Convergence rates of a Hermite generalization of Floater–Hormann interpolants

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Abstract

Cirillo and Hormann [2] introduce an iterative approach to the Hermite interpolation problem, which, starting from the Lagrange interpolant, successively adds m corrections terms to interpolate the data up to the m-th derivative. The method is general enough to be applied to any interpolant in linear form with a sufficiently continuous set of basis functions, but Cirillo and Hormann focus their attention on Floater–Hormann interpolants, a family of barycentric rational interpolants that are based on a particular blend of local polynomial interpolants of degree d. They show that the resulting iterative rational Hermite interpolants converge at the rate of $O(h^{(m+1)(d+1)})$ as the mesh size h converges to zero for $m=1,2$, and their numerical results suggest that the same rate holds for $m>2$. In this paper we prove this convergence rate for any $m \geq 1$.

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1 Introduction

Let $m \geq 1$ and $f \in C^m[a,b]$ be an m times continuously differentiable, real-valued function over the interval $[a,b]$, and consider the $n+1$ distinct *interpolation nodes*

$$a = x_0 < x_1 < \cdots < x_n = b.$$

The *Hermite interpolation problem* then consists in finding a function $r_m \in C^m[a,b]$, such that

$$r_m^{(k)}(x_i) = f_i^{(k)} = f^{(k)}(x_i), \qquad i=0,\dots,n, \quad k=0,\dots,m. \tag{1}$$

Cirillo and Hormann [2] recently proposed a general method for defining such Hermite interpolants in an iterative way. Given a set of $n+1$ basis functions $b_0, b_1 \dots, b_n \in C^m[a,b]$ that satisfy the *Lagrange property* $b_i(x_j) = \delta_{i,j}$, they suggest to start with the associated Lagrange interpolant

$$r_0(x) = \sum_{i=0}^n b_i(x) f_i^{(0)}$$

and to iteratively define

$$\begin{aligned} r_k(x) &= r_{k-1}(x) + \sum_{i=0}^n (x-x_i)^k b_i(x)^{k+1} g_{i,k} \\ &= \sum_{i=0}^n \sum_{j=0}^k (x-x_i)^j b_i(x)^{j+1} g_{i,j} \end{aligned}$$

for $k=1,\dots,m$, where

$$g_{i,0} = f_i^{(0)}, \qquad g_{i,j} = \frac{f_i^{(j)} - r_{j-1}^{(j)}(x_i)}{j!}. \tag{2}$$

They show that the resulting function r_m satisfies (1) and that this approach reproduces the classical polynomial Hermite interpolant if the Lagrange basis polynomials are taken as basis functions b_i. Moreover, they apply this construction to the basis functions

$$b_i(x) = \frac{w_i}{x-x_i} \bigg/ W(x), \qquad i=0,\dots,n,$$

of the barycentric rational Floater–Hormann interpolants [5] for $d \le n$ with *barycentric weights*

$$w_i = (-1)^{i+d} \sum_{j=\max(0,i-d)}^{\min(i,n-d)} \prod_{k=j,k\neq i}^{j+d} \frac{1}{|x_i - x_k|}, \qquad i = 0,\dots,n, \tag{3}$$

and common denominator

$$W(x) = \sum_{i=0}^{n} \frac{w_i}{x - x_i}. \tag{4}$$

The corresponding *iterative rational Hermite interpolant* can be expressed as

$$r_m(x) = \frac{1}{W(x)^{m+1}} \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{{w_i}^{j+1}}{x - x_i} W(x)^{m-j} g_{i,j}, \tag{5}$$

with $g_{i,j}$ as in (2).

Cirillo and Hormann [2] show that the interpolant r_m has no poles in $\mathbb{R}$ and can be evaluated with $O(m^2 n)$ operations, based on the barycentric form [14]. The advantages of this family of interpolants over classical approaches, such as polynomials and splines, are twofold. On the one hand, the barycentric rational interpolants in (5) compare favourably to polynomials when the nodes $x_0,\dots,x_n$ are equidistant, as they do not exhibit the Runge phenomenon in this setting. Moreover, in the special case $m = 1$, the Lebesgue constant is bounded from above by a constant [3], compared to the exponential growth of interpolating Hermite polynomials [13]. On the other hand, the interpolant r_m is infinitely often differentiable, while splines only have a finite number of continuous derivatives. Nevertheless, splines come with the advantage of constant-time evaluation [4], and polynomials are probably the best choice in the case of Chebyshev nodes or similar non-uniform node distributions [16].

Apart from polynomials and splines, also rational functions have been considered for solving the Hermite interpolation problem. The most relevant related construction is described by Schneider and Werner [14], who were the first to study rational Hermite interpolants with the barycentric approach. They provide an algorithm for computing the weights of the barycentric form of such interpolants and derive formulas for determining their derivatives. Despite the advantages of their approach, the main difficulty remains to find sets of weights that guarantee the absence of poles. Schneider and Werner suggest to prescribe a positive denominator for the rational Hermite interpolant and to use their algorithm to get the corresponding barycentric weights. However, the denominator suggested in [14] can lead to huge approximation errors near the centre of the interpolation interval. A different approach is to determine the barycentric weights by solving a nonlinear optimization problem, as proposed by Zhao et al. [17], but the resulting weights are then not independent of f.

In order to get barycentric rational Hermite interpolants with no real poles and favourable approximation rates, Floater–Hormann interpolants have been generalized in two ways. Jing, Kang, and Zhu [11] focus on the interpolation of f and its first derivative and propose to define r_1 as a blend of local polynomial Hermite interpolants of degree $2d+1$. The resulting rational functions have the same degrees as the interpolants in (5) but lower convergence rates. Instead, Floater and Schulz [6] derive a Hermite version of the Floater–Hormann interpolant by considering multiple interpolation nodes, giving interpolants with the same degree and approximation order as r_m, but larger maximum approximation errors [2].

Regarding the approximation order of the interpolants in (5), Cirillo and Hormann [2] observe that r_m converges to the function f at the rate of $O(h^{(m+1)(d+1)})$, where

$$h = \max_{i=1,\dots,n} (x_i - x_{i-1}), \tag{6}$$

is the global mesh size of the nodes, but they prove this behaviour only for $m = 1,2$. The key argument in their proof is the identity

$${w_i}^m g_{i,m} = \sum_{i_1=0}^{n} w_{i_1} \sum_{i_2=0}^{n} w_{i_2} \cdots \sum_{i_m=0}^{n} w_{i_m} f[x_i, x_{i_1},\dots,x_{i_m}], \qquad i = 0,\dots,n, \tag{7}$$

which they show for $m = 1,2$, but only conjecture to be true for $m > 2$.

The main goal of this work is to prove this convergence rate for $m > 2$, but instead of proving (7) for $m > 2$ and then concluding the convergence rate, we pursue a different strategy in this paper. We first show that the iterative rational Hermite interpolant in (5) can be expressed in an alternative way, which in turn allows us to write the error in a more convenient form (Section 2). We then use this result to derive the expected convergence rate (Theorem 5) and to confirm the conjecture that (7) holds for any $m \ge 1$ (Section 3).

2 A new formula for the iterative rational Hermite interpolant

Let
$$q_m(x) = \frac{1}{W(x)^{m+1}} \sum_{i_0=0}^{n} w_{i_0} \sum_{i_1=0}^{n} w_{i_1} \cdots \sum_{i_m=0}^{n} w_{i_m} \sum_{j=0}^{m} \frac{f[x_{i_0},\dots,x_{i_j}]}{\prod_{k=j}^{m}(x-x_{i_k})}. \tag{8}$$
The main goal of this section is to prove that q_m coincides with the interpolant r_m in (5). We first show that q_m interpolates the function f and its first m derivatives at the interpolation nodes. To this end, we consider the error
$$e_m(x) = f(x) - q_m(x) \tag{9}$$
and start by expressing it in terms of
$$A_m(x) = \sum_{i_0=0}^{n} w_{i_0} \sum_{i_1=0}^{n} w_{i_1} \cdots \sum_{i_m=0}^{n} w_{i_m} f[x, x_{i_0},\dots,x_{i_m}]. \tag{10}$$

Lemma 1. *The error in* (9) *can be written as*
$$e_m(x) = \frac{A_m(x)}{W(x)^{m+1}}.$$

Proof. By Newton's error formula [8] for the polynomial interpolant of the values $f_{i_0},\dots,f_{i_m}$ at the nodes $x_{i_0},\dots,x_{i_m}$,
$$f(x) - \sum_{j=0}^{m} f[x_{i_0},\dots,x_{i_j}] \prod_{k=0}^{j-1}(x-x_{i_k}) = f[x,x_{i_0},\dots,x_{i_m}] \prod_{k=0}^{m}(x-x_{i_k}),$$
we get
$$\sum_{j=0}^{m} \frac{f[x_{i_0},\dots,x_{i_j}]}{\prod_{k=j}^{m}(x-x_{i_k})} = \frac{f(x)}{\prod_{k=0}^{m}(x-x_{i_k})} - f[x,x_{i_0},\dots,x_{i_m}],$$
and the statement then follows, because
$$\begin{aligned}
e_m(x) &= f(x) - \frac{1}{W(x)^{m+1}} \sum_{i_0=0}^{n} w_{i_0} \sum_{i_1=0}^{n} w_{i_1} \cdots \sum_{i_m=0}^{n} w_{i_m} \left(\frac{f(x)}{\prod_{k=0}^{m}(x-x_{i_k})} - f[x,x_{i_0},\dots,x_{i_m}] \right) \\
&= f(x) - \frac{1}{W(x)^{m+1}} \big(W(x)^{m+1} f(x) - A_m(x) \big) \\
&= \frac{A_m(x)}{W(x)^{m+1}}.
\end{aligned}$$
□

Before we proceed to prove that q_m is indeed a Hermite interpolant of f, we need an auxiliary result.

Lemma 2. *Let*
$$\omega_i(x) = (x-x_i)W(x), \qquad i = 0,\dots,n,$$
and
$$\Omega_{i,j}(x) = \omega_i(x)^{j+1}, \qquad j \ge 0. \tag{11}$$
Then,
$$|\Omega_{i,j}^{(k)}(x_i)| \le \frac{(k+j)!}{j!} \max_{l=0,\dots,k} |\vartheta_{i,l}|^{j+1},$$
for any $k \ge 0$, *where*
$$\vartheta_{i,0} = -w_i, \qquad \vartheta_{i,j} = \sum_{\substack{l=0\\ l\ne i}}^{n} \frac{w_l}{(x_i-x_l)^j}, \qquad j \ge 1.$$

Proof. By the general Leibniz rule for higher order derivatives of a product of several functions,

$$\Omega_{i,j}^{(k)}(x) = \sum_{|\boldsymbol{\gamma}|=k} \binom{k}{\gamma_0,\dots,\gamma_j} \prod_{l=0}^{j} \omega_i^{(\gamma_l)}(x),$$

where the sum ranges over all $(j+1)$-dimensional multi-indices $\boldsymbol{\gamma} = (\gamma_0,\dots,\gamma_j)$ whose non-negative integer components sum up to k. Since

$$\omega_i(x) = w_i + (x - x_i) \sum_{\substack{l=0 \\ l\neq i}}^{n} \frac{w_l}{x - x_l} \tag{12}$$

and therefore

$$\omega_i^{(p)}(x_i) = (-1)^{p+1} p!\, \vartheta_{i,p}, \qquad p \geq 0,$$

and since there are exactly $\binom{k+j}{j}$ possible $\boldsymbol{\gamma}$'s whose components sum up to k, we conclude that

$$|\Omega_{i,j}^{(k)}(x_i)| = k! \left| \sum_{|\boldsymbol{\gamma}|=k} \prod_{l=0}^{j} \vartheta_{i,\gamma_l} \right| \leq k! \binom{k+j}{j} \max_{l=0,\dots,k} |\vartheta_{i,l}|^{j+1}.$$

□

We are now ready to show that q_m is a Hermite interpolant of order m.

Proposition 3. *If $f \in C^{2m+1}[a,b]$, then*

$$e_m^{(k)}(x_i) = 0, \qquad i = 0,\dots,n, \quad k = 0,\dots,m.$$

Proof. We start by fixing the index i and, using Lemma 1, rewriting the error e_m as

$$e_m(x) = \phi_m(x) A_m(x) B_m(x), \tag{13}$$

with $A_m(x)$ as in (10) and

$$\phi_m(x) = (x - x_i)^{m+1}, \qquad B_m(x) = \frac{1}{\Omega_{i,m}(x)}.$$

Applying the Leibniz rule twice, we have

$$\begin{aligned} e_m^{(k)}(x_i) &= \sum_{j=0}^{k} \binom{k}{j} \phi_m^{(k-j)}(x_i) (A_m B_m)^{(j)}(x_i) \\ &= \sum_{j=0}^{k} \binom{k}{j} \phi_m^{(k-j)}(x_i) \sum_{l=0}^{j} \binom{j}{l} A_m^{(j-l)}(x_i) B_m^{(l)}(x_i), \end{aligned}$$

and since $\phi_m^{(j)}(x_i) = 0$ for $j = 0,\dots,m$, it remains to show that $A_m^{(j)}(x_i)$ and $B_m^{(j)}(x_i)$ are both bounded for $j = 0,\dots,m$.

On the one hand, using the derivative formula for divided differences [1, 10], we have

$$A_m^{(j)}(x_i) = j! \sum_{i_0=0}^{n} w_{i_0} \sum_{i_1=0}^{n} w_{i_1} \cdots \sum_{i_m=0}^{n} w_{i_m} f[(x_i)^{j+1}, x_{i_0}, \dots, x_{i_m}],$$

where $(\cdot)^k$ indicates a k-fold argument, and hence,

$$|A_m^{(j)}(x_i)| \leq j!(n+1)^{m+1} \max_{l=0,\dots,n} |w_l|^{m+1} \|f^{(m+j+1)}\|,$$

with $\|\cdot\|$ denoting the maximum norm. This upper bound is finite under the assumption that $f \in C^{2m+1}[a,b]$.

On the other hand, it follows from Hoppe's formula [9, 12] and the relation

$$\Omega_{i,m}(x)^q = \begin{cases} 1, & \text{if } q = 0, \\ \Omega_{i,q(m+1)-1}(x), & \text{if } q > 0. \end{cases}$$

that

$$B_m^{(j)}(x_i) = \sum_{p=0}^{j} \frac{(-1)^p}{\Omega_{i,m}(x_i)^{p+1}} \sum_{q=0}^{p} \binom{p}{q} (-1)^{p-q} \Omega_{i,m}(x_i)^{p-q} \big((\Omega_{i,m})^q\big)^{(j)}(x_i)$$
$$= \sum_{p=0}^{j} \left(\frac{\delta_{j,0}}{\Omega_{i,m}(x_i)} + \sum_{q=1}^{p} \binom{p}{q} (-1)^q \frac{\Omega_{i,q(m+1)-1}^{(j)}(x_i)}{\Omega_{i,(q+1)(m+1)-1}(x_i)} \right).$$

Recalling from (11) and (12) that

$$\Omega_{i,m}(x_i) = w_i^{\,m+1},$$

we then have

$$|B_m^{(j)}(x_i)| \le \frac{j+1}{|w_i|^{m+1}} + \sum_{p=0}^{j} \sum_{q=1}^{p} \binom{p}{q} \frac{|\Omega_{i,q(m+1)-1}^{(j)}(x_i)|}{|w_i|^{(q+1)(m+1)}},$$

and it follows from Lemma 2 that this upper bound is finite. □

The main result of this section now follows after rewriting q_m and r_m in rational form.

Theorem 4. *The interpolants q_m and r_m coincide for any $f \in C^{2m+1}[a,b]$.*

Proof. We first recall from [5] that $W(x)$ in (4) can be expressed as

$$W(x) = \sum_{i=0}^{n-d} \frac{(-1)^i}{(x-x_i)\cdots(x-x_{i+d})}.$$

Therefore, after multiplying numerator and denominator of r_m in (5) by $\mu(x)^{m+1}$, where

$$\mu(x) = \prod_{j=0}^{n} (x-x_j),$$

we can rewrite the interpolant in rational form as

$$r_m(x) = \sum_{i=0}^{n} \sum_{j=0}^{m} \mu(x)^j w_i^{\,j+1} Q(x)^{m-j} g_{i,j} \prod_{\substack{l=0\\ l\ne i}}^{n} (x-x_l) \Big/ Q(x)^{m+1},$$

where

$$Q(x) = \mu(x)W(x) = \sum_{i=0}^{n-d} \left(\prod_{j=0}^{i-1} (x_j - x) \prod_{j=i+d+1}^{n} (x-x_j) \right)$$

is a polynomial of degree at most $(n-d)$.

Likewise, multiplying numerator and denominator of q_m in (8) by $\mu(x)^{m+1}$ gives

$$q_m(x) = \sum_{i_0=0}^{n} w_{i_0} \sum_{i_1=0}^{n} w_{i_1} \cdots \sum_{i_m=0}^{n} w_{i_m} \sum_{j=0}^{m} \mu(x)^j f[x_{i_0},\dots,x_{i_j}] \prod_{k=j}^{m} \prod_{\substack{l=0\\ l\ne i_k}}^{n} (x-x_l) \Big/ Q(x)^{m+1}.$$

We observe that r_m and q_m share the same denominator and both numerators are of degree at most $(m+1)(n+1)-1$. As the coefficients of both numerator polynomials are uniquely determined by the $(m+1)(n+1)$ conditions required to solve the Hermite interpolation problem, they must coincide. □

3 Approximation error

Let us now bound the approximation error of the iterative rational Hermite interpolant r_m in (5). To this end, as in [5], we need to assume that in the case $d=0$ the local mesh ratio

$$\beta = \max_{1\le i\le n-2} \min\left\{ \frac{x_{i+1}-x_i}{x_i - x_{i-1}}, \frac{x_{i+1}-x_i}{x_{i+2}-x_{i+1}} \right\}$$

remains bounded as $h \to 0$.

Theorem 5. *Suppose that $m \geq 1$, $d \geq 0$, and $f \in C^{(m+1)(d+2)}[a,b]$, and let h be as in (6). Then,*

$$\|f - r_m\| \leq C h^{(m+1)(d+1)}, \tag{14}$$

where the constant C depends only on m, d, the derivatives of f, the interval length $b-a$, and, only in the case $d=0$, on the local mesh ratio β.

Proof. In order to prove this result, we follow the arguments used in the proof of Theorem 3 in [2]. By Lemma 1 and Theorem 4, we have

$$f(x) - r_m(x) = \frac{A_m(x)}{W(x)^{m+1}},$$

and, since r_m interpolates f at the interpolation nodes, it is sufficient to consider $x \in [a,b] \setminus \{x_0, \dots, x_n\}$. As in the proof of Theorem 2 in [5], we now derive an upper bound for the numerator $A_m(x)$ and a lower bound for the denominator $W(x)^{m+1}$.

Regarding the numerator, we recall Lemma 2 in [2], which states that the barycentric weights in (3) satisfy

$$\sum_{i=0}^{n} w_i f[x, x_i] = \sum_{i=0}^{n-d} (-1)^i f[x, x_i, \dots, x_{i+d}]$$

for any $x \in \mathbb{R}$, and apply this fact $(m+1)$ times to A_m in (10) to obtain

$$A_m(x) = \sum_{i_0=0}^{n-d} (-1)^{i_0} \sum_{i_1=0}^{n-d} (-1)^{i_1} \cdots \sum_{i_m=0}^{n-d} (-1)^{i_m} f[x, x_{i_0}, \dots, x_{i_0+d}, \dots, x_{i_m}, \dots, x_{i_m+d}].$$

Let us first assume that $n-d$ is odd, so that the number of terms in all sums is even. Focussing on the last sum with index i_m, we combine the first and second terms, the third and fourth, and so on, to get

$$\begin{aligned} &\sum_{i_m=0}^{n-d} (-1)^{i_m} f[x, x_{i_0}, \dots, x_{i_0+d}, \dots, x_{i_{m-1}}, \dots, x_{i_{m-1}+d}, x_{i_m}, \dots, x_{i_m+d}] = \\ &\qquad - \sum_{\substack{i_m=0 \\ i_m \text{ even}}}^{n-d-1} (x_{i_m+d+1} - x_{i_m}) f[x, x_{i_0}, \dots, x_{i_0+d}, \dots, x_{i_{m-1}}, \dots, x_{i_{m-1}+d}, x_{i_m}, \dots, x_{i_m+d+1}] \end{aligned}$$

and, after applying the same strategy to all remaining sums, we have

$$\begin{aligned} A_m(x) = (-1)^{m+1} &\sum_{\substack{i_0=0 \\ i_0 \text{ even}}}^{n-d-1} (x_{i_0+d+1} - x_{i_0}) \sum_{\substack{i_1=0 \\ i_1 \text{ even}}}^{n-d-1} (x_{i_1+d+1} - x_{i_1}) \cdots \\ &\cdots \sum_{\substack{i_m=0 \\ i_m \text{ even}}}^{n-d-1} (x_{i_m+d+1} - x_{i_m}) f[x, x_{i_0}, \dots, x_{i_0+d+1}, \dots, x_{i_m}, \dots, x_{i_m+d+1}]. \end{aligned}$$

Recalling from the proof of Theorem 2 in [5] that

$$\sum_{i=0}^{n-d-1} (x_{i+d+1} - x_i) \leq (d+1)(b-a),$$

we conclude

$$|A_m(x)| \leq (d+1)^{m+1} (b-a)^{m+1} \frac{\|f^{((m+1)(d+2))}\|}{((m+1)(d+2))!}. \tag{15}$$

If $n-d$ is even, then the terms in the sums do not come in pairs anymore, and we need to be a bit more careful. Splitting off the last term from the sum with index i_m and combining the other terms as in the previous case we get

$$\sum_{i_m=0}^{n-d}(-1)^{i_m} f[x,x_{i_0},\dots,x_{i_0+d},\dots,x_{i_{m-1}},\dots,x_{i_{m-1}+d},x_{i_m},\dots,x_{i_m+d}] =$$

$$-\sum_{\substack{i_m=0\\ i_m \text{ even}}}^{n-d-2}(x_{i_m+d+1}-x_{i_m}) f[x,x_{i_0},\dots,x_{i_0+d},\dots,x_{i_{m-1}},\dots,x_{i_{m-1}+d},x_{i_m},\dots,x_{i_m+d+1}]$$

$$+ f[x,x_{i_0},\dots,x_{i_0+d},\dots,x_{i_{m-1}},\dots,x_{i_{m-1}+d},x_{n-d},\dots,x_n].$$

Repeating this procedure for all remaining sums, we find that

$$A_m(x) = \sum_{k=0}^{m+1}(-1)^{m+1-k}\binom{m+1}{k}\sum_{\substack{i_0=0\\ i_0 \text{ even}}}^{n-d-2}(x_{i_0+d+1}-x_{i_0})\sum_{\substack{i_1=0\\ i_1 \text{ even}}}^{n-d-2}(x_{i_1+d+1}-x_{i_1})\cdots$$

$$\cdots\sum_{\substack{i_{m-k}=0\\ i_{m-k} \text{ even}}}^{n-d-2}(x_{i_{m-k}+d+1}-x_{i_{m-k}}) f[x,x_{i_0},\dots,x_{i_0+d+1},\dots,x_{i_{m-k}},\dots,x_{i_{m-k}+d+1},(x_{n-d},\dots,x_n)^k],$$

and therefore

$$|A_m(x)| \le \sum_{k=0}^{m+1}(d+1)^{m+1-k}(b-a)^{m+1-k}\binom{m+1}{k}\frac{\|f^{((m+1)(d+2)-k)}\|}{((m+1)(d+2)-k)!}. \tag{16}$$

For the denominator, we remember from [5] that

$$|W(x)| \ge \frac{1}{d!\,h^{d+1}}$$

if $d \ge 1$ and

$$|W(x)| \ge \frac{1}{(1+\beta)h}$$

if $d = 0$. The statement then follows by combining these bounds. □

As in [2], Equations (15) and (16) allow us to deduce the degree of polynomial reproduction of r_m.

Corollary 6. *The iterative rational Hermite interpolant r_m reproduces polynomials of degree $(m+1)(d+1)-1$ and even of degree $(m+1)(d+2)-1$, if $n-d$ is odd.*

We conclude this paper by confirming the conjecture of Cirillo and Hormann [2] regarding the validity of (7) for any $m \ge 1$.

Corollary 7. *If $m \ge 1$ and $f \in C^{2m+1}[a,b]$, then*

$$w_i{}^m g_{i,m} = \sum_{i_1=0}^{n} w_{i_1}\sum_{i_2=0}^{n} w_{i_2}\cdots\sum_{i_m=0}^{n} w_{i_m} f[x_i,x_{i_1},\dots,x_{i_m}], \qquad i=0,\dots,n.$$

Proof. Using (2), Theorem 4, and (13), we can express $g_{i,m}$ as

$$g_{i,m} = \frac{1}{m!}\big(f^{(m)}(x_i) - r_{m-1}^{(m)}(x_i)\big) = \frac{1}{m!}e_{m-1}^{(m)}(x_i) = \frac{1}{m!}(\phi_{m-1}A_{m-1}B_{m-1})^{(m)}(x_i),$$

and applying the Leibniz rule twice gives

$$\begin{aligned} g_{i,m} &= \frac{1}{m!}\sum_{j=0}^{m}\binom{m}{j}A_{m-1}^{(m-j)}(x_i)(\phi_{m-1}B_{m-1})^{(j)}(x_i) \\ &= \frac{1}{m!}\sum_{j=0}^{m}\binom{m}{j}A_{m-1}^{(m-j)}(x_i)\sum_{l=0}^{j}\binom{j}{l}\phi_{m-1}^{(j-l)}(x_i)B_{m-1}^{(l)}(x_i). \end{aligned}$$

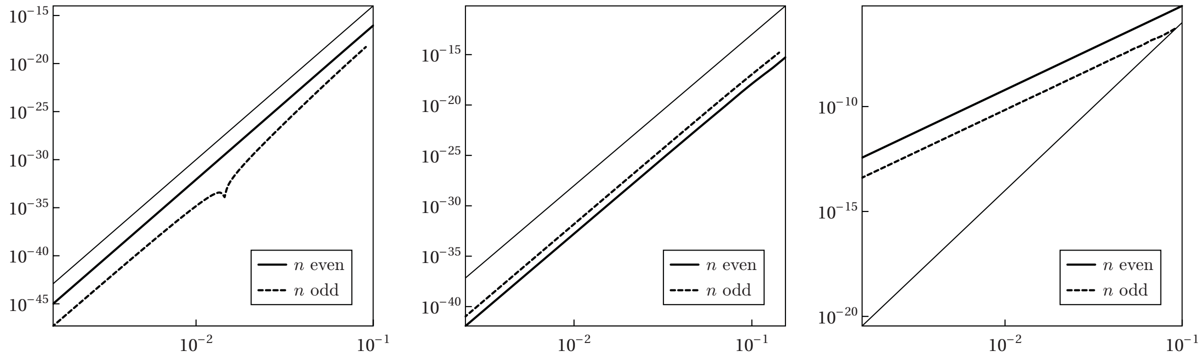

Figure 1: Log-log plot of the error with respect to h for Experiment 1, 2, and 3 (from left to right).

Experiment	m	d	f	x_i
1	3	3	$\exp(-(18x-9)^2/64)/3$	i/n
2	4	2	$\sqrt{256-81(2x-1)^2}/18-1/2$	$(1-\cos(i\pi/n))/2$
3	3	1	$\lvert 2x-1\rvert(2x-1)^3$	i/n

Table 1: Parameters m and d, functions f, and interpolation nodes x_i used in our numerical experiments.

Since

$$\phi_{m-1}^{(j)}(x_i)=\begin{cases} m!, & \text{if } j=m, \\ 0, & \text{otherwise,} \end{cases}$$

and we can show, using the same arguments as in the proof of Proposition 3, that $A_{m-1}^{(j)}(x_i)$ and $B_{m-1}^{(j)}(x_i)$ are bounded for $j=0,\dots,m$, as long as $f\in C^{2m}[a,b]$, we conclude

$$w_i{}^m g_{i,m}=\frac{w_i{}^m}{m!}A_{m-1}(x_i)\,m!\,B_{m-1}(x_i)=A_{m-1}(x_i),$$

which proves the statement. □

4 Numerical results

We implemented the iterative rational Hermite interpolants r_m and carried out various experiments to numerically verify the approximation order stated in Theorem 5. Table 1 lists the individual settings of three exemplary experiments that we decided to report in detail. The plots in Figure 1 show the expected convergence rate $O(h^{(m+1)(d+1)})$ as a thin straight line and the behaviour of the approximation error $\|e_m\|$ for different values of n, obtained by sampling $e_m = f - r_m$ at 100 equidistant points in each of the n subintervals $[x_{i-1}, x_i]$, $i=1,\dots,n$ and determining the maximum of these samples. Since the error behaves differently for even and odd values of n, we separate both cases and show two graphs, one for $n=2k$ and one for $n=2k+1$, with $k=5,\dots,320$ in both cases. All computations were performed in *C++* using the multiple-precision library *MPFR* [7].

The first two experiments support the statement of Theorem 5, and Experiment 3 confirms that the order of continuity of f is crucial for the approximation order. Since $m=3$ and $d=1$, we would expect an approximation order of $O(h^8)$, but this order is guaranteed by Theorem 5 only if $f\in C^{12}$, while the function considered in this experiment is just three times continuously differentiable, and the approximation order is indeed merely $O(h^4)$. More generally, we observed that the approximation order appears to be $O(h^{k+1})$ if $f\in C^k$ for some $k<(m+1)(d+1)$, so it might be possible to relax the condition on f in Theorem 5 to $f\in C^{(m+1)(d+1)-1}$, but so far we do not know how to prove it.

Overall, the experiments show that the error is smaller for $n-d$ even, and we observed this pattern in other experiments, too. This is contrary to the upper bounds derived in the proof of Theorem 5, since the upper bound for $n-d$ odd in (15) is clearly smaller than the upper bound in (16) for $n-d$ even. However, we do not claim that these bounds are sharp, and it remains interesting future work to further investigate this phenomenon.

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