# Symmetric four-directional bivariate pseudo-spline symbols

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**Abstract**

Univariate pseudo-splines are a generalization of uniform B-splines and interpolatory $2n$-point subdivision schemes. Each pseudo-spline is characterized as the limit of the subdivision scheme with least possible support among all schemes with specific degrees of polynomial generation and reproduction. In this paper we propose a formula for the symbols of the bivariate counterpart of pseudo-splines.

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## 1 Introduction

Univariate pseudo-splines [5] are the limits of subdivision schemes with least possible support among all schemes with specific degrees of polynomial generation and reproduction, and they neatly fill the gap between uniform B-splines and interpolatory $2n$-point schemes. As a first step towards the generalization of this concept to the bivariate setting, we propose a family of symmetric four-directional bivariate symbols $a_n^l(\boldsymbol{z})$, $0 \le l < n$ and prove that the members of this family satisfy the algebraic properties for polynomial generation and reproduction up to degree $2n-1$ and $2l+1$, respectively. We further show that the special cases $a_n^0(\boldsymbol{z})$ and $a_n^{n-1}(\boldsymbol{z})$ are the symbols of the four-directional box splines and the minimally supported interpolatory schemes by Han and Jia [8], respectively. Hence, our family fills the gap between these schemes, akin to univariate pseudo-splines.

All methods we use are of purely algebraic nature and work directly on the bivariate subdivision *symbol* defined by the finitely supported *subdivision mask* $A = \{a_{\boldsymbol{\alpha}} \in \mathbb{R} : \boldsymbol{\alpha} \in \mathbb{Z}^2\}$ as

$$a(\boldsymbol{z}) = \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^2} a_{\boldsymbol{\alpha}} \boldsymbol{z}^{\boldsymbol{\alpha}}, \qquad \boldsymbol{z}^{\boldsymbol{\alpha}} = z_1^{\alpha_1} z_2^{\alpha_2} \qquad \boldsymbol{z} = (z_1, z_2) \in (\mathbb{C} \setminus \{0\})^2.$$

In the four-directional setting that we consider, the symbol is called *symmetric* if

$$a(z_1, z_2) = a(1/z_1, z_2) = a(z_1, 1/z_2) = a(z_2, z_1).$$

The *support* of the mask, the symbol, and the scheme is defined as the convex hull of the set $\{\boldsymbol{\alpha} \in \mathbb{Z}_2 : a_{\boldsymbol{\alpha}} \neq \boldsymbol{0}\}$, and the *size* of the support is the area of this convex hull.

Similar to the univariate setting, the generation and reproduction degrees of a bivariate subdivision scheme are closely related to the behaviour of the symbol and its derivatives at $\boldsymbol{z} \in E$, where $E = \{(1,1), (-1,1), (1,-1), (-1,-1)\}$. For example, the generation and reproduction of constant functions is guaranteed, if $a(1,1) = 4$ and $a(\boldsymbol{z}) = 0$ for $\boldsymbol{z} \in E'$, where $E' = E \setminus \{(1,1)\}$, which in turn is a necessary condition for the convergence of the scheme. Regarding higher degrees of generation and reproduction, Cavaretta et al. [1] show that a convergent bivariate scheme *generates* polynomials up to degree $m \ge 1$, if

$$(D^{\boldsymbol{k}} a)(\boldsymbol{z}) = 0, \qquad \boldsymbol{z} \in E', \quad \boldsymbol{k} \in \mathbb{N}_0^2, \quad 0 \le |\boldsymbol{k}| \le m, \tag{1}$$

which is also known as the *sum rule of order* $m+1$. Moreover, Charina et al. [2] prove that a non-singular primal scheme that generates polynomials up to degree $m$ further *reproduces* polynomials up to degree $m' \ge 1$ with $m' \le m$, if

$$(D^{\boldsymbol{k}} a)(1,1) = 0, \qquad \boldsymbol{k} \in \mathbb{N}_0^2, \quad 0 < |\boldsymbol{k}| \le m'. \tag{2}$$

In [9], Sauer shows that

$$\mathcal{J}_k = \left\langle 1 - \boldsymbol{z}^2 \right\rangle^k = \left\langle (1-z_1^2)^{\alpha_1} (1-z_2^2)^{\alpha_2} : \boldsymbol{\alpha} \in \mathbb{N}_0^2, |\boldsymbol{\alpha}| = k \right\rangle, \qquad k \ge 1, \tag{3}$$

is the ideal of all bivariate polynomials $p$ which satisfy

$$(D^{\boldsymbol{k}} p)(\boldsymbol{z}) = 0, \qquad \boldsymbol{z} \in E, \quad \boldsymbol{k} \in \mathbb{N}_0^2, \quad 0 \le |\boldsymbol{k}| < k,$$

and that the bivariate polynomials which satisfy only (1) for $m = k-1$ belong to the quotient ideal

$$\mathcal{I}_k = \mathcal{J}_k : \langle 1-\boldsymbol{z}\rangle^k, \qquad k \geq 1. \tag{4}$$

Consequently, a convergent scheme with symbol $a \in \mathcal{I}_k$ generates polynomials up to degree $k-1$. However, $a \in \mathcal{J}_k$ does not imply polynomial reproduction of degree $k-1$, because $a(1,1) = 0$ in this case, and hence such a scheme is not even convergent [7]. But if $a$ reproduces polynomials up to degree $k-1$ and $b \in \mathcal{J}_k$, then the reproduction degree of $a+b$ is also $k-1$. Note that the indices of our versions of $\mathcal{I}_k$ in (4) and $\mathcal{J}_k$ in (3) are shifted by one with respect to those in [9] for convenience, so that

$$a \in \mathcal{I}_k, \quad b \in \mathcal{I}_l \qquad \Longrightarrow \qquad a \cdot b \in \mathcal{I}_{k+l},$$

and similarly for $\mathcal{J}_k$.

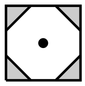
In what follows it will be useful to define the bivariate analogues of $\sigma(z) = \frac{(1+z)^2}{4z}$ and $\delta(z) = \frac{(1-z)^2}{4z}$, their difference, and their product as

$$\boldsymbol{\sigma}(\boldsymbol{z}) = \sigma(z_1)\sigma(z_2), \qquad \boldsymbol{\delta}(\boldsymbol{z}) = \delta(z_1)\delta(z_2), \qquad \boldsymbol{\gamma}(\boldsymbol{z}) = \boldsymbol{\sigma}(\boldsymbol{z}) - \boldsymbol{\delta}(\boldsymbol{z}), \qquad \boldsymbol{\pi}(\boldsymbol{z}) = \boldsymbol{\sigma}(\boldsymbol{z})\boldsymbol{\delta}(\boldsymbol{z}).$$

We further introduce the notation

$$\boldsymbol{\pi}(\boldsymbol{z})^{\boldsymbol{\alpha}} = \big(\sigma(z_1)\delta(z_1)\big)^{\alpha_1}\big(\sigma(z_2)\delta(z_2)\big)^{\alpha_2} = \frac{\delta(z_1^2)^{\alpha_1}\delta(z_2^2)^{\alpha_2}}{4^{\alpha_1+\alpha_2}} \in \mathcal{J}_{2|\boldsymbol{\alpha}|} \subset \mathcal{I}_{2|\boldsymbol{\alpha}|}. \tag{5}$$

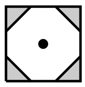
Besides the degrees of polynomial generation and reproduction, we are also interested in the *support of a symbol*, and we frequently use the graphical notation




to denote the octagonal region $\{\boldsymbol{\alpha} : |\alpha_1| \leq m, |\alpha_2| \leq n, |\alpha_1| + |\alpha_2| \leq m+n-l\}$, or rather the rectangle $[-m, m] \times [-n, n]$, minus the triangular regions with side length $l$ in each corner. Following these conventions, we write the symbol of a *primal symmetric four-directional box spline* as

$$B_{j,k}(\boldsymbol{z}) = \left(\frac{1+z_1}{2}\right)^{2j}\left(\frac{1+z_2}{2}\right)^{2j}\left(\frac{1+z_1z_2}{2}\right)^{k}\left(\frac{1+z_1/z_2}{2}\right)^{k}\frac{1}{z_1^{j+k}z_2^{j}} = \boldsymbol{\sigma}(\boldsymbol{z})^j\boldsymbol{\gamma}(\boldsymbol{z})^k \tag{6}$$

and recall from Charina et al. [3] that this symbol is contained in $\mathcal{I}_{2m}$, where $m = 2j+k-\max(j,k)$. Note that for given $m$ the support




of $B_{j,k} \in \mathcal{I}_{2m}$ is minimal if and only if $j = \lceil m/2 \rceil$ and $k = \lfloor m/2 \rfloor$.

## 2 Symmetric four-directional bivariate pseudo-spline symbols

We are now ready to propose our family of symmetric four-directional bivariate pseudo-spline symbols.

**Definition 2.1.** For any $n \geq 1$ and $0 \leq l < n$, let

$$a_n^l(\boldsymbol{z}) = \sum_{i=0}^{l} \tilde{a}_{n-i}(\boldsymbol{z}) b_n^i(\boldsymbol{z}), \qquad b_n^i(\boldsymbol{z}) = \sum_{j=0}^{i} c_n^{(i,j)} \boldsymbol{\pi}(\boldsymbol{z})^{(i-j,j)}, \tag{7}$$

with

$$\tilde{a}_n(\boldsymbol{z}) = 4\boldsymbol{\sigma}(\boldsymbol{z})^{\lceil n/2 \rceil}\boldsymbol{\gamma}(\boldsymbol{z})^{\lfloor n/2 \rfloor},$$

$\boldsymbol{\pi}(\boldsymbol{z})^{\boldsymbol{\alpha}}$ as in (5), and real coefficients

$$c_n^{(i,j)} = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \binom{\lfloor \frac{n-i}{2} \rfloor + k - 1}{k}\binom{n+i-2j-1}{i-j-k}\binom{n+2j-i-1}{j-k}, \qquad 0 \leq j \leq i < n. \tag{8}$$

We first notice that this family contains the symbols of certain scaled four-directional box splines.

**Proposition 2.2.** *For $l=0$, the symbols in (7) are the symbols of the scaled primal symmetric four-directional box splines with minimal support,*

$$a_n^0(\boldsymbol{z}) = 4B_{\lceil n/2\rceil,\lfloor n/2\rfloor}(\boldsymbol{z}), \qquad n \geq 1.$$

*Proof.* The statement follows immediately from (6) and by noting that $c_n^{(0,0)} = 1$ for $n \geq 1$. □

Moreover, this family contains the symbols of the interpolatory schemes by Han and Jia [8], and we discovered that they can be represented nicely in terms of the symbols of the univariate $2n$-point schemes [6],

$$u_n^l(z) = 2\sigma(z)^n \sum_{i=0}^{l} \binom{n+i-1}{i} \delta(z)^i, \qquad 0 \leq l < n.$$

**Proposition 2.3.** *For $l = n-1$, the symbols in (7), which can be written as*

$$a_n^{n-1}(\boldsymbol{z}) = \sum_{i=0}^{n-1} u_{n-i}^{n-i-1}(z_1) u_{i+1}^i(z_2) - \sum_{i=0}^{n-2} u_{n-i-1}^{n-i-2}(z_1) u_{i+1}^i(z_2), \qquad n \geq 1, \tag{9}$$

*are the symbols of the minimally supported interpolatory schemes by Han and Jia [8].*

*Proof.* The proof consists of three main steps. We first denote the right hand side in (9) by $\hat{a}_n(\boldsymbol{z})$ and show that

$$\hat{a}_n(\boldsymbol{z}) = 4\boldsymbol{\sigma}(\boldsymbol{z}) \left( d_n^{n-1}(\boldsymbol{z}) + \boldsymbol{\gamma}(\boldsymbol{z}) \sum_{k=0}^{n-2} \boldsymbol{\sigma}^{n-2-k}(\boldsymbol{z}) d_n^k(\boldsymbol{z}) \right),$$

where

$$d_n^l(\boldsymbol{z}) = \sum_{j=0}^{l} \binom{n+l-2j-1}{l-j} \binom{n+2j-l-1}{j} \boldsymbol{\pi}(\boldsymbol{z})^{(l-j,j)}, \qquad 0 \leq l < n. \tag{10}$$

To this purpose we first derive two helpful identities for the univariate pseudo-splines $u_n^l$. On the one hand,

$$u_n^l(z) - u_n^{l-1}(z) = 2\sigma(z)^n \binom{n+l-1}{l} \delta(z)^l, \tag{11}$$

and, on the other hand,

$$\begin{aligned}
u_n^l(z) - u_{n-1}^l(z) &= 2\sigma(z)^{n-1} \left[ (1-\delta(z)) \sum_{i=0}^{l} \binom{n+i-1}{i} \delta(z)^i - \sum_{i=0}^{l} \binom{n+i-2}{i} \delta(z)^i \right] \\
&= 2\sigma(z)^{n-1} \left[ \sum_{i=0}^{l} \binom{n+i-1}{i} \delta(z)^i - \sum_{i=1}^{l+1} \binom{n+i-2}{i-1} \delta(z)^i - \sum_{i=0}^{l} \binom{n+i-2}{i} \delta(z)^i \right] \\
&= 2\sigma(z)^{n-1} \left[ \sum_{i=0}^{l} \binom{n+i-2}{i-1} \delta(z)^i - \sum_{i=1}^{l+1} \binom{n+i-2}{i-1} \delta(z)^i \right] \\
&= -2\sigma(z)^{n-1} \binom{n+l-1}{l} \delta(z)^{l+1},
\end{aligned} \tag{12}$$

where both identities hold for $0 \leq l < n$, if we extend the definition of $u_n^l$ by letting $u_0^0(z) = 2$ and $u_n^{-1}(z) = 0$ for $n > 0$. We then conclude from (11) that

$$\sum_{j=0}^{l} \left[ u_{n-j}^{l-j}(z_1) - u_{n-j}^{l-j-1}(z_1) \right] \left[ u_{n-l+j}^{j}(z_2) - u_{n-l+j}^{j-1}(z_2) \right] = 4\boldsymbol{\sigma}(\boldsymbol{z})^{n-l} d_n^l(\boldsymbol{z})$$

and from (12) that

$$\sum_{j=0}^{l-1} \left[ u_{n-j}^{l-1-j}(z_1) - u_{n-j-1}^{l-1-j}(z_1) \right] \left[ u_{n-l+j+1}^{j}(z_2) - u_{n-l+j}^{j}(z_2) \right] = 4\boldsymbol{\sigma}(\boldsymbol{z})^{n-l} \boldsymbol{\delta}(\boldsymbol{z}) d_n^{l-1}(\boldsymbol{z}).$$

By letting $e_n^l(\boldsymbol{z}) = \sum_{j=0}^{l} u_{n-j}^{l-j}(z_1) u_{n-l+j}^{j}(z_2)$ and omitting the argument $(\boldsymbol{z})$ for brevity, we get

$$4\boldsymbol{\sigma}^{n-l}(d_n^l - \boldsymbol{\delta} d_n^{l-1}) = e_n^l - e_{n-1}^{l-1} - e_n^{l-1} + e_{n-1}^{l-2}$$

and further

$$\begin{aligned}\hat{a}_n &= \sum_{k=0}^{n-1}(e_n^k - e_{n-1}^{k-1}) - \sum_{k=0}^{n-2}(e_n^k - e_{n-1}^{k-1}) = 4\boldsymbol{\sigma}^n + \sum_{k=1}^{n-1}(e_n^k - e_{n-1}^{k-1}) - \sum_{k=1}^{n-1}(e_n^{k-1} - e_{n-1}^{k-2}) \\ &= 4\boldsymbol{\sigma}^n + 4\sum_{k=1}^{n-1}\boldsymbol{\sigma}^{n-k}(d_n^k - \boldsymbol{\delta} d_n^{k-1}) = 4\boldsymbol{\sigma}\left(d_n^{n-1} + \boldsymbol{\gamma}\sum_{k=0}^{n-2}\boldsymbol{\sigma}^{n-k-2} d_n^k\right).\end{aligned}$$

In the second step, we prove that this representation of $\hat{a}_n$ is identical to $a_n^{n-1}$ in (7). To this end, we first observe that

$$\begin{aligned}\hat{a}_n - 4\boldsymbol{\sigma} d_n^{n-1} &= 4\boldsymbol{\sigma}\boldsymbol{\gamma}\sum_{i=0}^{n-2}\boldsymbol{\sigma}^i d_n^{n-2-i} = 4\boldsymbol{\sigma}\boldsymbol{\gamma}\sum_{i=0}^{n-2}\boldsymbol{\sigma}^{\lceil \frac{i}{2}\rceil}(\boldsymbol{\gamma}+\boldsymbol{\delta})^{\lfloor \frac{i}{2}\rfloor} d_n^{n-2-i} \\ &= 4\boldsymbol{\sigma}\boldsymbol{\gamma}\sum_{i=0}^{n-2}\boldsymbol{\sigma}^{\lceil \frac{i}{2}\rceil}\sum_{k=0}^{\lfloor \frac{i}{2}\rfloor}\binom{\lfloor \frac{i}{2}\rfloor}{k}\boldsymbol{\gamma}^{\lfloor \frac{i}{2}\rfloor - k}\boldsymbol{\delta}^k d_n^{n-2-i} \\ &= 4\sum_{i=0}^{n-2}\sum_{k=0}^{\lfloor \frac{i}{2}\rfloor}\binom{\lfloor \frac{i}{2}\rfloor}{k}\boldsymbol{\sigma}^{\lceil \frac{i}{2}\rceil - k+1}\boldsymbol{\gamma}^{\lfloor \frac{i}{2}\rfloor - k+1}\pi^{(k,k)} d_n^{n-2-i}.\end{aligned}$$

We now rearrange the summation order, substitute $(i,k)$ with $(i+2k,k)$, and use the fact that $\tilde{a}_{i+2} = 4\boldsymbol{\sigma}^{\lceil \frac{i}{2}\rceil+1}\boldsymbol{\gamma}^{\lfloor \frac{i}{2}\rfloor+1}$ to get

$$\begin{aligned}\hat{a}_n - 4\boldsymbol{\sigma} d_n^{n-1} &= \sum_{i=0}^{n-2}\tilde{a}_{i+2}\sum_{k=0}^{\lfloor \frac{n-i}{2}\rfloor - 1}\binom{\lfloor \frac{i}{2}\rfloor + k}{k}\pi^{(k,k)} d_n^{n-2-i-2k} \\ &= \sum_{i=0}^{n-2}\tilde{a}_{n-i}\sum_{k=0}^{\lfloor \frac{i}{2}\rfloor}\binom{\lfloor \frac{n-i}{2}\rfloor + k - 1}{k}\pi^{(k,k)} d_n^{i-2k}.\end{aligned}$$

Substituting $d_n^{i-2k}$ according to (10), we then have

$$\begin{aligned}\hat{a}_n - 4\boldsymbol{\sigma} d_n^{n-1} &= \sum_{i=0}^{n-2}\tilde{a}_{n-i}\sum_{k=0}^{\lfloor \frac{i}{2}\rfloor}\binom{\lfloor \frac{n-i}{2}\rfloor + k - 1}{k}\sum_{j=0}^{i-2k}\binom{n+i-2k-2j-1}{i-2k-j}\binom{n+2j-i+2k-1}{j}\pi^{(i-k-j,j+k)} \\ &= \sum_{i=0}^{n-2}\tilde{a}_{n-i}\sum_{k=0}^{\lfloor \frac{i}{2}\rfloor}\binom{\lfloor \frac{n-i}{2}\rfloor + k - 1}{k}\sum_{j=k}^{i-k}\binom{n+i-2j-1}{i-j-k}\binom{n+2j-i-1}{j-k}\pi^{(i-j,j)},\end{aligned}$$

and noticing that the last sum does not change if we let $j$ range from 0 to $i$, because the first binomial coefficient vanishes for $j > i-k$ and the second vanishes for $j < k$, we conclude that

$$\hat{a}_n - 4\boldsymbol{\sigma} d_n^{n-1} = \sum_{i=0}^{n-2}\tilde{a}_{n-i}\sum_{j=0}^{i} c_n^{(i,j)}\pi^{(i-j,j)}.$$

We finally observe that

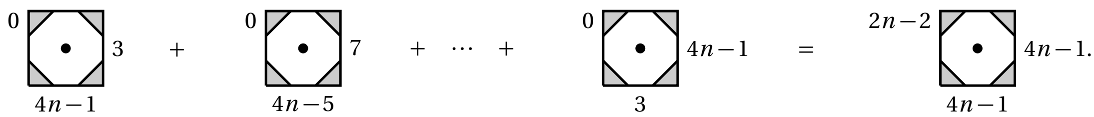
$$4\boldsymbol{\sigma} d_n^{n-1} = \tilde{a}_1\sum_{j=1}^{n-1}\binom{2n-2-2j}{n-1-j}\binom{2j}{j}\pi^{(n-1-j,j)} = \tilde{a}_1\sum_{j=1}^{n-1} c_n^{(n-1,j)}\pi^{(n-1-j,j)},$$

because for $i = n-1$, the first binomial coefficient in (8) is 1 for $k=0$ and 0 for $k>0$.

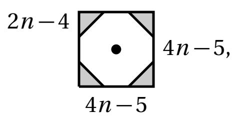
In the third step, we show that $\hat{a}_n$ is interpolatory and minimally supported. We first note that the univariate $2n$-point schemes satisfy $u_n^{n-1}(z)+u_n^{n-1}(-z)=2$, because they are interpolatory [7]. Consequently,

$$
\begin{aligned}
&\hat{a}_n(z_1,z_2)+\hat{a}_n(z_1,-z_2)+\hat{a}_n(-z_1,z_2)+\hat{a}_n(-z_1,-z_2)\\
&\qquad=\sum_{i=0}^{n-1} u_{n-i}^{n-i-1}(z_1)\Big(u_{i+1}^i(z_2)+u_{i+1}^i(-z_2)\Big)+\sum_{i=0}^{n-1} u_{n-i}^{n-i-1}(-z_1)\Big(u_{i+1}^i(z_2)+u_{i+1}^i(-z_2)\Big)\\
&\qquad\quad-\sum_{i=0}^{n-2} u_{n-i-1}^{n-i-2}(z_1)\Big(u_{i+1}^i(z_2)+u_{i+1}^i(-z_2)\Big)-\sum_{i=0}^{n-2} u_{n-i-1}^{n-i-2}(-z_1)\Big(u_{i+1}^i(z_2)+u_{i+1}^i(-z_2)\Big)\\
&\qquad=2\sum_{i=0}^{n-1}\Big(u_{n-i}^{n-i-1}(z_1)+u_{n-i}^{n-i-1}(-z_1)\Big)-2\sum_{i=0}^{n-2}\Big(u_{n-i-1}^{n-i-2}(z_1)+u_{n-i-1}^{n-i-2}(-z_1)\Big)\\
&\qquad=4n-4(n-1)=4,
\end{aligned}
$$

which implies that the schemes $\hat{a}_n$ are interpolatory, too [4]. Therefore, the degrees of polynomial generation and reproduction are the same [2, Proposition 3.4], and the generation degree follows as a special case from Theorem 2.4 for $l=n-1$. As for the support, we remember that $u_n^{n-1}$ is supported on $[-2n+1,2n-1]$. Hence, the supports of the symbols in the first sum in (9) are rectangular and add up like




Similarly, the supports of the symbols in the second sum add up to




which is contained in the support form the first sum, so that the support of $\hat{a}_n$ matches the minimal support reported in [8]. □

Let us now turn to the announced algebraic properties of the symbols in Definition 2.1.

**Theorem 2.4.** *The symbols in* (7) *are symmetric, and they satisfy condition* (1) *for* $m=2n-1$ *and condition* (2) *for* $m'=2l+1$.

*Proof.* We first observe that $\boldsymbol{\sigma}$, $\boldsymbol{\delta}$, and $\boldsymbol{\gamma}$, and therefore $\tilde{a}_n$ are symmetric. Moreover, since

$$
\pi(z_1,z_2)^{(\alpha_1,\alpha_2)}=\pi(z_2,z_1)^{(\alpha_2,\alpha_1)}\qquad\text{and}\qquad c_n^{(i,j)}=c_n^{(i,i-j)},\qquad j=0,\dots,i,
$$

we conclude that $b_n^i$ is symmetric, hence also $a_n^l$. Regarding condition (1), we know that $\tilde{a}_{n-i}\in\mathcal{I}_{2n-2i}$ and it follows from (5) that $b_n^i\in\mathcal{I}_{2i}$, because $\pi(\boldsymbol{z})^{(i-j,j)}\in\mathcal{J}_{2i}\subset\mathcal{I}_{2i}$, for $j=0,\dots,i$. Altogether, we thus get $a_n^l\in\mathcal{I}_{2n}$. To prove condition (2) for $m'=2l+1$, we first note that for $a_n^{n-1}$ this condition follows from the fact that it satisfies condition (1) for $m=2n-1$, because it is an interpolatory scheme [2]. Hence,

$$
D^{\boldsymbol{\alpha}}a_n^{n-1}(1,1)=0,\qquad 0<|\boldsymbol{\alpha}|<2n.
$$

Then, using the recursion

$$
a_n^{l-1}(\boldsymbol{z})=a_n^l(\boldsymbol{z})-a_{n-l}^0(\boldsymbol{z})b_n^l(\boldsymbol{z}),\qquad 0<l<n,
$$

which follows directly from (7), we conclude by induction that

$$
\begin{aligned}
D^{\boldsymbol{\alpha}}a_n^{l-1}(1,1)&=D^{\boldsymbol{\alpha}}a_n^l(1,1)-D^{\boldsymbol{\alpha}}\Big(a_{n-l}^0(1,1)b_n^l(1,1)\Big)\\
&=D^{\boldsymbol{\alpha}}a_n^l(1,1)-\sum_{\boldsymbol{\beta}\le\boldsymbol{\alpha}}D^{\boldsymbol{\alpha}-\boldsymbol{\beta}}a_{n-l}^0(1,1)D^{\boldsymbol{\beta}}b_n^l(1,1)=0,\qquad 0<|\boldsymbol{\alpha}|<2l,
\end{aligned}
$$

because $D^{\boldsymbol\beta} b_n^l(1,1) = 0$ for $0 < |\boldsymbol\beta| < 2l$. In fact, since the $\beta$-th derivative of $(\sigma(z)\delta(z))^l$ vanishes at $z = 1$ for $\beta \le 2l-1$, we see that

$$D^{\boldsymbol\beta}\pi(\boldsymbol z)^{(l-j,j)}\Big|_{\boldsymbol z=(1,1)} = 0 \qquad \text{for} \quad \beta_1 \le 2(l-j)-1 \quad \text{or} \quad \beta_2 \le 2j-1.$$

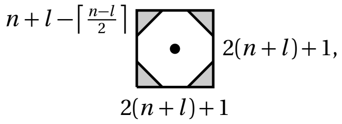
Therefore,

$$D^{\boldsymbol\beta} b_n^l(\boldsymbol z) = \sum_{j=0}^{l} c_n^{(l,j)} D^{\boldsymbol\beta}\pi(\boldsymbol z)^{(l-j,j)}$$

can be different from zero at $\boldsymbol z = (1,1)$ only if $\beta_1 \ge 2(l-j)$ and $\beta_2 \ge 2j$, that is, for $|\boldsymbol\beta| \ge 2l$. □

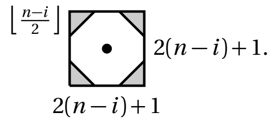
We continue with the analysis of two further properties of the symbols in Definition 2.1. The first is about the support size of the symbols, which we conjecture to be minimal, and the second concerns the necessary conditions for convergence.

**Proposition 2.5.** *The support of the symbols in* (7) *is*

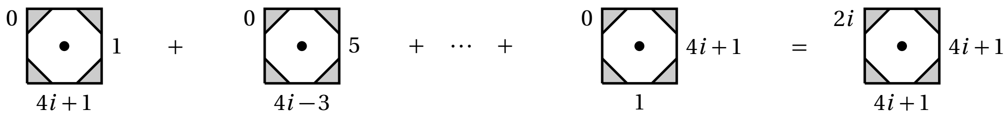



$$0 \le l < n.$$

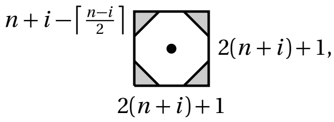
*Proof.* We first notice that the support of the box spline $\tilde a_{n-i}$ is




Moreover, the supports of the symbols in the sum of $b_n^i$ are rectangular and add up like




to a diamond-shaped domain with vertices $(\pm 2i, 0)$ and $(0, \pm 2i)$. Hence, the support of $\tilde a_{n-i} b_n^i$ is




which is contained in the support of $\tilde a_{n-l} b_n^l$ for $i \le l$. □

**Conjecture 2.6.** *The symbols in* (7) *are minimally supported.*

This conjecture is clearly true for the special cases $l = 0$ and $l = n-1$ [8], and we verified it numerically for $0 < l < n-1$ and $3 \le n \le 20$.

**Proposition 2.7.** *The symbols in* (7) *satisfy the necessary conditions for convergence,*

$$a_n^l(1,1) = 4, \qquad a_n^l(\boldsymbol z) = 0, \quad \boldsymbol z \in E', \qquad 0 \le l < n.$$

*Proof.* The proof relies on a key property of box splines, whose symbols satisfy

$$\tilde a_n(1,1) = 4, \qquad \tilde a_n(\boldsymbol z) = 0, \quad \boldsymbol z \in E', \qquad 0 \le l < n.$$

From this property we conclude that

$$a_n^l(1,1) = 4\sum_{i=0}^{l} b_n^i(1,1), \qquad a_n^l(\boldsymbol z) = 0 \quad \boldsymbol z \in E', \qquad 0 \le l < n,$$

and using the fact that $(\sigma(z)\delta(z))^i$ vanishes at $z = 1$ for $i > 0$, we get

$$\sum_{i=0}^{l} b_n^i(1,1) = b_n^0(1,1) = c_n^{(0,0)} = 1, \qquad 0 \le l < n.$$

□

To conclude this section, we want to point out that the family of bivariate four-directional pseudo-spline symbols in (7) is not the unique possible bivariate extension of the univariate case. For example, if $n-l$ is odd, then the symbol

$$\breve{a}_n^l(\boldsymbol{z}) = a_n^l(\boldsymbol{z}) + \tilde{a}_{n-l-1}(\boldsymbol{z}) \sum_{j=1}^{l} \mu_j \pi(\boldsymbol{z})^{(l+1-j,j)}$$

for any set of weights $\mu_1,\dots,\mu_l \in \mathbb{R}$ with $\mu_j = \mu_{l+1-j}$, $j = 1,\dots,l$ is symmetric, has the same support as $a_n^l$ and satisfies the same algebraic properties for polynomial generation and reproduction. In fact, the statement about the support follows as in the proof of Proposition 2.5, and since $\tilde{a}_{n-l-1} \in \mathcal{I}_{2n-2l-2}$ and $\pi(\boldsymbol{z})^{(l+1-j,j)} \in \mathcal{J}_{2l+2} \subset \mathcal{I}_{2l+2}$ for $j = 1,\dots,l$, the difference $\breve{a}_n^l - a_n^l$ is both in $\mathcal{I}_{2n}$ and in $\mathcal{J}_{2l+2}$, and it is clear that $\breve{a}_n^l$ satisfies condition (1) for $m = 2n-1$ and condition (2) for $m' = 2l+1$, just like $a_n^l$. However, in the special case of $l = n-1$ the symbol $\breve{a}_n^l$ is interpolatory only if all weights $\mu_j$ are zero. Our numerical investigations further indicate that for $n-l$ even, the members of our family with symbols $a_n^l$ are the unique minimally supported symbols that satisfy the algebraic properties for generation degree $2n-1$ and reproduction degree $2l+1$.

## 3 Examples

We now present some examples of the subdivision masks $A_n^l$ associated with the symbols $a_n^l$ in (7) for $n = 3$ and show the graphs of the corresponding basic limit functions obtained after three subdivision steps. For $n = 3$, our family contains the four-directional box spline with symbol $a_3^0 = 4B_{2,1}$ and the four-directional bivariate analogue of the interpolatory 6-point scheme with symbol $a_3^2$. Both schemes have polynomial generation degree 5 and reproduction degrees 1 and 5, respectively. The third family member with symbol $a_3^1$ satisfies the algebraic properties for generating polynomials up to degree 5 and reproducing polynomials up to degree 3. It fills the gap between the special cases $a_3^0$ and $a_3^2$ not only regarding the reproduction degree, but also regarding the support and the shape of the basic limit function (see Figure 1).

The examples in Figure 1, as well as many numerical experiments performed for $0 \le l < n \le 20$, suggest that the subdivision schemes with symbols in (7) are convergent and that the regularity decreases with $l$ for fixed $n$ and increases in $n$ for fixed $l$, as in the case of univariate pseudo-splines. However, a systematic and theoretical analysis of convergence, though very crucial, is beyond the scope of this paper, and so are further investigations regarding other properties such as stability and degree of smoothness of the basic limit functions.

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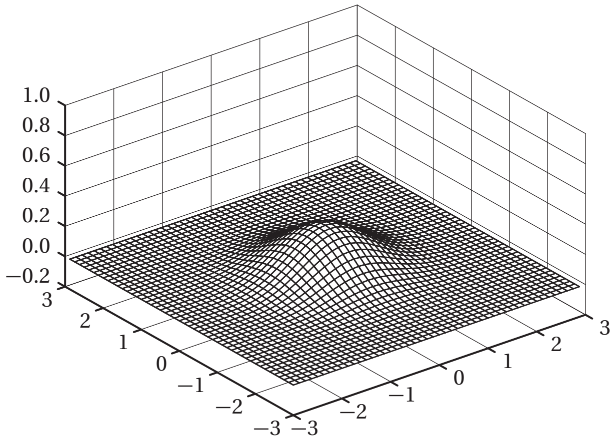
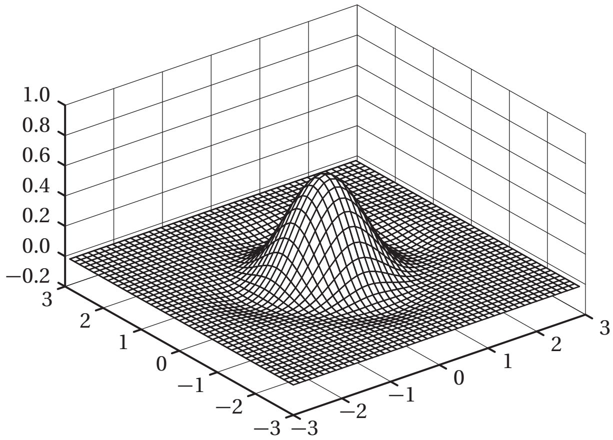
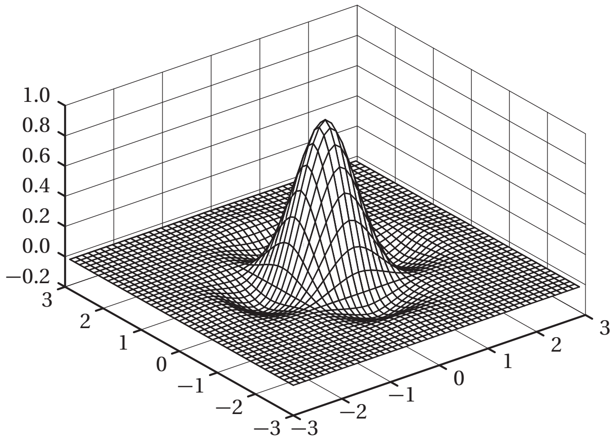



$$A_3^0 = \frac{1}{256}\begin{bmatrix} 0 & 1 & 4 & 6 & 4 & 1 & 0 \\ 1 & 8 & 23 & 32 & 23 & 8 & 1 \\ 4 & 23 & 56 & 74 & 56 & 23 & 4 \\ 6 & 32 & 74 & 96 & 74 & 32 & 6 \\ 4 & 23 & 56 & 74 & 56 & 23 & 4 \\ 1 & 8 & 23 & 32 & 23 & 8 & 1 \\ 0 & 1 & 4 & 6 & 4 & 1 & 0 \end{bmatrix}$$




$$A_3^1 = \frac{1}{256}\begin{bmatrix} 0 & 0 & 0 & -3 & -6 & -3 & 0 & 0 & 0 \\ 0 & 0 & -2 & -8 & -12 & -8 & -2 & 0 & 0 \\ 0 & -2 & -4 & 14 & 32 & 14 & -4 & -2 & 0 \\ -3 & -8 & 14 & 80 & 122 & 80 & 14 & -8 & -3 \\ -6 & -12 & 32 & 122 & 168 & 122 & 32 & -12 & -6 \\ -3 & -8 & 14 & 80 & 122 & 80 & 14 & -8 & -3 \\ 0 & -2 & -4 & 14 & 32 & 14 & -4 & -2 & 0 \\ 0 & 0 & -2 & -8 & -12 & -8 & -2 & 0 & 0 \\ 0 & 0 & 0 & -3 & -6 & -3 & 0 & 0 & 0 \end{bmatrix}$$




$$A_3^2 = \frac{1}{512}\begin{bmatrix} 0 & 0 & 0 & 0 & 3 & 6 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -27 & -50 & -27 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & -27 & 0 & 174 & 300 & 174 & 0 & -27 & 0 & 3 \\ 6 & 6 & -50 & 0 & 300 & 512 & 300 & 0 & -50 & 6 & 6 \\ 3 & 0 & -27 & 0 & 174 & 300 & 174 & 0 & -27 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -27 & -50 & -27 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 6 & 3 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**Figure 1:** Masks and graphs of the basic limit functions for the pseudo-splines with $n = 3$.

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