Convergence rates of derivatives of Floater–Hormann interpolants for well-spaced nodes

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Abstract

Floater–Hormann interpolants constitute a family of barycentric rational interpolants which are based on blending local polynomial interpolants of degree d. Recent results suggest that the k-th derivatives of these interpolants converge at the rate of $O(h^{d+1-k})$ for $k \le d$ as the mesh size h converges to zero. So far, this convergence rate has been proven for $k = 1, 2$ and for $k \ge 3$ under the assumption of equidistant or quasi-equidistant interpolation nodes. In this paper we extend these results and prove that Floater–Hormann interpolants and their derivatives converge at the rate of $O(h_j^{d+1-k})$, where h_j is the local mesh size, for any $k \ge 0$ and any set of well-spaced nodes.

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1 Introduction

The barycentric rational interpolants that were introduced by Berrut [1] and later generalized by Floater and Hormann [7] received a lot of attention in recent years, because they possess a number of interesting properties. In contrast to general rational interpolation, they are linear in the data and guaranteed to have no poles in $\mathbb{R}$. They are particularly suited for interpolation at equidistant nodes [3, 4], and one of their advantages over splines is that they are infinitely smooth. Moreover, they are known to have arbitrarily high approximation orders [7], and in this paper we study the convergence rates of their derivatives.

Given a function $f : [a, b] \to \mathbb{R}$, a set $X_n = \{x_0, x_1, \dots, x_n\}$ of $n+1$ interpolation nodes with

$$a = x_0 < x_1 < \cdots < x_n = b,$$

and a parameter d with $0 \le d \le n$, the Floater–Hormann interpolant [7] is defined as

$$r(x) = \sum_{i=0}^{n-d} \lambda_i(x) p_i(x) \bigg/ \sum_{i=0}^{n-d} \lambda_i(x),$$

where p_i denotes the unique polynomial of degree at most d that interpolates f locally,

$$p_i(x_j) = f(x_j), \qquad j = i, \dots, i+d,$$

and

$$\lambda_i(x) = \frac{(-1)^i}{(x - x_i) \cdots (x - x_{i+d})}.$$

Floater and Hormann [7] show that for sufficiently smooth f the error

$$e(x) = f(x) - r(x) \tag{1}$$

satisfies

$$|e(x)| \le C h^{d+1}, \qquad x \in [a, b], \tag{2}$$

where

$$h = \max_{0 \le i \le n-1} h_i, \qquad h_i = h_{i+1,i}, \qquad h_{i,j} = |x_i - x_j|.$$

Hence, the convergence rate of the error e is $O(h^{d+1})$ as the global mesh size h converges to 0. Since r is a blend of local polynomial interpolants of degree at most d, it is reasonable to expect the convergence rate of the k-th derivative of e to be $O(h^{d+1-k})$ for $k \le d$, and Berrut et al. [2] prove it for $k = 1, 2$. For the special case of equidistant or quasi-equidistant nodes, Klein and Berrut [10] establish this convergence rate for $k \ge 3$, but only at the nodes and not at intermediate points.

The goal of this paper is to generalize these results in two directions. We first show that the convergence rate of $e^{(k)}$ is $O(h^{d+1-k})$ for any $k \ge 1$ in the case of well-spaced interpolation nodes.

Definition 1. For each $n \in \mathbb{N}$, let X_n be a set of interpolation nodes. We then say that $X = (X_n)_{n\in\mathbb{N}}$ is a family of *well-spaced nodes*, if there exist constants $R_1, R_2 \geq 1$, independent of n, such that the two conditions

$$\frac{1}{R_1} \leq \frac{h_i}{h_{i-1}} \leq R_1, \qquad i = 1, \dots, n-1, \tag{3}$$

and

$$\begin{aligned} \frac{h_i}{h_{i+1,j}} &\leq \frac{R_2}{i+1-j}, \qquad j = 0, \dots, i, \quad i = 0, \dots, n-1, \\ \frac{h_i}{h_{j,i}} &\leq \frac{R_2}{j-i}, \qquad j = i+1, \dots, n, \quad i = 0, \dots, n-1, \end{aligned} \tag{4}$$

hold for every set of nodes X_n.

While condition (3) bounds the mesh ratio locally, condition (4) bounds the mesh ratio globally in the sense that it limits the factor by which the length h_i of an interval can be larger than the average lengths $(h_{i-k} + h_{i-k+1} + \cdots + h_i)/(k+1)$ and $(h_i + h_{i+1} + \cdots + h_{i+k})/(k+1)$ of neighbouring intervals to the left and to the right for all valid k. These nodes were introduced by Bos et al. [5] and include not only equidistant and quasi-equidistant nodes, but also Chebyshev–Gauss–Lobatto nodes [11] and extended Chebyshev nodes [6]. An example of not well-spaced nodes are nodes in geometric progression,

$$x_i = \frac{\mu^i - 1}{\mu^n - 1}, \qquad i = 0, \dots, n,$$

for $\mu > 1$, which satisfy (3), but not (4).

We also show that the error depends on the local mesh size instead of the global mesh size. More precisely, we establish the following upper bounds on the error and its derivatives.

Theorem 1. *For any set of well-spaced interpolation nodes, any k with $0 \leq k \leq d$, and $f \in C^{d+2+k}[a,b]$,*

$$|e^{(k)}(x)| \leq C h^{d+1-k}, \qquad x \in [a,b],$$

and more specifically,

$$|e^{(k)}(x)| \leq C h_j^{d+1-k}, \qquad x \in [x_j, x_{j+1}], \qquad j = 0, \dots, n-1. \tag{5}$$

Note that in Theorem 1 and throughout the text we denote by C a generic constant depending only on k, d, the derivatives of f, the interval length $b-a$, and the constants R_1 and R_2 from Definition 1. To establish the bounds in Theorem 1, we first analyse the error at the nodes x_j (Section 2) and then at intermediate points $x \in (x_j, x_{j+1})$ (Section 3). We conclude the paper with several numerical examples which confirm the bound in (5) and highlight the dependence on the local mesh size h_j (Section 4).

2 Error at the nodes

In what follows, it helps to remember from [7] that the error (1) can be written as

$$e(x) = \frac{A(x)}{B(x)} \tag{6}$$

where

$$A(x) = \sum_{i=0}^{n-d} (-1)^i f[x_i, x_{i+1}, \dots, x_{i+d}, x]$$

and

$$B(x) = \sum_{i=0}^{n-d} \lambda_i(x).$$

We are now ready to study the convergence rate of the derivatives of e at the interpolation nodes.

Lemma 1. *For any set of well-spaced interpolation nodes, any* k *with* $1 \le k \le d$*, and*[1] $f \in C^{d+2+k}[a,b]$*,*

$$|e^{(k)}(x_j)| \le C h_j^{d+1-k}, \qquad j = 0, \dots, n-1$$

and

$$|e^{(k)}(x_j)| \le C h_{j-1}^{d+1-k}, \qquad j = 1, \dots, n.$$

Proof. Throughout this proof we consider only the first statement, because the second statement can be established analogously by taking into account that $h_j \le R_1 h_{j-1}$, according to (3). We also point out that the proof is largely inspired by the proof of Theorem 2.1 in [10], except that we utilize (3) and (4) to derive local error bounds in h_j instead of the global error bounds in h that were considered by Klein and Berrut [10]. Moreover, we resort to Hoppe's formula in (7) as a generalization of the chain rule to higher derivatives instead of Faà di Bruno's formula, which was used in [10] for the same purpose, because the latter does not lead to our local error bounds.

We start by fixing the index j and expressing the error in (6) as

$$e(x) = \phi(x)\hat{e}(x),$$

where

$$\phi(x) = x - x_j, \qquad \hat{e}(x) = \frac{A(x)}{D(x)}, \qquad D(x) = \phi(x)B(x).$$

By the Leibniz rule, we have

$$e^{(k)}(x) = \phi(x)\hat{e}^{(k)}(x) + k\phi'(x)\hat{e}^{(k-1)}(x) \tag{$*$}$$

and

$$e^{(k)}(x_j) = k\hat{e}^{(k-1)}(x_j). \tag{$**$}$$

Again, we use the Leibniz rule to obtain

$$\hat{e}^{(k-1)}(x_j) = \sum_{l=0}^{k-1} \binom{k-1}{l} A^{(k-1-l)}(x_j)(D^{-1})^{(l)}(x_j).$$

Since Lemma 2 in [2] guarantees that the absolute values of A and its derivatives are bounded by some constant over $[a,b]$ for[2] $f \in C^{d+1+k}[a,b]$, it remains to show that

$$|(D^{-1})^{(l)}(x_j)| \le C h_j^{d-l}, \qquad l = 0, \dots, d-1.$$

Using Hoppe's formula [8, 9] we obtain

$$(D^{-1})^{(l)}(x) = \sum_{p=0}^{l} \frac{(-1)^p}{D^{p+1}(x)} \sum_{m=0}^{p} \binom{p}{m} (-1)^{p-m} D^{p-m}(x)(D^m)^{(l)}(x), \tag{7}$$

so that

$$|(D^{-1})^{(l)}(x_j)| \le \sum_{p=0}^{l}\sum_{m=0}^{p} \binom{p}{m} \frac{|(D^m)^{(l)}(x_j)|}{|D^{m+1}(x_j)|},$$

and the final step now is to prove by induction over m that

$$\frac{|(D^m)^{(l)}(x_j)|}{|D^{m+1}(x_j)|} \le C h_j^{d-l}, \qquad l = 0, \dots, d-1 \tag{8}$$

for any $m \ge 0$.

We obtain this result by first deriving a lower bound for $|D(x_j)|$ and an upper bound for $|D^{(l)}(x_j)|$, and to this end it helps to write $D(x)$ as

$$D(x) = E(x) + \phi(x)F(x)$$

[1] Note that the differentiability class of f is misprinted as C^{d+1+k} in the published version of this article.

[2] Here is the reason for the misprint in the statement above: to conclude ($**$) from ($*$) we need to bound $\hat{e}^{(k)}(x_j)$, which can be done following the reasoning here and below, but only if $f \in C^{d+2+k}[a,b]$.

with

$$E(x) = \sum_{i \in I_j} (-1)^i \prod_{\substack{k=i \\ k \neq j}}^{i+d} \frac{1}{x - x_k}, \qquad F(x) = \sum_{i \in I \setminus I_j} (-1)^i \prod_{k=i}^{i+d} \frac{1}{x - x_k},$$

where

$$I = \{0, 1, \dots, n-d\}, \qquad I_j = \{i \in I : j - d \leq i \leq j\}.$$

Berrut et al. [2] show that

$$|D(x_j)| = |E(x_j)| \geq \prod_{\substack{k=i \\ k \neq j}}^{i+d} h_{j,k}^{-1}, \qquad i \in I_j \tag{9}$$

and continue to bound the right hand side from below by $C h^{-d}$. Instead, we use (3) to conclude

$$h_{j,k} \leq \sum_{m=\max(0,j-d)}^{\min(n-1,j+d-1)} h_m \leq \sum_{m=\max(0,j-d)}^{\min(n-1,j+d-1)} R_1^{|j-m|} h_j \leq 2d R_1^d h_j \tag{10}$$

for all $h_{j,k}$ in (9), which leads to the lower bound

$$|D(x_j)| \geq C h_j^{-d}. \tag{11}$$

For the upper bounds on the derivatives of D at x_j, we assume $l \geq 1$ for the moment, follow [10], and use the relation

$$D^{(l)}(x_j) = E^{(l)}(x_j) + l F^{(l-1)}(x_j)$$

and the Leibniz rule to get

$$E^{(l)}(x) = \sum_{i \in I_j} (-1)^{i+l} l! \sum_{|\boldsymbol{\alpha}_{i,j}|=l} \prod_{\substack{k=i \\ k \neq j}}^{i+d} \frac{1}{(x - x_k)^{1+\alpha_k}},$$

where the second sum ranges over all d-dimensional multi-indices $\boldsymbol{\alpha}_{i,j} = (\alpha_i, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_{i+d})$ whose non-negative integer components add up to l. By (3) and (4),

$$\begin{aligned} |E^{(l)}(x_j)| &\leq \sum_{i \in I_j} l! \sum_{|\boldsymbol{\alpha}_{i,j}|=l} \prod_{\substack{k=i \\ k \neq j}}^{i+d} \frac{1}{h_{j,k}^{1+\alpha_k}} \\ &\leq \sum_{i \in I_j} l! \sum_{|\boldsymbol{\alpha}_{i,j}|=l} \prod_{\substack{k=i \\ k \neq j}}^{i+d} \left(\frac{R_1 R_2}{h_j |j-k|} \right)^{1+\alpha_k} \\ &\leq C h_j^{-(d+l)}, \end{aligned} \tag{12}$$

and the same upper bound can be derived analogously for $|F^{(l-1)}(x_j)|$, so that overall

$$|D^{(l)}(x_j)| \leq C h_j^{-(d+l)}, \qquad l = 1, \dots, d-1. \tag{13}$$

Let us now return to (8) and observe that the base case $m = 0$ and the special case $l = 0$ follow directly from (11). For the induction step assume that (8) holds for an arbitrary value of $m \geq 0$ and apply again the Leibniz rule to get

$$\frac{(D^{m+1})^{(l)}(x)}{D^{m+2}(x)} = \frac{\sum_{k=0}^{l} \binom{l}{k} (D^m)^{(l-k)}(x) D^{(k)}(x)}{D^{m+1}(x) D(x)}.$$

Using the induction hypothesis as well as the bounds in (11) and (13), we then have

$$\begin{aligned} \frac{|(D^{m+1})^{(l)}(x_j)|}{|D^{m+2}(x_j)|} &\leq \frac{\sum_{p=0}^{l} \binom{l}{p} |(D^m)^{(l-p)}(x_j)| |D^{(p)}(x_j)|}{|D^{m+1}(x_j)| |D(x_j)|} \\ &= \frac{|(D^m)^{(l)}(x_j)|}{|D^{m+1}(x_j)|} + \sum_{p=1}^{l} \binom{l}{p} \frac{|(D^m)^{(l-p)}(x_j)|}{|D^{m+1}(x_j)|} \frac{|D^{(p)}(x_j)|}{|D(x_j)|} \\ &\leq C_1 h_j^{d-l} + \sum_{p=1}^{l} \binom{l}{p} C_1 h_j^{d-l+p} \frac{C_2 h_j^{-(d+p)}}{C_3 h_j^{-d}} \\ &\leq C h_j^{d-l} \end{aligned}$$

for $l = 1, \dots, d-1$. □

Lemma 1 generalizes Theorem 2.1 in [10] in two ways. On the one hand, it covers well-spaced interpolation nodes, which includes equidistant and quasi-equidistant nodes as special cases. On the other hand, it provides an error bound in terms of the local mesh size h_j instead of the global mesh size h. The special cases $k=1$ and $k=2$ also appear as Theorems 1 and 2 in [2], which are more general than Lemma 1 in the sense that they do not require the nodes to be well-spaced, but as in [10] the error bound is given in terms of the global mesh size only.

3 Error at intermediate points

We now consider the convergence rate of the derivatives of e at the intermediate points between the interpolation nodes.

Lemma 2. *For any set of well-spaced interpolation nodes, any k with $0 \le k \le d$, and $f \in C^{d+2+k}[a,b]$,*

$$|e^{(k)}(x)| \le C h_j^{d+1-k}, \qquad x \in (x_j, x_{j+1}), \qquad j = 0, \dots, n-1.$$

Proof. The proof of this lemma is largely inspired by Theorem 5 in [2] and roughly follows the same reasoning as the proof of Lemma 1. Hence, we expect the reader to already be familiar with the main arguments and keep the exposition brief.

We start by fixing the index j and writing the error in (6) as

$$e(x) = \psi(x)\tilde{e}(x),$$

where

$$\psi(x) = (x - x_j)(x - x_{j+1}), \qquad \tilde{e}(x) = \frac{A(x)}{D(x)}, \qquad D(x) = \psi(x)B(x).$$

By the Leibniz rule, we have

$$e^{(k)}(x) = \psi(x)\tilde{e}^{(k)}(x) + k\psi'(x)\tilde{e}^{(k-1)}(x) + \frac{k(k-1)}{2}\psi''(x)\tilde{e}^{(k-2)}(x),$$

and since it follows from the definition of ψ that

$$|\psi(x)| \le h_j^2, \qquad |\psi'(x)| \le 2h_j, \qquad |\psi''(x)| \le 2, \tag{14}$$

it remains to show that $|\tilde{e}^{(k)}(x)| \le C h_j^{d-1-k}$. As in the proof of Lemma 1, we use the Leibniz rule to obtain

$$\tilde{e}^{(k)}(x) = \sum_{l=0}^{k} \binom{k}{l} A^{(k-l)}(x)(D^{-1})^{(l)}(x)$$

and since the absolute values of A and its derivatives are bounded by some constant over $[a,b]$ for $f \in C^{d+2+k}$, it is sufficient to prove

$$|(D^{-1})^{(l)}(x)| \le C h_j^{d-1-l}, \qquad l = 0, \dots, d.$$

Using again Hoppe's formula and the same reasoning as in the previous proof, the final step now is to prove by induction over m that

$$\frac{|(D^m)^{(l)}(x)|}{|D^{m+1}(x)|} \le C h_j^{d-1-l}, \qquad l = 0, \dots, d \tag{15}$$

for any $m \ge 0$, and the crucial ingredients are a lower bound for $|D(x)|$ and an upper bound for $|D^{(l)}(x)|$.

For the lower bound, we recall from [2] that

$$|D(x)| \ge \prod_{\substack{k=i \\ k \ne j, j+1}}^{i+d} \frac{1}{|x - x_k|}, \qquad i \in \{p \in I : j - d + 1 \le p \le j\},$$

but instead of further bounding this from below by $C h^{-(d-1)}$, we use (10) to obtain

$$|D(x)| \ge \prod_{k=i}^{j-1} h_{j+1,k}^{-1} \prod_{k=j+2}^{i+d} h_{j,k}^{-1} \ge C h_j^{-(d-1)}. \tag{16}$$

For the upper bounds on the derivatives of D, we assume $l \geq 1$ for the moment, split $D^{(l)}(x)$ into five parts as in [2],

$$D^{(l)}(x) = E_1^{(l)}(x) + E_2^{(l)}(x) + E_3^{(l)}(x) + E_4^{(l)}(x) + E_5^{(l)}(x),$$

where

$$E_1(x) = \psi(x) \sum_{i=0}^{j-d-1} \lambda_i(x), \qquad E_2(x) = \psi(x)\lambda_{j-d}(x), \qquad E_3(x) = \psi(x) \sum_{i=j-d+1}^{j} \lambda_i(x),$$

$$E_4(x) = \psi(x)\lambda_{j+1}(x), \qquad E_5(x) = \psi(x) \sum_{i=j+2}^{n-d} \lambda_i(x),$$

and derive separate upper bounds for each of the terms $E_i^{(l)}(x)$.

For $E_1^{(l)}(x)$, we let

$$F_1(x) = \sum_{i=0}^{j-d-1} \lambda_i(x) = \sum_{i=0}^{j-d-1} (-1)^i \prod_{k=i}^{i+d} \frac{1}{x - x_k}$$

and use the Leibniz rule to get

$$E_1^{(l)}(x) = \psi(x)F_1^{(l)}(x) + l\psi'(x)F_1^{(l-1)}(x) + \frac{l(l-1)}{2}\psi''(x)F_1^{(l-2)}(x).$$

Using the Leibniz rule again we further find that

$$F_1^{(l)}(x) = \sum_{i=0}^{j-d-1} (-1)^{i+l} l! \sum_{|\boldsymbol{\beta}_i|=l} \prod_{k=i}^{i+d} \frac{1}{(x - x_k)^{1+\beta_k}},$$

where the second sum ranges over all $(d+1)$-dimensional multi-indices $\boldsymbol{\beta}_i = (\beta_i, \ldots, \beta_{i+d})$ whose non-negative integer components sum up to l. Since $x \in (x_j, x_{j+1})$, the terms of the first sum alternate in sign and increase in absolute value, so that $|F_1^{(l)}(x)|$ is bounded from above by the absolute value of the last term. With the same reasoning as in (12) we then have

$$\begin{aligned} |F_1^{(l)}(x)| &\leq l! \sum_{|\boldsymbol{\beta}_{j-d-1}|=l} \prod_{k=j-d-1}^{j-1} \frac{1}{|x - x_k|^{1+\beta_k}} \\ &\leq l! \sum_{|\boldsymbol{\beta}_{j-d-1}|=l} \prod_{k=j-d-1}^{j-1} \frac{1}{h_{j,k}^{1+\beta_k}} \\ &\leq C h_j^{-(d+1+l)}, \end{aligned}$$

and together with (14) we conclude

$$|E_1^{(l)}(x)| \leq C h_j^{-(d-1+l)}, \qquad l = 1, \ldots, d. \tag{17}$$

For $E_2^{(l)}(x)$, we let

$$F_2(x) = (x - x_j)\lambda_{j-d}(x) = (-1)^{j-d} \prod_{k=j-d}^{j-1} \frac{1}{x - x_k},$$

so that

$$E_2^{(l)}(x) = (x - x_{j+1})F_2^{(l)}(x) + l F_2^{(l-1)}(x)$$

and

$$F_2^{(l)}(x) = (-1)^{j-d+l} l! \sum_{|\boldsymbol{\beta}_{j-d}|=l} \prod_{k=j-d}^{j-1} \frac{1}{(x - x_k)^{1+\beta_k}},$$

with $\boldsymbol{\beta}_{j-d}$ defined as before. Therefore,

$$|F_2^{(l)}(x)| \leq l! \sum_{|\boldsymbol{\beta}_{j-d}|=l} \prod_{k=j-d}^{j-1} \frac{1}{h_{j,k}^{1+\beta_k}} \leq C h_j^{-(d+l)},$$

and

$$|E_2^{(l)}(x)| \le C h_j^{-(d-1+l)}, \qquad l = 1, \dots, d. \tag{18}$$

For $E_3^{(l)}(x)$, we notice that

$$E_3(x) = \sum_{i=j-d+1}^{j} (-1)^i \prod_{\substack{k=i \\ k\neq j,j+1}}^{i+d} \frac{1}{x - x_k},$$

hence

$$E_3^{(l)}(x) = \sum_{i=j-d+1}^{j} (-1)^{i+l} l! \sum_{|\boldsymbol{\beta}_i|=l} \prod_{\substack{k=i \\ k\neq j,j+1}}^{i+d} \frac{1}{(x - x_k)^{1+\beta_k}}$$

and

$$|E_3^{(l)}(x)| \le \sum_{i=j-d+1}^{j} l! \sum_{|\boldsymbol{\beta}_i|=l} \prod_{\substack{k=i \\ k\neq j,j+1}}^{i+d} \frac{1}{h_{j,k}^{1+\beta_k}} \le C h_j^{-(d-1+l)}, \qquad l = 1, \dots, d. \tag{19}$$

Combining (17), (18), (19), and noting that the error bounds for $E_4^{(l)}(x)$ and $E_5^{(l)}(x)$ can be derived similarly as the bounds for $E_2^{(l)}(x)$ and $E_1^{(l)}(x)$, respectively, we finally conclude

$$|D^{(l)}(x)| \le C h_j^{-(d-1+l)}, \qquad l = 1, \dots, d. \tag{20}$$

We now observe that the base case $m = 0$ of (15) and the special case $l = 0$ follow directly from (16) and the induction step follows from (16) and (20) with the same arguments as in the proof of Lemma 1. □

While the special cases $k = 0, 1, 2$ were already covered by [7] and [2], Lemma 2 generalizes the result to general $0 \le k \le d$ and provides a local instead of a global error bound. However, this comes at the cost of having to assume that the interpolation nodes are well-spaced. Theorem 2 in [7] and Theorem 3 in [2], which cover the cases $k = 0 < d$ and $k = 1 < d$, respectively, hold for any nodes. Theorem 3 in [7] and Theorems 4–6 in [2], which handle the cases $k = 0 = d$, $k = 1 = d$, $k = 2 < d$, and $k = 2 = d$, respectively, only require that the mesh ratio is bounded, which is basically the first condition (3) of well-spaced nodes.

4 Numerical examples

To confirm our theoretical results, we prepared four numerical examples, using the formulas by Schneider and Werner [12] for evaluating the derivatives of the rational interpolant r both at the nodes and at intermediate points. In the first two examples we investigated the behaviour of the maximum norm $\|e^{(k)}\| = \max_{a\le x\le b} |e^{(k)}(x)|$ of the error and its derivatives in dependence of the global mesh size h. For both examples we used Matlab with double precision and approximated $\|e^{(k)}\|$ by evaluating $|e^{(k)}(x)|$ at 100 equidistant points in each interval $[x_j, x_{j+1}]$, $j = 0, \dots, n-1$. Instead, the last two examples were prepared with Maple using a precision of 30 digits and illustrate the pointwise values $|e^{(k)}(x)|$ for various x and k with respect to the local mesh size.

In our first example we study the Floater–Hormann interpolant with $d = 3$ for Runge's test function,

$$f_1(x) = \frac{1}{1 + 25(2x-1)^2}, \quad x \in [0, 1],$$

sampled at $n + 1$ equidistant nodes. Table 1 reports the maximum norm and estimated approximation order of the error and its derivatives for several values of n. Figure 1 shows the maximum norm of the error and its derivatives in dependence of h for all even n from $n = 10$ to $n = 500$. We did not include the values for odd n in the plot, because they follow the same trend, but are always a bit smaller, so that including them would have resulted in slightly confusing zigzag curves. The data clearly supports the first bound in Theorem 1.

In our second example we consider the Floater–Hormann interpolant with $d = 2$ for the function

$$f_2(x) = \sin(\pi x), \quad x \in [0, 1],$$

sampled at the Chebyshev–Gauss–Lobatto nodes.[3] Table 2 and Figure 2 are similar to those of the first example. Again, the data supports our theoretical results, and the case $k = 3$ shows that the expected bound

[3] Note that this example is for illustration purposes only. We do not advocate the use of Floater–Hormann interpolation for these nodes, for which polynomial interpolation is better in every respect [13].

n	$\|e\|$	order	$\|e'\|$	order	$\|e''\|$	order	$\|e'''\|$	order
10	4.03e–02		4.22e+00		1.57e+02		2.88e+03	
20	1.81e–03	4.48	3.59e–01	3.56	2.80e+01	2.49	1.01e+03	1.51
40	2.85e–06	9.31	1.11e–03	8.34	1.77e–01	7.30	1.34e+01	6.23
80	3.43e–08	6.38	2.66e–05	5.38	8.60e–03	4.37	1.33e+00	3.33
160	2.03e–09	4.08	3.14e–06	3.08	2.04e–03	2.08	6.40e–01	1.06
320	1.23e–10	4.04	3.81e–07	3.04	4.97e–04	2.04	3.14e–01	1.03
640	7.58e–12	4.02	4.69e–08	3.02	1.23e–04	2.02	1.55e–01	1.01

Table 1: Norm and approximation order of the error and its derivatives for f_1, using equidistant nodes and $d = 3$. Compare Figure 1.

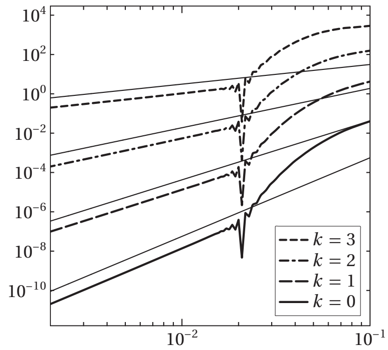

Figure 1: Log-log plot of $\|e^{(k)}\|$ over h for f_1, using equidistant nodes and $d = 3$. The straight reference lines represent the expected $O(h^{d+1-k})$ behaviour.

n	$\|e\|_\infty$	order	$\|e'\|_\infty$	order	$\|e''\|_\infty$	order	$\|e'''\|_\infty$	order
10	2.13e–04		4.90e–03		2.85e–01		1.10e+01	
20	2.71e–05	3.03	1.27e–03	1.98	6.87e–02	2.09	9.58e+00	0.20
40	3.44e–06	2.99	3.22e–04	1.99	3.31e–02	1.06	9.24e+00	0.05
80	4.30e–07	3.00	8.10e–05	1.99	1.65e–02	1.00	9.15e+00	0.01
160	5.39e–08	3.00	2.03e–05	2.00	8.27e–03	1.00	9.13e+00	0.00
320	6.74e–09	3.00	5.07e–06	2.00	4.14e–03	1.00	9.17e+00	–0.01
640	8.42e–10	3.00	1.27e–06	2.00	2.07e–03	1.00	9.25e+00	–0.01

Table 2: Norm and approximation order of the error and its derivatives for f_2, using Chebyshev nodes and $d = 2$. Compare Figure 2.

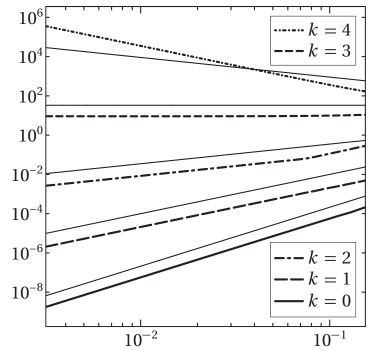

Figure 2: Log-log plot of $\|e^{(k)}\|$ over h for f_2, using Chebyshev nodes and $d = 2$. The straight reference lines represent the expected $O(h^{d+1-k})$ behaviour.

also holds for $k = d + 1$. But since we only get boundedness and not convergence in this case, we did not include it in the statement of Theorem 1.

In our third example we sample the function

$$f_3(x) = \exp(x^2), \quad x \in [0, 1],$$

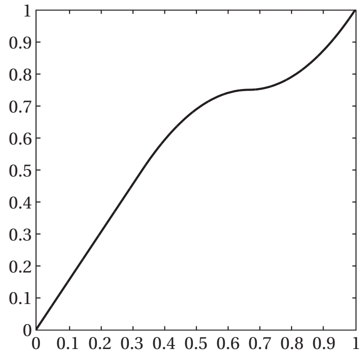

Figure 3: Plot of the regular distribution function $g(x)$ in (21).

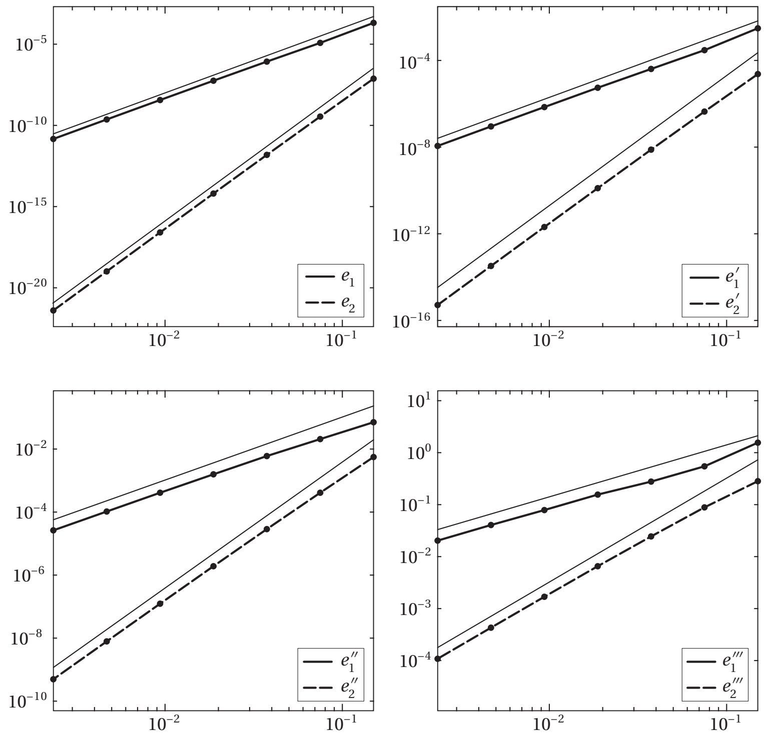

Figure 4: Log-log plot of $e_1^{(k)} = |e^{(k)}(1/4)|$ and $e_2^{(k)} = |e^{(k)}(3/4)|$ over h for f_3, using the well-spaced nodes generated by $g(x)$ in Figure 3 and $d = 3$. The straight reference lines represent the expected $O(h^{d+1-k})$ and $O(h^{2(d+1-k)})$ behaviours.

at the interpolation nodes generated by the function (see Figure 3)

$$g(x) = \begin{cases} \frac{3}{2}x, & x \in \left[0, \frac{1}{3}\right), \\ -\frac{9}{4}x^2 + 3x - \frac{1}{4}, & x \in \left[\frac{1}{3}, \frac{2}{3}\right), \\ \frac{9}{4}x^2 - 3x + \frac{7}{4}, & x \in \left[\frac{2}{3}, 1\right], \end{cases} \tag{21}$$

that is, $x_i = g(i/n)$, $i = 0, \ldots, n$. These nodes are well-spaced, because g is a *regular distribution function* [5]. For this function g, the local mesh size h_j around $x = 1/4$ and $x = 3/4$ behaves differently, namely like $O(h)$ and $O(h^2)$, respectively. Therefore, the expected convergence rates of $e^{(k)}(1/4)$ and $e^{(k)}(3/4)$, according to Theorem 1, are $O(h^{d+1-k})$ and $O(h^{2(d+1-k)})$, respectively. This is confirmed by the plots in Figure 4 for the case $d = 3$.

In our last example we go back to Chebyshev–Gauss–Lobatto nodes, consider the Floater–Hormann interpolant with $d = 1$ for the function

$$f_4(x) = \tfrac{3}{4}e^{-(9x-2)^2/4} + \tfrac{3}{4}e^{-(9x+1)^2/49} + \tfrac{1}{2}e^{-(9x-7)^2/4} + \tfrac{1}{5}e^{-(9x-4)^2}, \quad x \in [0,1],$$

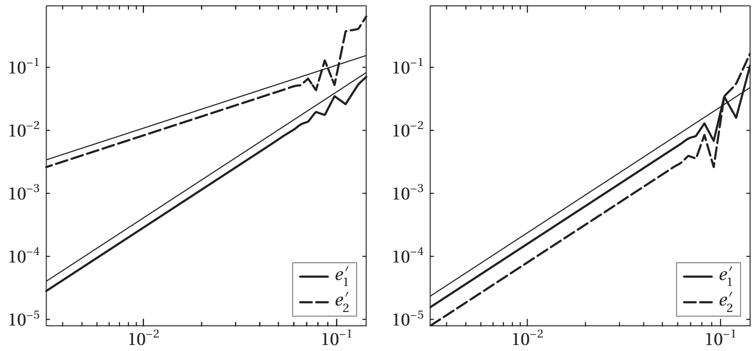

Figure 5: Log-log plot of $e_1' = |e'(0)|$ and $e_2' = |e'(1/2)|$ over h for f_4, using $n+1$ Chebyshev nodes and $d=1$ for even n (left) and odd n (right). The straight reference lines represent the $O(h)$ and the $O(h^2)$ behaviour.

and study the convergence rate of $e'(x)$ at the start and the centre of the interpolation interval. According to Theorem 4 in [10], the expected convergence rates of $e'(0)$ and $e'(1/2)$ with respect to the global mesh size are both $O(h)$, but while the left plot in Figure 5 confirms this rate for $e'(1/2)$, it also illustrates that $e'(0)$ converges at the rate of $O(h^2)$. Theorem 1 explains this result, because the local mesh size at $x=0$ behaves like $O(h^2)$, while the local mesh size at $x=1/2$ behaves like $O(h)$. However, the right plot in Figure 5 shows that $e'(1/2)$ converges at the rate of $O(h^2)$, too, if restricted to odd n, so that $x=1/2$ is not an interpolation node, and it remains future work to better understand the underlying reason.

Acknowledgements

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