j+1)^{C_{N_{\max}}}}{(2j-1)^{\mu C_{F_{\min}}}} \approx \sum_{j=1}^{\infty} \frac{(2j)^{C_{N_{\max}}}}{(2j)^{\mu C_{F_{\min}}}} = \sum_{j=1}^{\infty} \frac{1}{(2j)^{\mu C_{F_{\min}} - C_{N_{\max}}}},$$

which converges for $\mu C_{F_{\min}} - C_{N_{\max}} > 1$. Thus, for $\mu > \frac{1+C_{N_{\max}}}{C_{F_{\min}}}$, the operator $M_\mu[f,\mathcal{F}]$ has approximation order $O(h^{C_{N_{\min}}})$. $\square$

Let us now study the approximation order of the operator $\tilde{M}_\mu[f,\mathcal{F}]$. To this end we set

$$\tilde{N}_k = \sum_{x_i \in F_k} \#(\mathcal{J}_i) - \sum_{x_i \in F_k} \#(\mathcal{S}_i) = \tilde{q}_k + 1$$

and

$$C_{\tilde{N}_{\max}} = \max_k \tilde{N}_k, \qquad C_{\tilde{N}_{\min}} = \min_k \tilde{N}_k.$$

Theorem 5. *Let Ω be an interval that contains X, $f \in C^{q_{\max}+1}(\Omega)$, $\mu > \frac{1+C_{\tilde{N}_{\max}}}{C_{F_{\min}}}$ and $\#(F_k) = const$ for each k. Then,*

$$|f(x) - \tilde{M}_\mu[f,\mathcal{F}](x)| \le C M h^{C_{\tilde{N}_{\min}}} \phi_{\max}$$

for any $x \in \Omega$, with C a positive constant which depends on F_k and μ.

Proof. Let

$$\tilde{e}(x) = |f(x) - \tilde{M}_\mu[f,\mathcal{F}](x)|.$$

By (17) and the fact that the basis functions $B_{k,\mu}(x)$ are non-negative and form a partition of unity,

$$\tilde{e}(x) \le \left| \sum_{k=1}^{m} B_{k,\mu}(x) f(x) - \sum_{k=1}^{m} B_{k,\mu}(x) \tilde{P}_k[f](x) \right| \le \sum_{k=1}^{m} |f(x) - \tilde{P}_k[f](x)| B_{k,\mu}(x).$$

Using Theorem 3 and (4), we then get

$$\tilde{e}(x) \le \sum_{k=1}^{m} (1+\|\tilde{P}_k\|_\infty) \frac{\prod_{x_i \in F_k} |x-x_i|^{\#(\mathcal{J}_i)}}{\prod_{x_i \in F_k} |x-x_i|^{\#(\mathcal{S}_i)}} \cdot \frac{|f^{(\tilde{q}_k+1)}(\xi_k)|}{(\tilde{q}_k+1)!} \cdot \frac{\prod_{x_i \in F_k} |x-x_i|^{-\mu}}{\sum_{l=1}^{m} \prod_{x_j \in F_l} |x-x_j|^{-\mu}}. \tag{29}$$

Let $F_{k_{\min}} \in \mathcal{F}$ be the subset such that

$$\prod_{x_i \in F_{k_{\min}}} |x-x_i| = \min_k \prod_{x_i \in F_k} |x-x_i|.$$

Then,

$$\prod_{x_i \in F_{k_{\min}}} |x-x_i| \le (Lh)^{\#(F_{k_0})},$$

since at least one set F_{k_0} has a node in T_0. Finally, we have to bound, for each $k = 1, \dots, m$,

$$\prod_{x_i \in F_k} |x-x_i|^{\#(\mathcal{J}_i)}.$$

For each F_k with at least one node in T_0, we have

$$\prod_{x_i \in F_k} |x-x_i|^{\#(\mathcal{J}_i)} \le \prod_{x_i \in F_k} (Lh)^{\#(\mathcal{J}_i)} \le (Lh)^{N_k},$$

and for each F_k with no nodes in $T_0 \cup \cdots \cup T_{j-1}$ and at least one node in T_j,

$$\prod_{x_i \in F_k} |x - x_i|^{\#(\mathcal{J}_i)} \leq \prod_{x_i \in F_k} \big((L+2j+1)h\big)^{\#(\mathcal{J}_i)} \leq \big((L+2j+1)h\big)^{N_k}.$$

Then we get

$$\begin{aligned}
\tilde{e}(x) &\leq \sum_{k=1}^{m} (1 + \|\tilde{P}_k\|_\infty) \frac{\prod_{x_i \in F_k} |x - x_i|^{\#(\mathcal{J}_i)}}{\prod_{x_i \in F_k} |x - x_i|^{\#(\mathcal{S}_i)}} \cdot \frac{|f^{(\tilde{q}_k+1)}(\xi_k)|}{(\tilde{q}_k+1)!} \cdot \frac{\prod_{x_i \in F_k} |x - x_i|^{-\mu}}{\prod_{x_i \in F_{k_{\min}}} |x - x_i|^{-\mu}} \\
&\leq \prod_{x_i \in F_{k_{\min}}} |x - x_i|^{\mu} \sum_{k=1}^{m} (1 + \|\tilde{P}_k\|_\infty) \frac{\prod_{x_i \in F_k} |x - x_i|^{\#(\mathcal{J}_i)}}{(\tilde{q}_k+1)!} |f^{(\tilde{q}_k+1)}(\xi_k)| \prod_{x_i \in F_k} |x - x_i|^{-\mu - \#(\mathcal{S}_i)}.
\end{aligned}$$

We denote by $\mathcal{I}_j$ the set of subsets F_k with at least one node in T_j and no nodes in $T_0 \cup \cdots \cup T_{j-1}$. By construction, $\bigcup_{j=0}^{N} \mathcal{I}_j = \mathcal{F}$ and $\bigcap_{j=0}^{N} \mathcal{I}_j = \emptyset$. Moreover, if we let

$$\begin{aligned}
\tilde{P}_{\max} &= \max_k \|\tilde{P}_k\|, \\
\phi_{\max} &= \max_{j=0,\ldots,q_{\max}} \|f^{(j)}\|_\infty,
\end{aligned}$$

then by bounding (29) we have

$$\begin{aligned}
\tilde{e}(x) &\leq \frac{(1+\tilde{P}_{\max})}{(\tilde{q}_{\min}+1)!} \phi_{\max} \prod_{x_i \in F_{k_{\min}}} |x - x_i|^{\mu} \sum_{k=1}^{m} \prod_{x_i \in F_k} |x - x_i|^{\#(\mathcal{J}_i) - \#(\mathcal{S}_i)} \prod_{x_i \in F_k} |x - x_i|^{-\mu} \\
&\leq \frac{(1+\tilde{P}_{\max})}{(\tilde{q}_{\min}+1)!} \phi_{\max} \prod_{x_i \in F_{k_{\min}}} |x - x_i|^{\mu} \left(\sum_{j=0}^{N} \sum_{F_k \in \mathcal{I}_j} \prod_{x_i \in F_k} |x - x_i|^{\#(\mathcal{J}_i) - \#(\mathcal{S}_i)} \prod_{x_i \in F_k} |x - x_i|^{-\mu} \right) \\
&\leq \frac{(1+\tilde{P}_{\max})}{(\tilde{q}_{\min}+1)!} \phi_{\max} \left(\sum_{F_k \in \mathcal{I}_0} \prod_{x_i \in F_k} |x - x_i|^{\#(\mathcal{J}_i) - \#(\mathcal{S}_i)} \left(\frac{\prod_{x_i \in F_{k_{\min}}} |x - x_i|}{\prod_{x_i \in F_k} |x - x_i|} \right)^{\mu} \right. \\
&\qquad \left. + \sum_{j=1}^{N} \sum_{F_k \in \mathcal{I}_j} \prod_{x_i \in F_k} |x - x_i|^{\#(\mathcal{J}_i) - \#(\mathcal{S}_i)} \left(\frac{\prod_{x_i \in F_{k_{\min}}} |x - x_i|}{\prod_{x_i \in F_k} |x - x_i|} \right)^{\mu} \right).
\end{aligned}$$

Since

$$\frac{\prod_{x_i \in F_{k_{\min}}} |x - x_i|}{\prod_{x_i \in F_k} |x - x_i|} \leq 1$$

for the subsets $F_k \in \mathcal{I}_0$ and

$$\frac{\prod_{x_i \in F_{k_{\min}}} |x - x_i|}{\prod_{x_i \in F_k} |x - x_i|} \leq \frac{(Lh)^{\#(F_{k_0})}}{\big[(2j-1)h\big]^{\#(F_k)}} = \frac{L^{\#(F_{k_0})}}{(2j-1)^{\#(F_k)}} h^{\#(F_{k_0}) - \#(F_k)}$$

for $F_k \in \mathcal{I}_j$, $j \geq 1$, then

$$\begin{aligned}
\tilde{e}(x) &\leq \frac{(1+\tilde{P}_{\max})}{(\tilde{q}_{\min}+1)!} \phi_{\max} \left(\sum_{F_k \in \mathcal{I}_0} \prod_{x_i \in F_k} |x - x_i|^{\#(\mathcal{J}_i) - \#(\mathcal{S}_i)} + \sum_{j=1}^{N} \sum_{F_k \in \mathcal{I}_j} \prod_{x_i \in F_k} |x - x_i|^{\#(\mathcal{J}_i) - \#(\mathcal{S}_i)} \left(\frac{L^{\#(F_{k_0})}}{(2j-1)^{\#(F_k)}} h^{\#(F_{k_0}) - \#(F_k)} \right)^{\mu} \right) \\
&\leq \frac{(1+\tilde{P}_{\max})}{(\tilde{q}_{\min}+1)!} \phi_{\max} \left(\sum_{F_k \in \mathcal{I}_0} (Lh)^{\tilde{N}_k} + \sum_{j=1}^{N} \sum_{F_k \in \mathcal{I}_j} \big((L+2j+1)h\big)^{\tilde{N}_k} \left(\frac{L^{\#(F_{k_0})}}{(2j-1)^{\#(F_k)}} h^{\#(F_{k_0}) - \#(F_k)} \right)^{\mu} \right) \\
&\leq \frac{(1+\tilde{P}_{\max})}{(\tilde{q}_{\min}+1)!} \phi_{\max} \left(\sum_{F_k \in \mathcal{I}_0} L^{\tilde{N}_k} h^{\tilde{N}_k} + \sum_{j=1}^{N} \sum_{F_k \in \mathcal{I}_j} (L+2j+1)^{\tilde{N}_k} h^{\tilde{N}_k} \left(\frac{L^{\#(F_{k_0})}}{(2j-1)^{\#(F_k)}} h^{\#(F_{k_0}) - \#(F_k)} \right)^{\mu} \right) \\
&\leq \frac{(1+\tilde{P}_{\max})}{(\tilde{q}_{\min}+1)!} \phi_{\max} h^{C_{\tilde{N}_{\min}}} \left(\sum_{F_k \in \mathcal{I}_0} L^{C_{\tilde{N}_{\max}}} + \sum_{j=1}^{N} \sum_{F_k \in \mathcal{I}_j} (L+2j+1)^{C_{\tilde{N}_{\max}}} \left(\frac{L^{C_{F_{\max}}}}{(2j-1)^{C_{F_{\min}}}} \right)^{\mu} h^{(\#(F_{k_0}) - \#(F_k))\mu} \right).
\end{aligned}$$

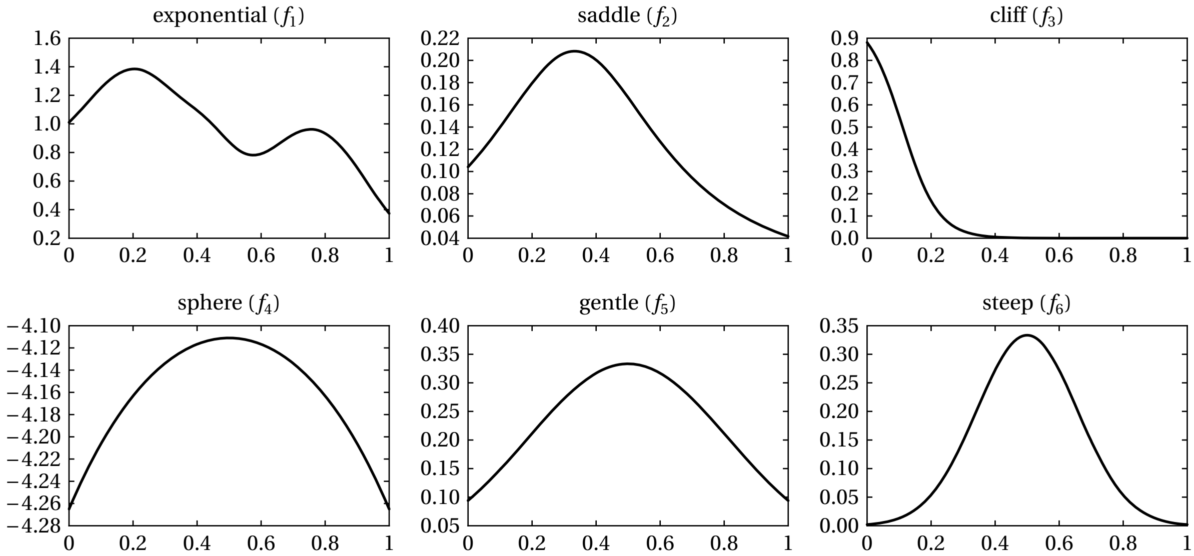

Figure 5: Test functions used in our numerical experiments. The definitions of the functions can be found in [6].

Let us assume that $\#(F_k) = const$ for each k. Then,

$$\begin{aligned}\tilde{e}(x) &\le \frac{(1+\tilde{P}_{\max})}{(\tilde{q}_{\min}+1)!}\phi_{\max} h^{C_{\tilde{N}_{\min}}}\left(M L^{C_{\tilde{N}_{\max}}} + M\sum_{j=1}^{N}(L+2j+1)^{C_{\tilde{N}_{\max}}}\left(\frac{L^{C_{F_{\max}}}}{(2j-1)^{C_{F_{\min}}}}\right)^{\mu}\right)\\ &\le \frac{(1+\tilde{P}_{\max})}{(\tilde{q}_{\min}+1)!}\phi_{\max} h^{C_{\tilde{N}_{\min}}} M\left(L^{C_{\tilde{N}_{\max}}} + \left(L^{C_{F_{\max}}}\right)^{\mu}\sum_{j=1}^{N}\frac{(L+2j+1)^{C_{\tilde{N}_{\max}}}}{(2j-1)^{\mu C_{F_{\min}}}}\right).\end{aligned}$$

Let us consider the series

$$\sum_{j=1}^{\infty}\frac{(L+2j+1)^{C_{\tilde{N}_{\max}}}}{(2j-1)^{\mu C_{F_{\min}}}} \approx \sum_{j=1}^{\infty}\frac{(2j)^{C_{\tilde{N}_{\max}}}}{(2j)^{\mu C_{F_{\min}}}} = \sum_{j=1}^{\infty}\frac{1}{(2j)^{\mu C_{F_{\min}} - C_{\tilde{N}_{\max}}}},$$

which converges for $\mu C_{F_{\min}} - C_{\tilde{N}_{\max}} > 1$. Thus, for $\mu > \frac{1+C_{\tilde{N}_{\max}}}{C_{F_{\min}}}$, the operator $\tilde{M}_{\mu}[f,\mathcal{F}]$ has approximation order $O(h^{C_{\tilde{N}_{\min}}})$. □

6 Numerical results

To numerically test the approximation order of the *multinode rational interpolation operators* predicted by Theorem 4, we carried out a series of experiments with different sets of equispaced nodes on $[0,1]$ and test functions (see Figure 5). We report these results in Section 6.1. In Section 6.2 we present numerical results on the approximation accuracy of the *multinode rational interpolation operators*.

6.1 Approximation order

Our first series of experiments numerically test the theoretical result on the approximation order in Theorem 4. With this aim we consider different coverings $\mathcal{F}$ of the nodeset X with an increasing number of subsets F_k (see Figure 6). For each of the six test functions f_i we constructed the multinode rational interpolant $M_{\mu}[f_i,\mathcal{F}](x)$, and we determined the maximum approximation error $e_{\max}$ by evaluating $|f_i(x) - M_4[f_i,\mathcal{F}](x)|$ at 100,000 random points $x \in [0,1]$ and recording the maximum value.

For the first experiment, the subsets are generated as follows:

1. we fix 3 equispaced points on the interval $[0,1]$ and associate to them the Birkhoff data $f(0)$, $f'(1/2)$, $f(1)$. We consider the subsets $F_1 = \{0,1/2\}$ and $F_2 = \{1/2,1\}$;
2. at each step we halve the distance between two successive nodes by adding equispaced points with associated Birkhoff data as shown in Figure 6.

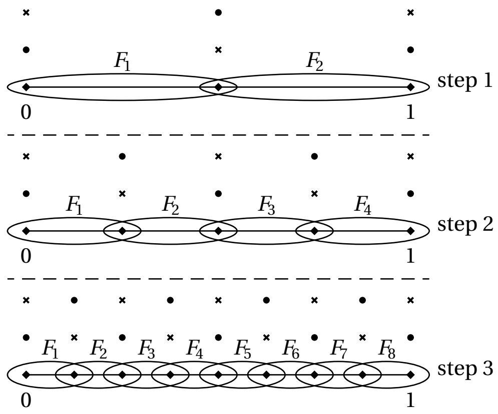

Figure 6: Three of the ten coverings $\mathcal{F}$ used in our numerical experiments with $n = 3$ and $\#(\mathcal{F}) = 2$ (top), $n = 5$ and $\#(\mathcal{F}) = 4$ (middle), $n = 9$ and $\#(\mathcal{F}) = 6$ (bottom).

n	m	h
3	2	1/2
5	4	1/4
9	8	1/8
17	16	1/16
33	32	1/32
65	64	1/64
129	128	1/128
257	256	1/256
513	512	1/512
1025	1024	1/1024

Table 1: Starting from $n = 3$ equispaced nodes on the unit interval, we generate coverings $\mathcal{F}$ with n nodes, m subsets F_k, and interval width h (compare Figure 6).

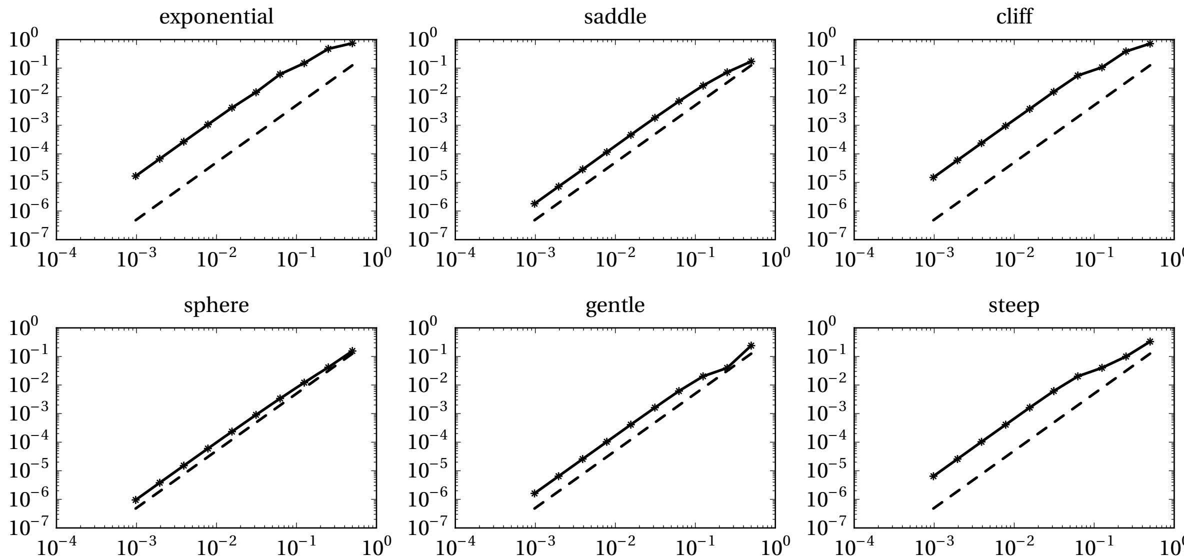

Figure 7: Log-log-plot of the approximation error $e_{\max}$ over the interval width for the six test functions in Figure 5. As reference, the dashed line indicates a perfect quadratic trend.

Table 1 lists the number of nodes and subsets, as well as the interval width h for the ten sets of nodes. In this case, $C_{N_{\min}} = 2$, $C_{N_{\max}} = 2$, and $C_{F_{\max}} = 2$. Figure 7 clearly demonstrates the quadratic approximation order of the operator $M_4[f_i, \mathcal{F}]$.

For the second experiment, the subsets are generated as follows:

1. we fix 4 equispaced points on the interval $[0,1]$ and associate to them the Birkhoff data $f(0)$, $f'(1/3)$, $f''(1/3)$, $f(2/3)$, $f'(1)$, $f''(1)$. In this case we consider the partition of X composed by the subsets $F_1 = \{0, 1/3\}$ and $F_2 = \{2/3, 1\}$;
2. at each step we divide by three the distance between two successive nodes by adding equispaced points with associated Birkhoff data as shown in Figure 8.

Table 2 lists the number of nodes and subsets, as well as the interval width h for the ten sets of nodes. In this case, $C_{N_{\min}} = 3$, $C_{N_{\max}} = 3$, and $C_{F_{\max}} = 2$. Figure 9 clearly demonstrates the cubic approximation order of the operator $M_4[f, \mathcal{F}]$.

6.2 Approximation accuracy

To test the effectiveness of the *multinode rational interpolation operators*, we compare them with the corresponding *combined Shepard operators*. The numerical results are obtained by locally considering the

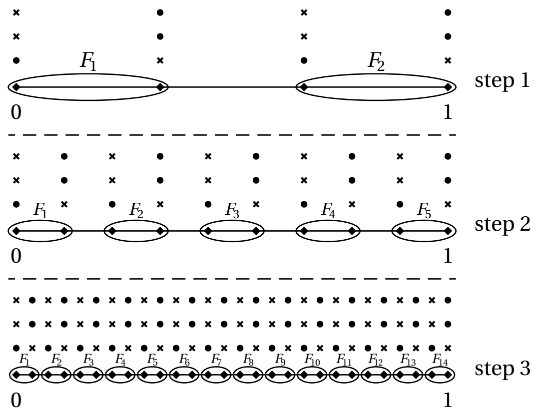

Figure 8: Three of the ten coverings $\mathcal{F}$ used in our numerical experiments with $n = 4$ and $\#(\mathcal{F}) = 2$ (top), $n = 10$ and $\#(\mathcal{F}) = 5$ (middle), $n = 28$ and $\#(\mathcal{F}) = 14$ (bottom).

n	m	h
4	2	1/3
10	5	1/9
28	14	1/27
82	41	1/81
244	122	1/243
730	365	1/729
2188	1094	1/2187
6562	3281	1/6561
19684	9842	1/19683
59050	29525	1/59049

Table 2: Starting from $n = 4$ equispaced nodes on the unit interval, we generate coverings $\mathcal{F}$ with n nodes, m subsets F_k, and interval width h (compare Figure 8).

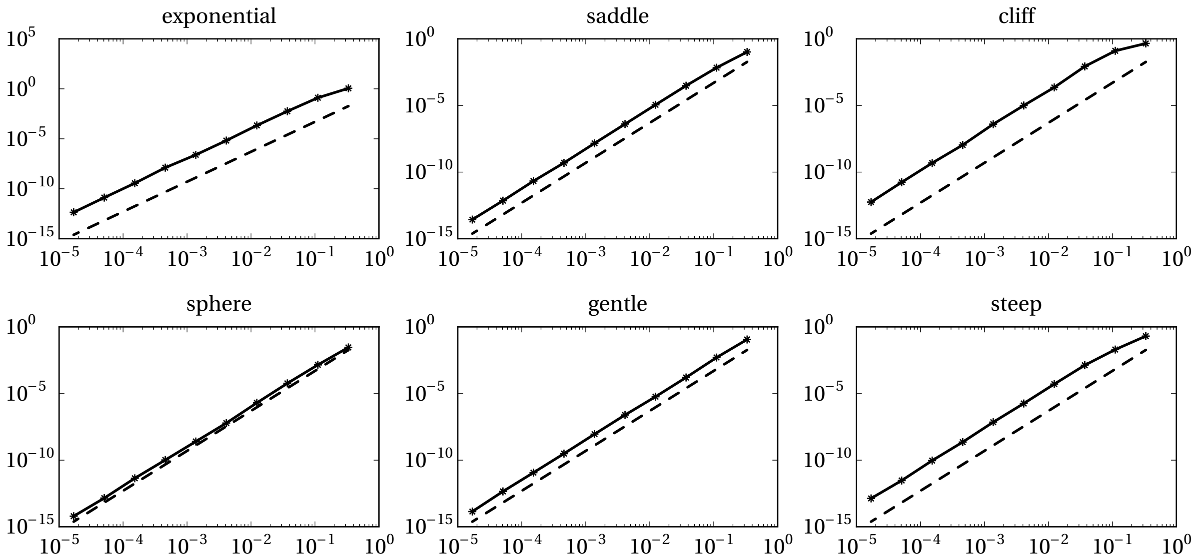

Figure 9: Log-log-plot of the approximation error $e_{\max}$ over the interval width for the six test functions in Figure 5. As reference, the dashed line indicates a perfect cubic trend.

famous cases of Hermite osculatory, Lidstone and Abel–Goncharov interpolation conditions on n nodes. We solve the local problems by cubic interpolating polynomials on two points with intersecting and subsequent subsets F_k. In Tables 3 and 4 we denote by:

1. S_4^H the Shepard–Hermite operator and M_4^H the multinode operator combined with Hermite polynomials of degree 3;
2. S_4^L the Shepard–Lidstone operator and M_4^L the multinode operator combined with Lidstone polynomials of degree 3.

We applied all six operators to the six test functions in Figure 5, using a grid of n equispaced points in $[0,1]$. For $M_4[f_i,\mathcal{F}]$, we considered both intersecting F_k and disjoint F_k. Tables 3 and 4 list the maximum error $e_{\max}$, the average error e_{mean}, and the mean square error e_{MS}. The pointwise errors e_i were determined in absolute value at $n_e = 1001$ points in $[0,1]$, and the errors were calculated by the formulas

$$e_{\max} = \max_{1\le i\le n_e} e_i, \qquad e_{\text{mean}} = \frac{1}{n_e}\sum_{i=1}^{n_e} e_i, \qquad e_{\text{MS}} = \sqrt{\frac{\sum_{i=1}^{n_e} e_i^2}{n_e}}.$$

The results show that the multinode global operator $M_\mu[f,\mathcal{F}]$ is comparable to the Shepard interpolation methods.

		S_4^H	M_4^H intersecting F_k	M_4^H $\mathcal{F}$ partition
f_1	$e_{\max}$	8.9291e-05	2.7545e-05	1.1463e-04
	e_{mean}	1.6655e-05	5.6201e-06	1.3466e-05
	e_{MS}	2.8505e-05	8.6650e-06	2.7513e-05
f_2	$e_{\max}$	4.5433e-06	9.9506e-07	6.1703e-06
	e_{mean}	4.4964e-07	1.4991e-07	3.8711e-07
	e_{MS}	9.3732e-07	2.5303e-07	9.8004e-07
f_3	$e_{\max}$	1.4480e-04	3.1294e-05	1.6588e-04
	e_{mean}	8.4623e-06	2.8993e-06	7.1336e-06
	e_{MS}	2.4384e-05	7.1505e-06	2.3047e-05
f_4	$e_{\max}$	4.2834e-07	6.5425e-08	2.3521e-07
	e_{mean}	1.3234e-07	1.5243e-08	3.7691e-08
	e_{MS}	1.9721e-07	2.2518e-08	7.0360e-08
f_5	$e_{\max}$	1.5330e-06	2.2286e-07	9.7610e-07
	e_{mean}	2.6925e-07	7.6150e-08	1.8635e-07
	e_{MS}	4.8507e-07	1.0319e-07	3.2550e-07
f_6	$e_{\max}$	1.4235e-05	3.4392e-06	1.8307e-05
	e_{mean}	2.3947e-06	7.9910e-07	2.0757e-06
	e_{MS}	4.2773e-06	1.2302e-06	4.4285e-06

Table 3: Comparison of the interpolation operators applied to the six test functions in Figure 5 using 33 equispaced interpolation nodes in $[0,1]$ in the case of Hermite-type data.

		S_4^L	M_4^L intersecting F_k	M_4^L $\mathcal{F}$ partition
f_1	$e_{\max}$	1.1326e-04	1.6864e-04	1.5177e-04
	e_{mean}	2.7617e-05	3.1229e-05	3.4990e-05
	e_{MS}	3.7231e-05	4.7020e-05	4.7786e-05
f_2	$e_{\max}$	3.3075e-06	5.2914e-06	5.5451e-06
	e_{mean}	7.6442e-07	7.9267e-07	8.8856e-07
	e_{MS}	1.1282e-06	1.3313e-06	1.3717e-06
f_3	$e_{\max}$	1.3568e-04	1.6064e-04	1.6126e-04
	e_{mean}	1.4280e-05	1.5709e-05	1.8896e-05
	e_{MS}	3.1622e-05	3.9120e-05	4.1129e-05
f_4	$e_{\max}$	5.7472e-07	3.9677e-07	4.2245e-07
	e_{mean}	1.4878e-07	8.0795e-08	9.1864e-08
	e_{MS}	2.0700e-07	1.2184e-07	1.2830e-07
f_5	$e_{\max}$	1.5514e-06	1.3012e-06	1.3699e-06
	e_{mean}	4.6157e-07	3.9876e-07	4.4639e-07
	e_{MS}	5.8894e-07	5.3260e-07	5.5324e-07
f_6	$e_{\max}$	1.6447e-05	1.9185e-05	2.0093e-05
	e_{mean}	3.8300e-06	4.2435e-06	4.7761e-06
	e_{MS}	5.5007e-06	6.4193e-06	6.6181e-06

Table 4: Comparison of the interpolation operators applied to the six test functions in Figure 5 using 33 equispaced interpolation nodes in $[0,1]$ in the case of Lidstone-type data.

7 Conclusions

In this paper we propose to split up a univariate unsolvable Hermite–Birkhoff interpolation problem in two or more solvable subproblems and to blend together the local solutions by using *multinode basis functions* [17] as blending functions. Numerical experiments are provided, which show a good accuracy of approximation and confirm the theoretical results on the approximation order discussed in the paper. It would be of interest to extend this approach to the case of $\mathbb{R}^2$, the sphere S^2, and other manifolds, taking into account the results of previously published papers on this topics (see, for example, [1, 2, 7, 9, 10, 12, 13, 14, 16] and the references therein).

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