Anisotropic Voronoi diagrams from distance graphs

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Abstract

We present a new type of an anisotropic Voronoi diagram, constructed from a distance graph, which is a set of distances between given points. Our anisotropic Voronoi diagram is a generalization of the Euclidean Voronoi diagram, using an anisotropic metric, which approximates a given distance graph best in the sense of least squares.

The anisotropic metric is based on a 2-dimensional, continuous one-to-one embedding into \( \mathbb{R}^m \) for \( m \geq 2 \). This embedding is constructed from the distance graph via a fitting procedure which is based on the Gauss-Newton algorithm.

Mathematical Background

Voronoi diagram (cf. [1])

Let \( m \in \mathbb{Z} \), with \( m \geq 2 \) and let \( P = \{p_1, p_2, \ldots \} \) with \( p_i \in \mathbb{R}^m \). Let \( D \) be a metric on \( \mathbb{R}^m \). Then we define the Voronoi cell \( V_D(p_i) \) of the point \( p_i \in \mathbb{R}^m \) as follows

\[
V_D(p_i) = \{ p \in \mathbb{R}^m : D(p, p_i) < D(p, p_j) \text{ for all } j \neq i \}.
\]

Then the Voronoi diagram \( V_D(P) \) is given by

\[
V_D(P) = \mathbb{R}^m \setminus \bigcup_{i} V_D(p_i).
\]

We denote the Voronoi diagram using the Euclidian metric by \( V(P) \). We call a Voronoi diagram orphan-free if each Voronoi cell is connected.

Anisotropic metric framework

Let \( m \in \mathbb{Z} \), with \( m \geq 2 \) and let \( x : \mathbb{R}^2 \to \mathbb{R}^m \) be a continuous one-to-one embedding with \( x(u, v) = (x_1(u, v), \ldots, x_m(u, v)) \). Let \( d(r) \) for \( r \geq 0 \) be a scalar-valued function with the following properties

- \( d(0) = 0 \),
- \( d(r) > 0 \) for \( r > 0 \),
- \( d(r) \geq d(s) \) for \( r \geq s \) is monotonic decreasing.

Then we define the distance \( D : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} \) between two points \( u_1 = (u_1, v_1) \) and \( u_2 = (u_2, v_2) \) as follows

\[
D(u_1, u_2) = d(||x(u_1, v_1) - x(u_2, v_2)||).
\]

Lemma

The distance \( D \), given by (1), defines a metric on \( \mathbb{R}^2 \).

Distance graph

Let \( n \in \mathbb{Z}^+ \), let \( I = [0, 1]^2 \) and let \( Q = \{q_1, \ldots, q_n\} \) be a set of points in \( I \). In addition, let \( Y \) be a set of double indices

\[
Y = \{(y, z) : y, z \in I \}
\]

satisfying \( y < z \) for all \( (y, z) \in Y \). Then a distance graph \( G \) is given by the points of \( Q \) and by the edges \( e_{ij} = (q_i, q_j) \) for \( (y, z) \in Y \) with assigned lengths \( l_{y,z} \). The lengths \( l_{y,z} \) are not the real lengths of the corresponding edges \( e_{ij} \) in \( I \), rather appropriate distances between the points \( q_i \) and \( q_j \) fulfilling the triangle inequality for existing triangles in the distance graph \( G \).

Goal: Construction of a continuous one-to-one embedding \( x(u, v) \), given by a B-spline surface of degree \( p_1, p_2 \), i.e.

\[
x(u, v) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} c_{i,j} M_i^{p_1}(u) N_j^{p_2}(v)
\]

with \( c_{i,j} \in \mathbb{R}^m \), which approximates a given distance graph best in the sense of least squares. For simplicity we choose \( d(r) = r \).

Construction of the embedding \( x(u, v) \)

We compute the unknown coefficients \( c = (c_{0,0}, \ldots) \) by solving the minimization problem

\[
c = \arg\min \sum_{(y, z) \in Y} ||x(q_i, q_j) - x(q_i, q_j)||^2 - l_{y,z}^2.
\]

We solve this non-linear optimization problem by using the Gauss-Newton algorithm, which minimizes in each iteration step the following objective function

\[
\left( \sum_{(y, z) \in Y} (R_e(c) + \nabla R_e(c)(\Delta c - c^*)^T) + \omega ||\Delta c - c^*||^2 \right)
\]

with respect to \( \Delta c \), where \( c^* \) denotes the solution from the last step. \( \Delta c \) at the update, \( \nabla R_e(c) \) is the row vector given by the partial derivatives of \( R_e(c) \), with respect to the control points and \( \omega > 0 \) is the parameter for the Tikhonov regularization term.

Anisotropic Voronoi diagram computation

By using an anisotropic metric \( D \), given by (1), we can construct an anisotropic Voronoi diagram \( V_D(P) \) in \( \mathbb{R}^2 \).

Computation of the anisotropic Voronoi diagram \( V_D(P) \)

- Given the points \( P = \{u_1, u_2, \ldots\} \) with \( u_i \in \mathbb{R}^m \), we first compute the corresponding points \( x(u_i) \).
- Then we construct for the set of points \( P_X = \{x(u_1), x(u_2), \ldots\} \) an Euclidian Voronoi diagram \( V(P_X) \) in \( \mathbb{R}^m \).

Lemma

Let \( m = 2 \). The anisotropic Voronoi diagram \( V_D(P) \) is orphan-free.

Lemma

Let \( m = 3 \) and let \( x(u, v) \) be also \( C^k \)-smooth. If the set of points \( P_X = \{x(u_1), x(u_2), \ldots\} \) is a 0.1%-sample of \( X = \mathbb{R}^3 \), then the resulting anisotropic Voronoi diagram \( V_D(P) \) is orphan-free.

Examples

![Examples of distance graphs.](example.png)

![Resulting embeddings into \( \mathbb{R}^2 \).](result.png)

References