Barycentric Coordinates on Surfaces

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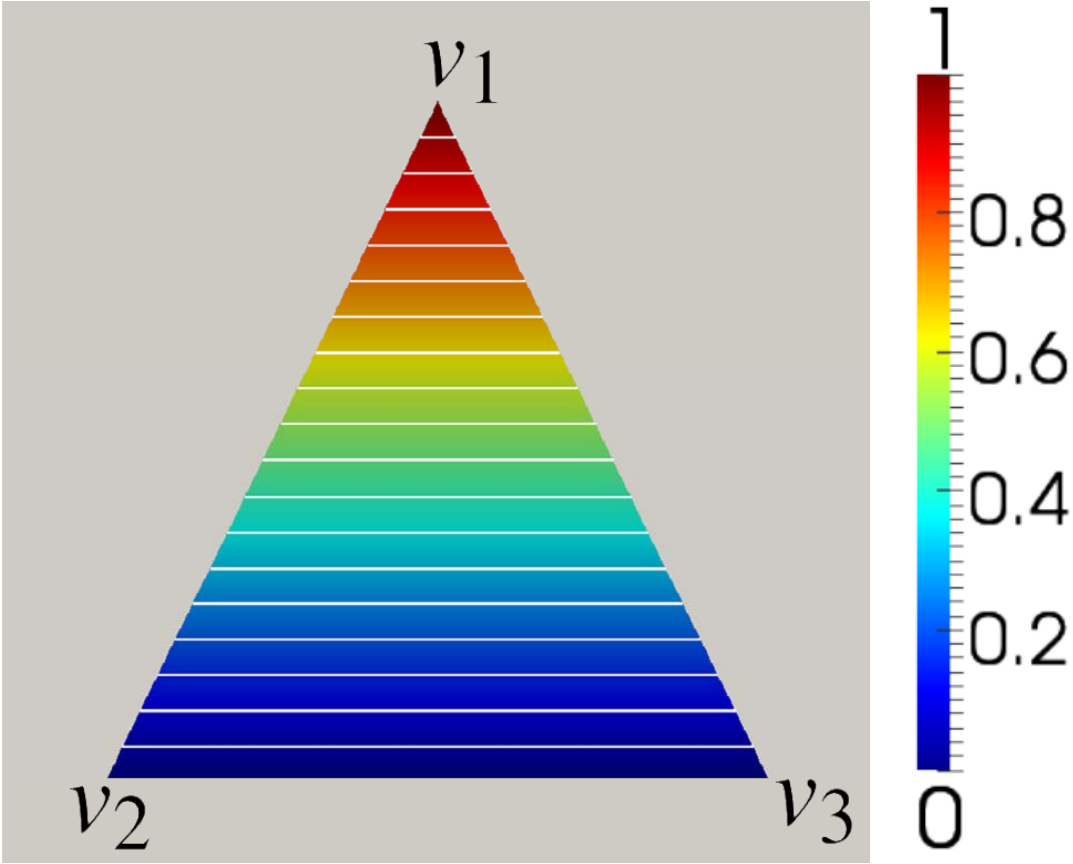
Motivation

Barycentric coordinates good for:

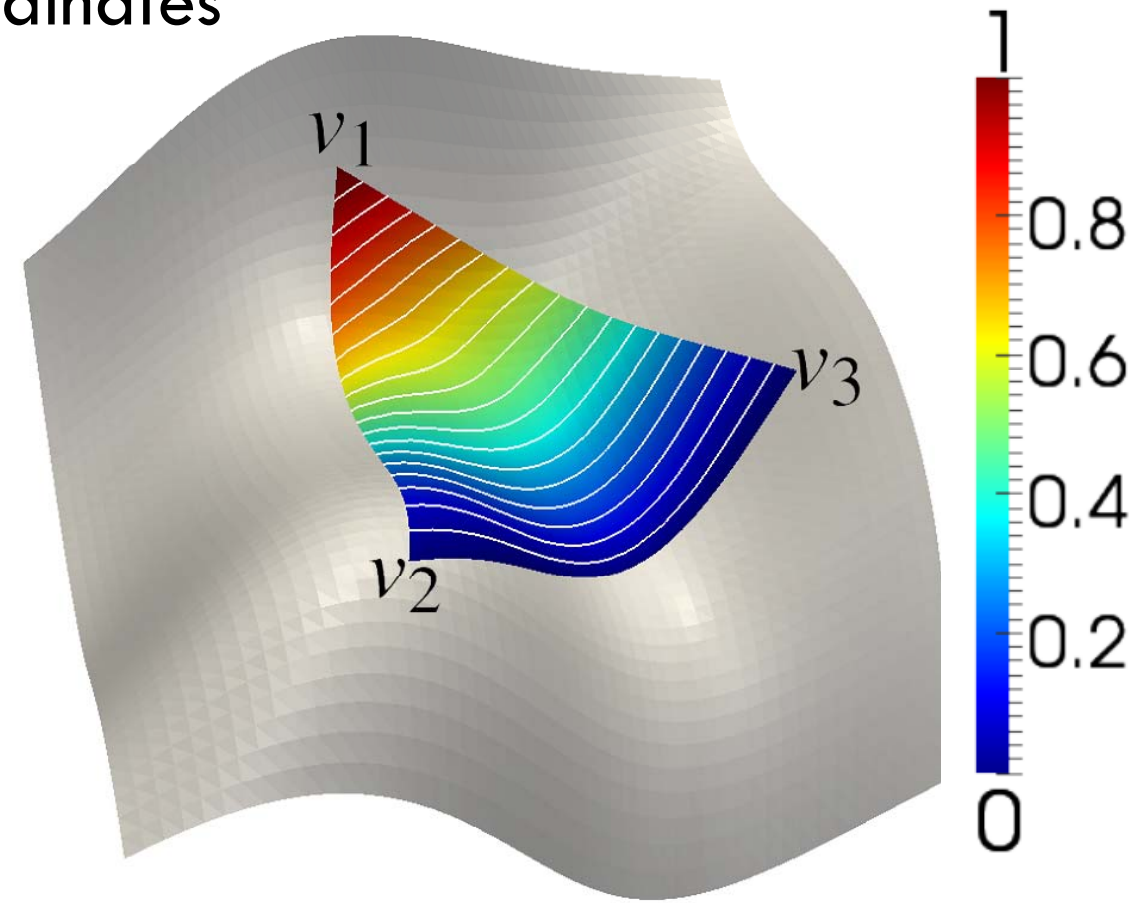
- interpolation
- shading
- deformation
- Bezier surfaces
- parameterization
- interior distance
- image cloning
- shape retrieval
- finite elements



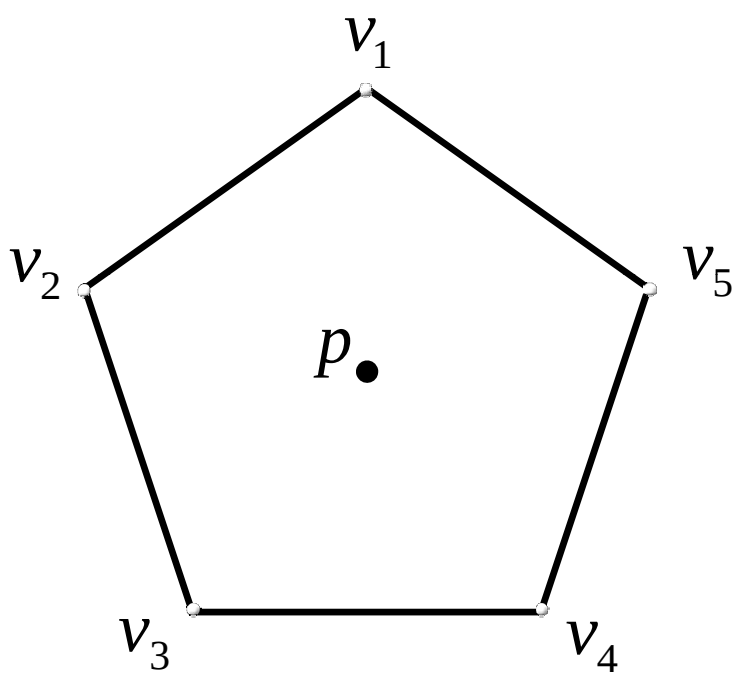
Goal

Define barycentric coordinates on surfaces:

- generalize planar
- intrinsic
- fast to compute



Definition on the plane



$$p = \text{affine comb of vertices}$$

$$p = \sum b_i v_i, \quad where \sum b_i = 1$$

Challenges on surfaces

$$p = \sum b_i v_i, \quad where \sum b_i = 1$$

p belongs to convex hull of vertices ✗

Challenges on surfaces

$$p = \sum b_i v_i, \quad \cancel{where \sum b_i = 1}$$

Mobius, Alfeld et al, Cabral et al	Spherical triangles
Ju et al	Spherical convex
Langer et al	Spherical all

Challenges on surfaces

$$p = \sum b_i v_i, \quad \cancel{where \sum b_i = 1}$$

involves Cartesian coords

not intrinsic

Our approach

$$p = \sum b_i v_i, \quad where \sum b_i = 1$$

$$p = \text{c.mass } \{v_i\} \text{ with masses } \{b_i\}$$

Our approach

$$\cancel{p = \sum b_i v_i,} \quad where \sum b_i = 1$$

Riemannian

$$p = \text{c.mass } \{v_i\} \text{ with masses } \{b_i\}$$

Outline

- Introduction
- Riemannian Center of Mass
- Construction
- Computation
- Properties
- Results
- Applications

Usual center of mass

$$p = \text{c.mass } \{v_i\} \text{ with masses } \{b_i\}$$

$$\Updownarrow$$

$$\boxed{p = \arg\min_x \sum b_i d^2(x, v_i)}$$

Euclidean distance

Riemannian center of mass

$$p = \text{c.mass}\ \{v_i\} \text{ with masses } \{b_i\}$$

$$\Updownarrow$$

$$\boxed{p = \arg\min_x \sum b_i d^2(x, v_i)}$$

Geodesic distance

Karcher's theorem

If the points are not "too far" from each other, the Riemannian center of mass is unique

$$U(x) = \sum b_i d^2(x, v_i)$$

- $U(x)$ has a unique minimum,
- which is the unique zero of $\nabla U(x)$

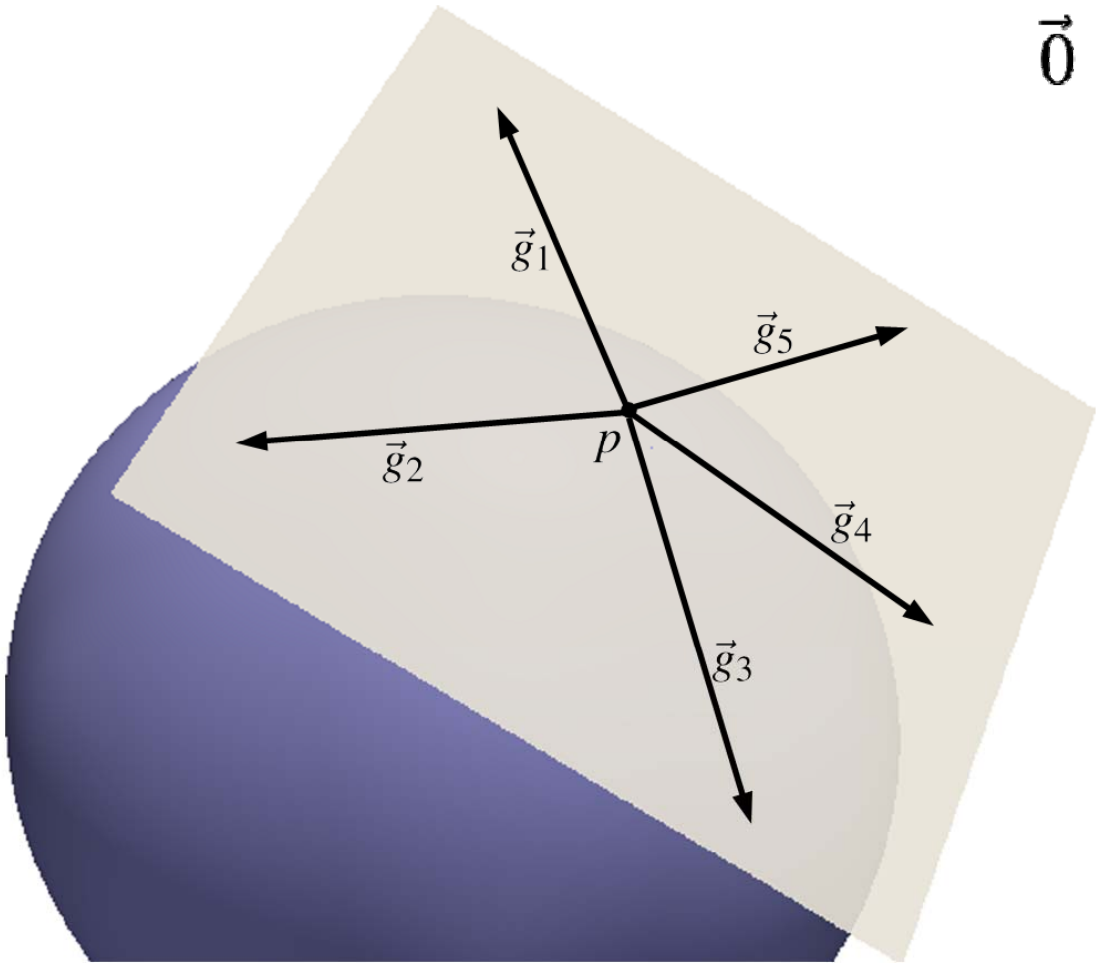
Construction

p has barycentric coordinates $\{b_i\}$

$p =$c.mass $\{v_i\}$ with masses $\{b_i\}$

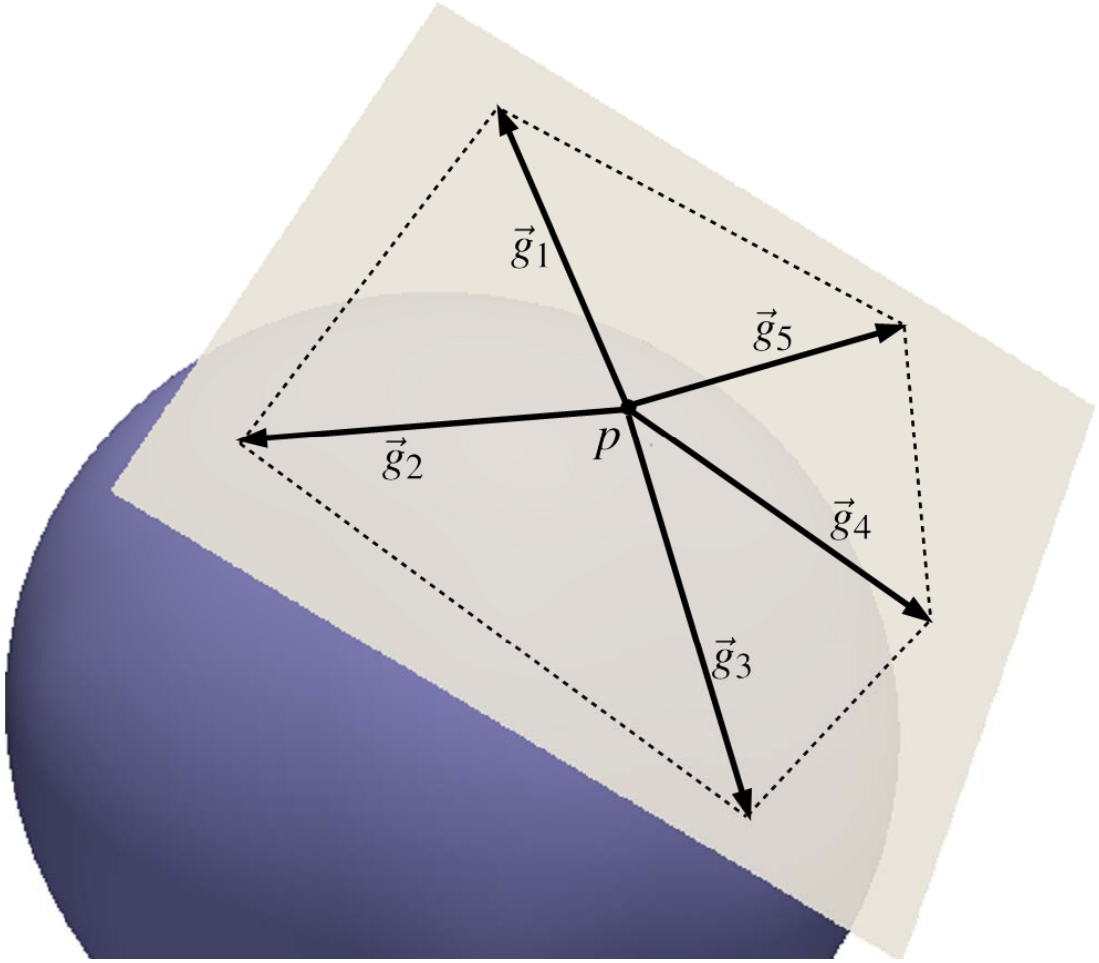
$\nabla U(x) = \vec{0}$ must hold at $x = p$

Construction



$$
\begin{aligned}
\vec{0} &= \nabla U(x)\Big|_{x=p} \\
&= \nabla\left(\sum b_i d^2(x, v_i)\right)\Big|_{x=p} \\
&= \sum b_i \nabla d^2(x, v_i)\Big|_{x=p} \\
&= \sum b_i \vec{g}_i
\end{aligned}
$$

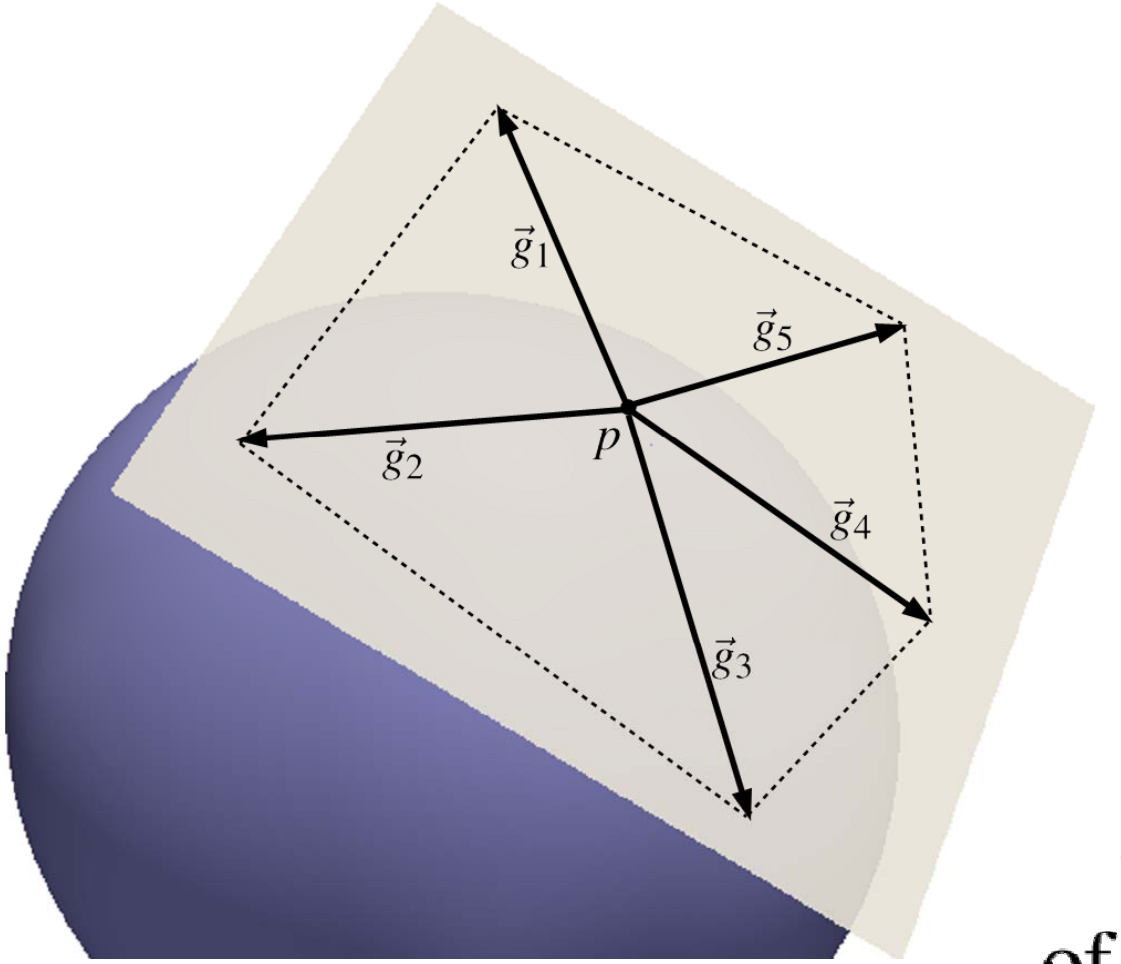
Construction



$$\sum b_i \vec{g}_i = \vec{0} \simeq p$$

$$\sum b_i = 1$$

Construction



$$\sum b_i \vec{g}_i = \vec{0} \simeq p$$

$$\sum b_i = 1$$

$$\Updownarrow$$

$$\{b_i\} =$$

planar baryc coords

of p wrt dashed polygon

Computation Summary

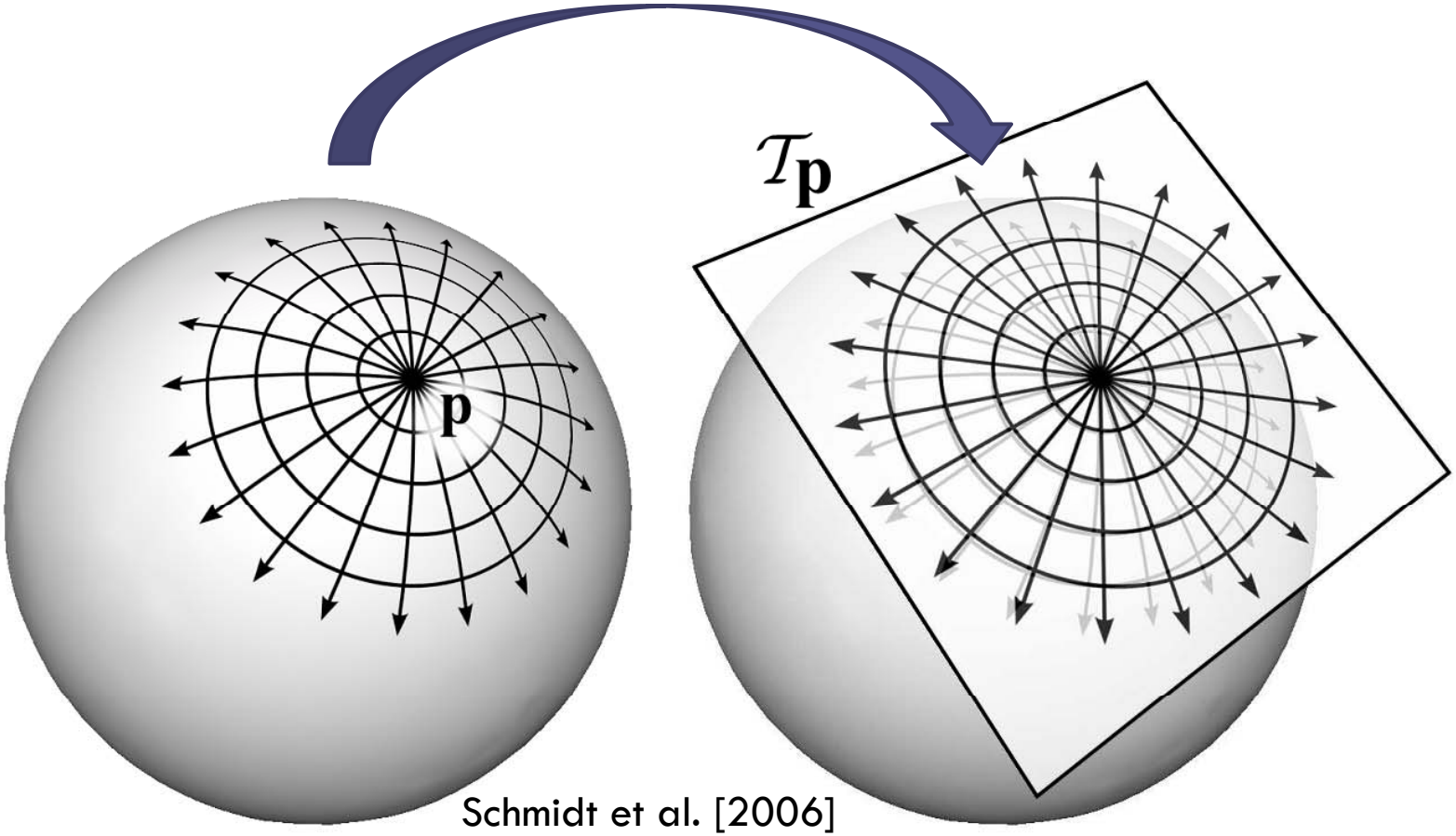
- Pick a kind of planar baryc coords
- E.g. Mean Value Coords

- Compute the gradient polygon at each point
- Explicit formula exists

- Surface baryc coords =
  = planar baryc coords wrt gradient polygon

Computation of gradients

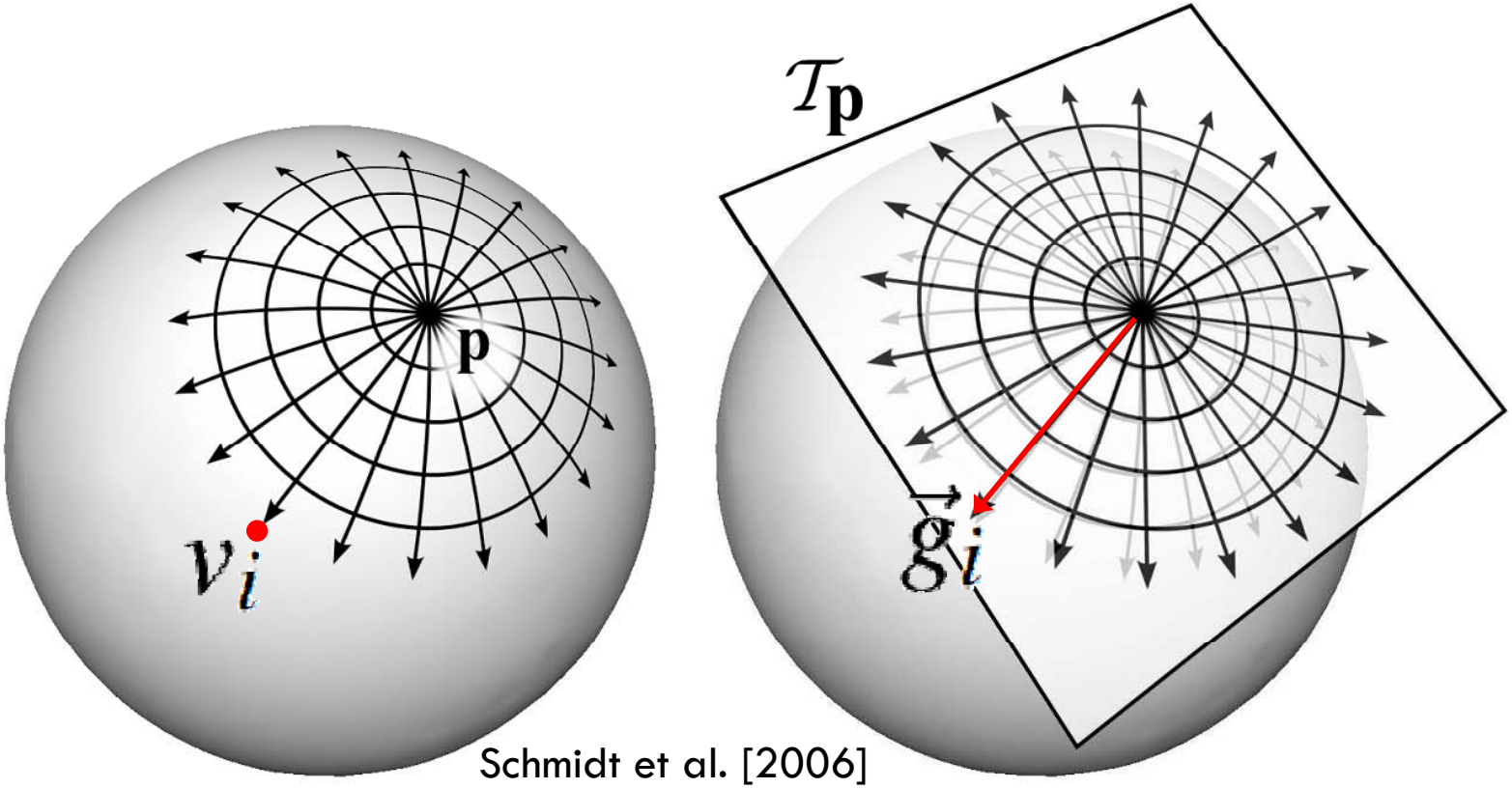
inverse exponential map



Schmidt et al. [2006]

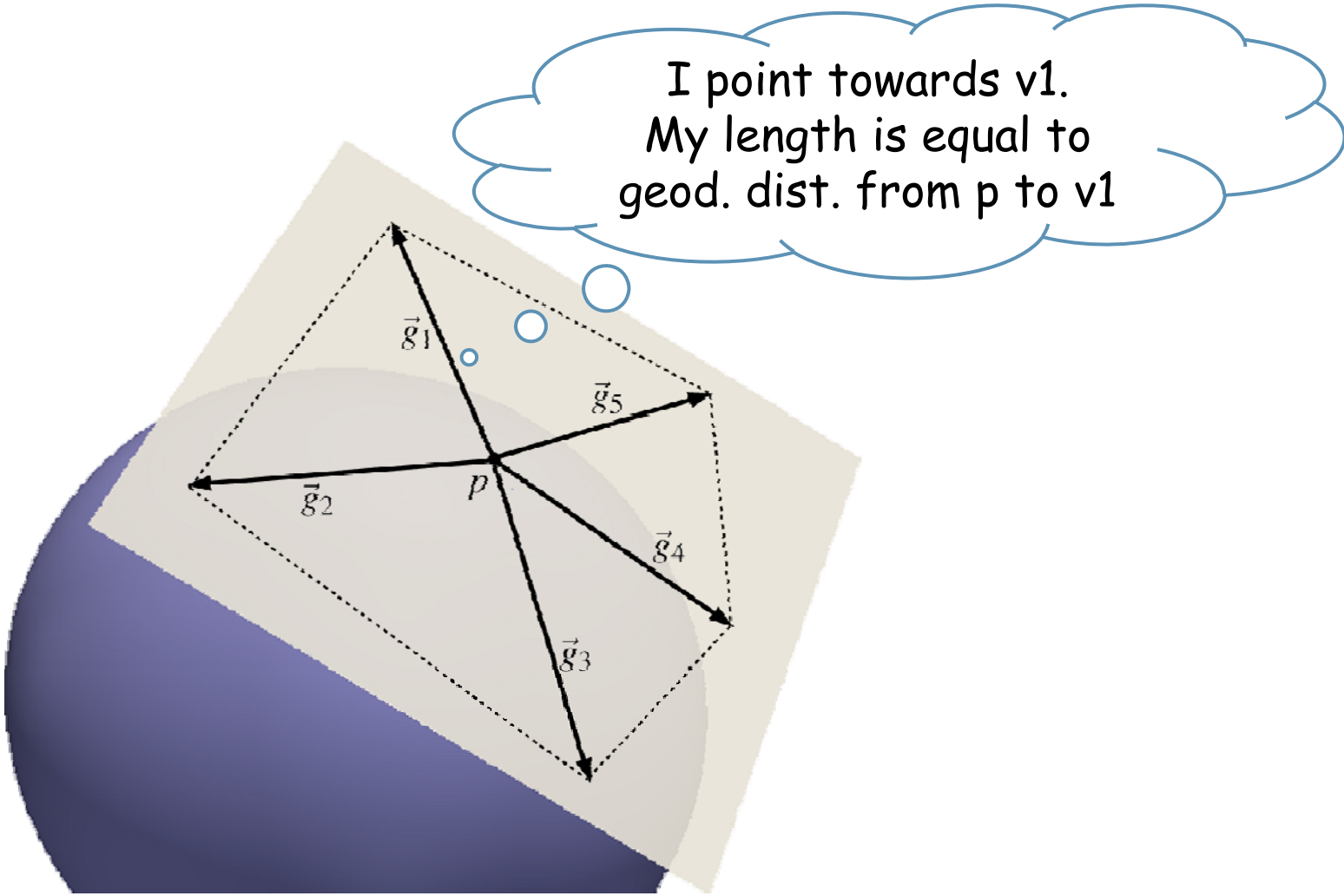
Computation of gradients

$$\vec{g}_i = \nabla d^2(x, v_i)|_{x=p} \simeq \exp_p^{-1}(v_i)$$



Schmidt et al. [2006]

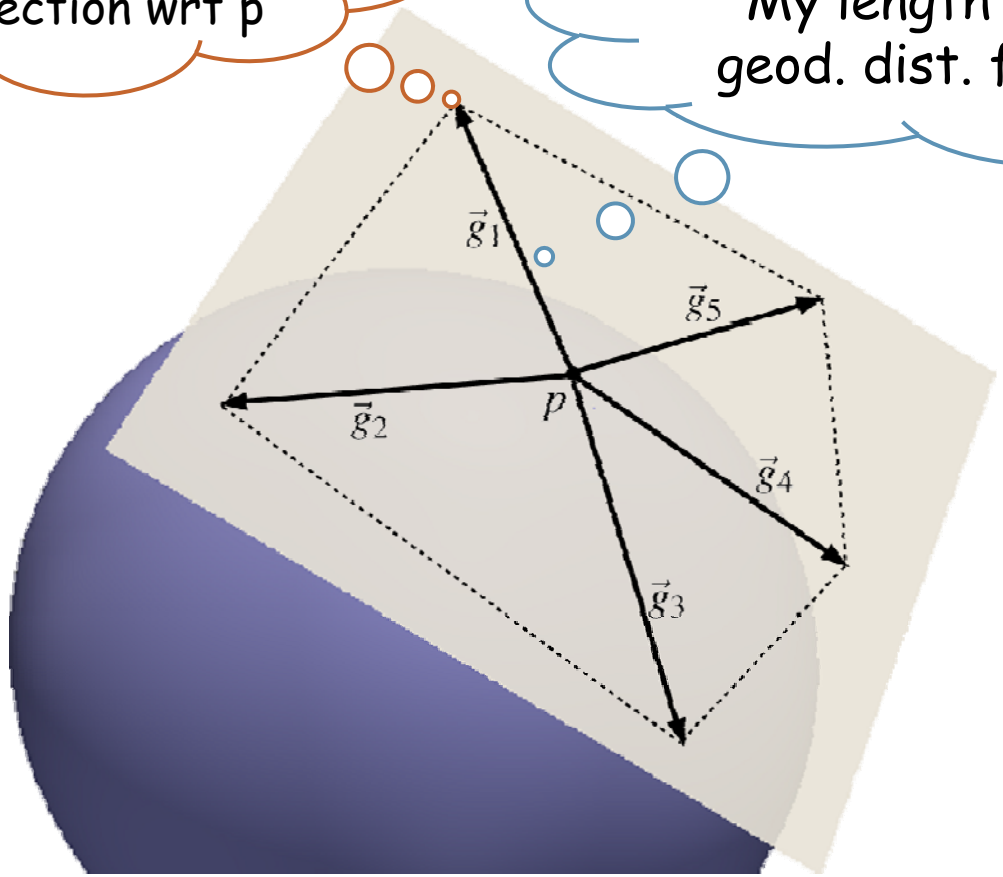
Intuitiveness



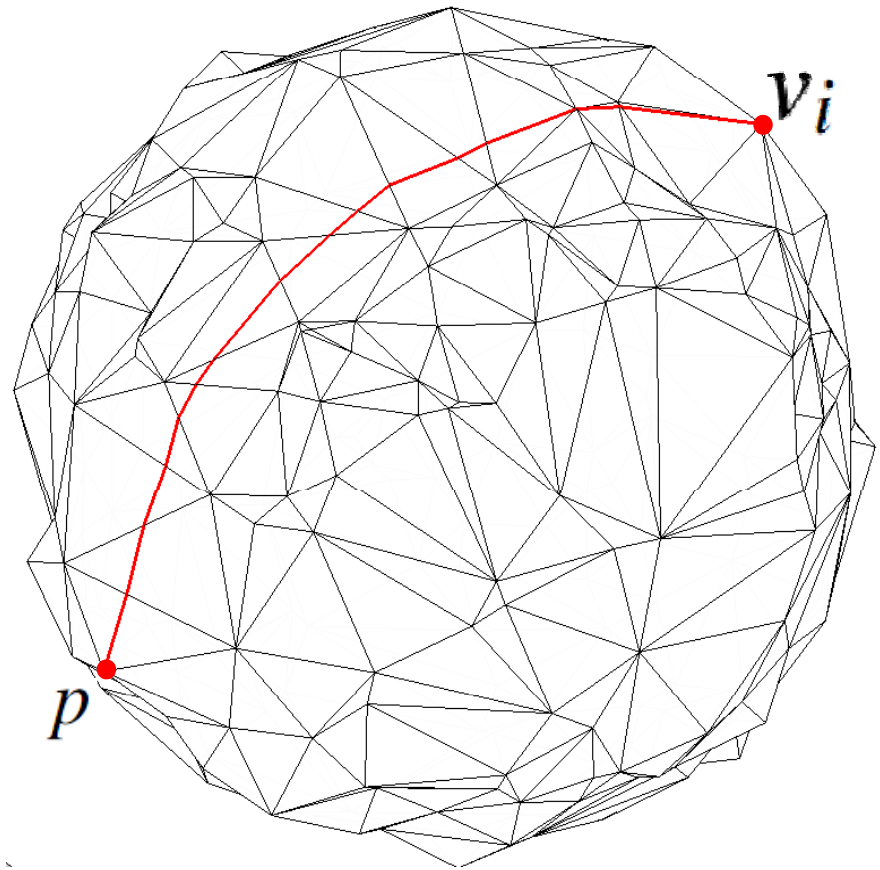
Intuitiveness

I am at the "correct" distance and direction wrt p

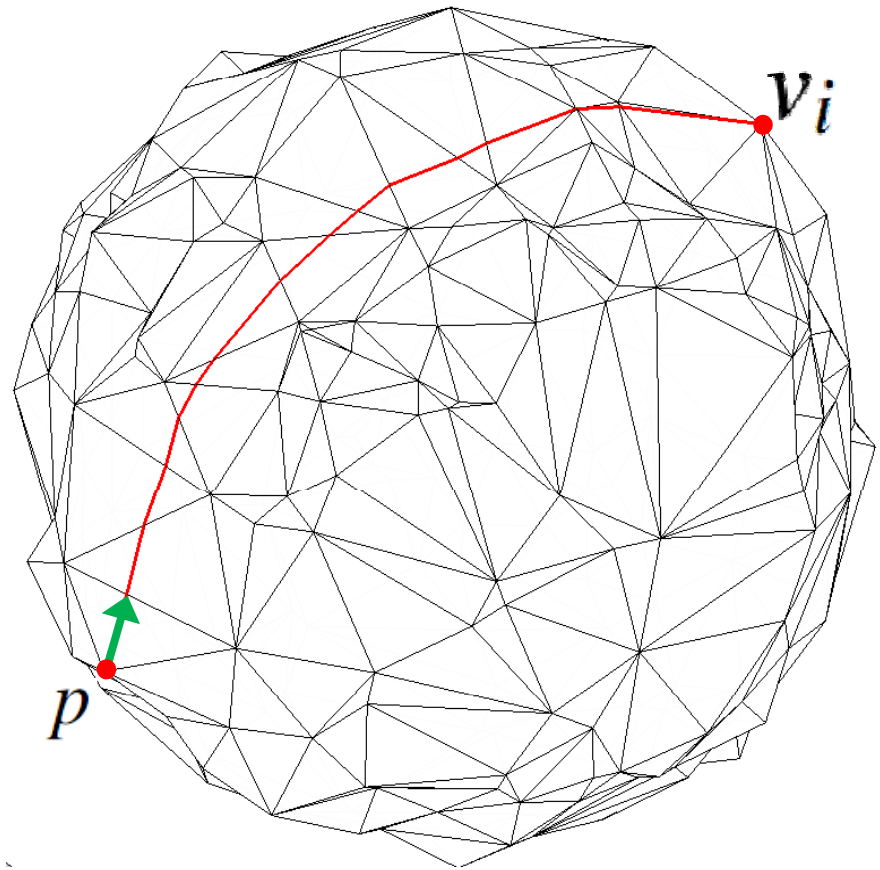
I point towards v1. My length is equal to geod. dist. from p to v1



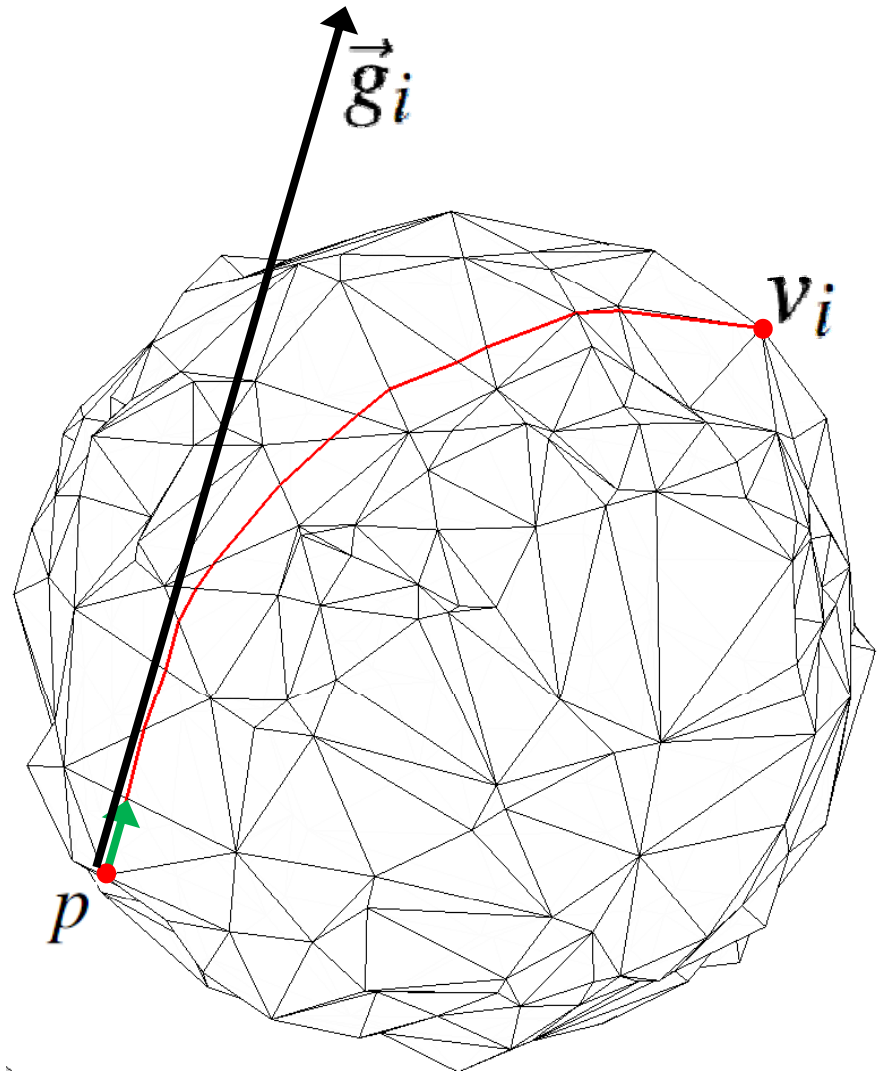
Discrete setting



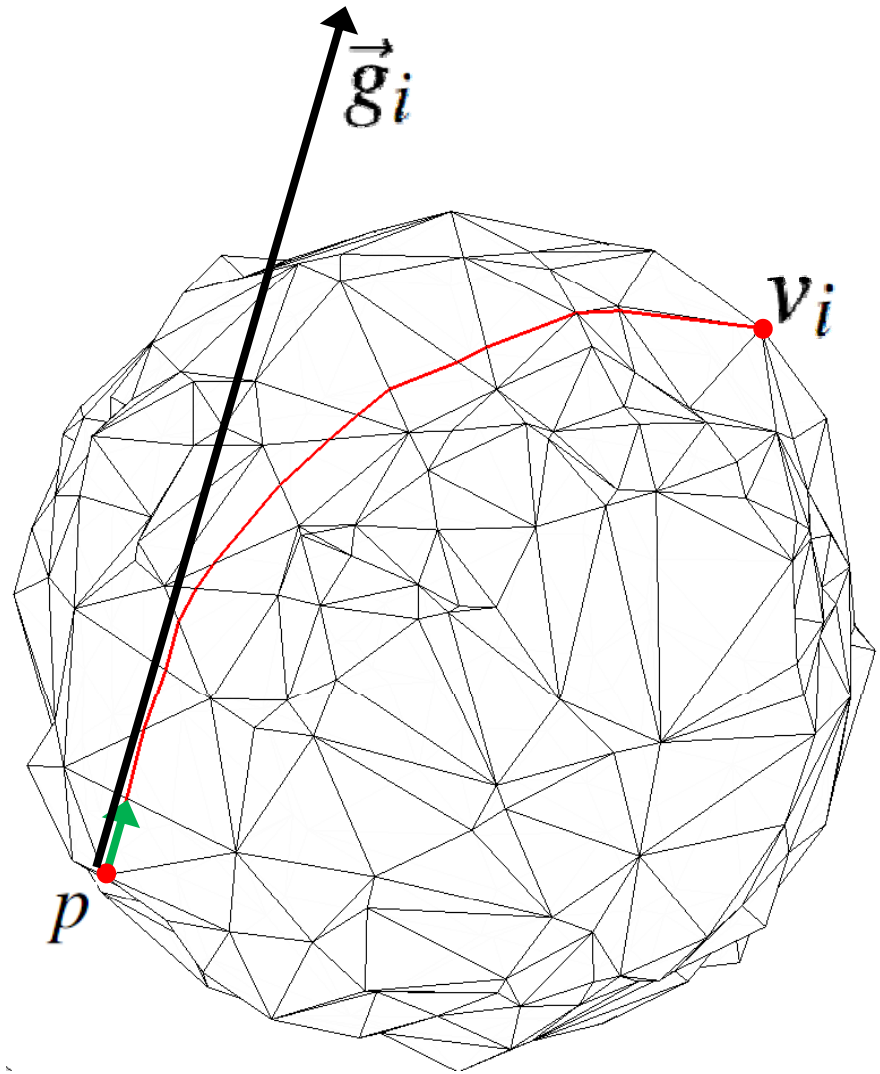
Discrete setting

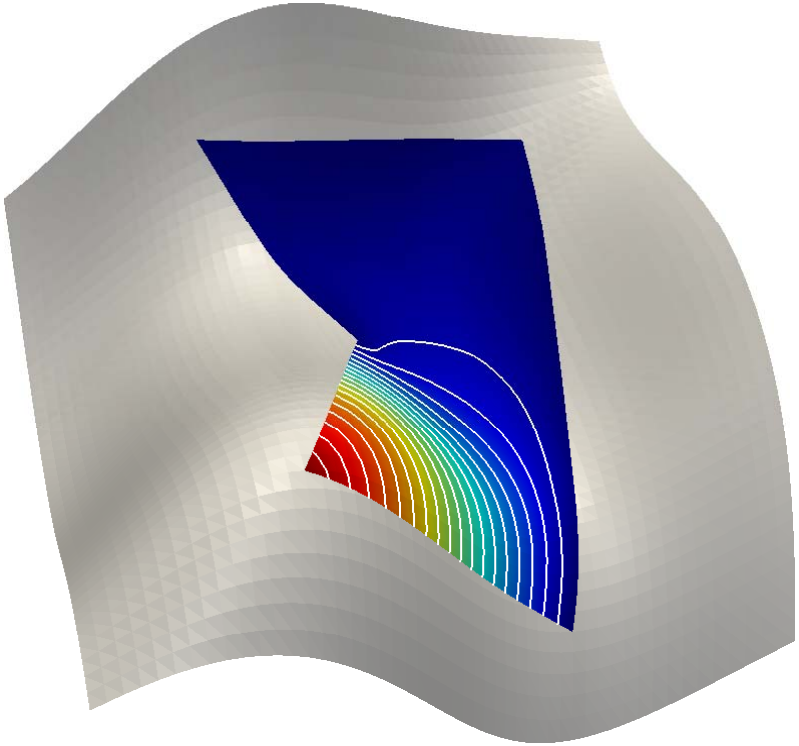


Discrete setting



Discrete setting





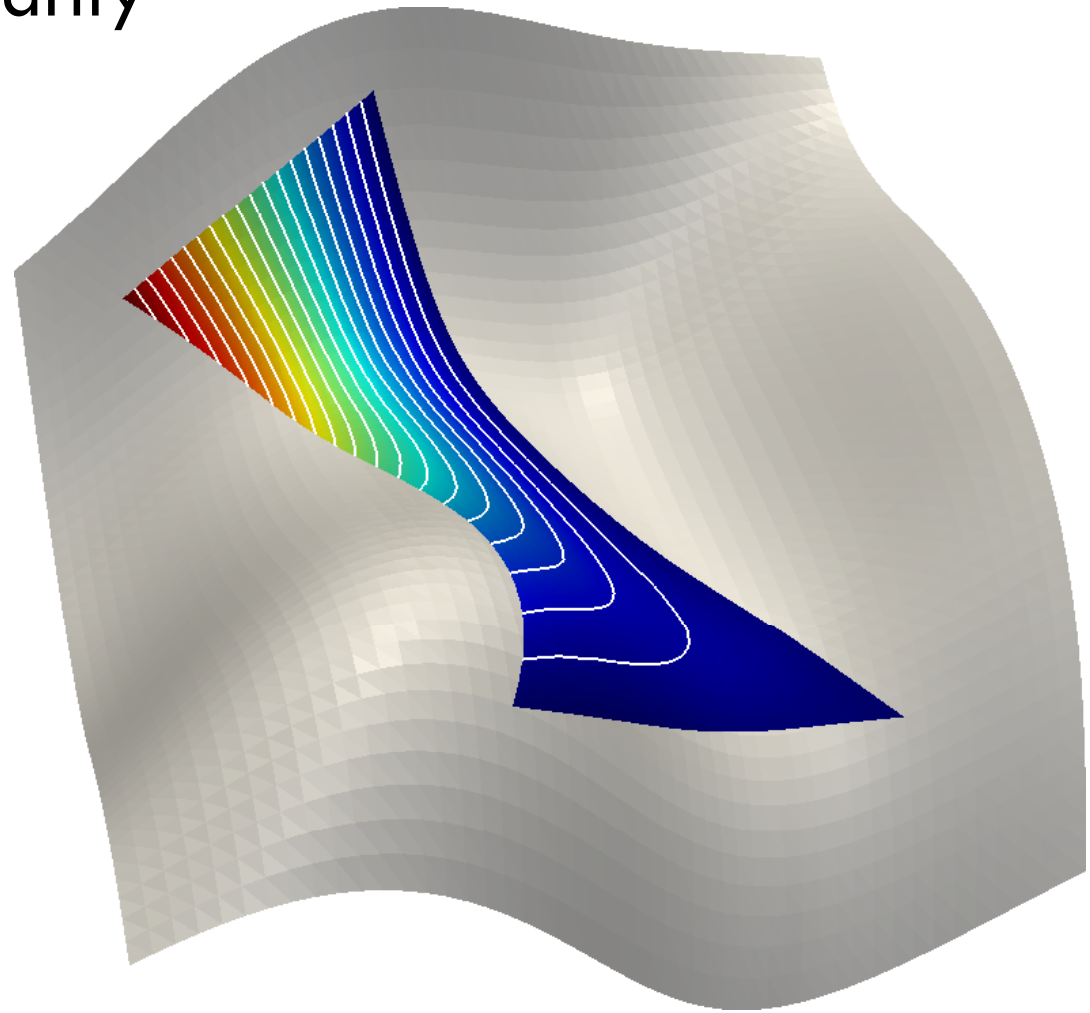
Pentagon
Need **five** instances of
"single source, all destinations"

Properties

- Defining properties
  - Lagrangian
  - Partition of unity
  - Riemannian center of mass
- Unique reconstruction from coordinates
  - Due to Karcher's theorem
- Planar reproduction
  - If surface is plane, get planar coordinates back
- Similarity invariance
- Smoothness

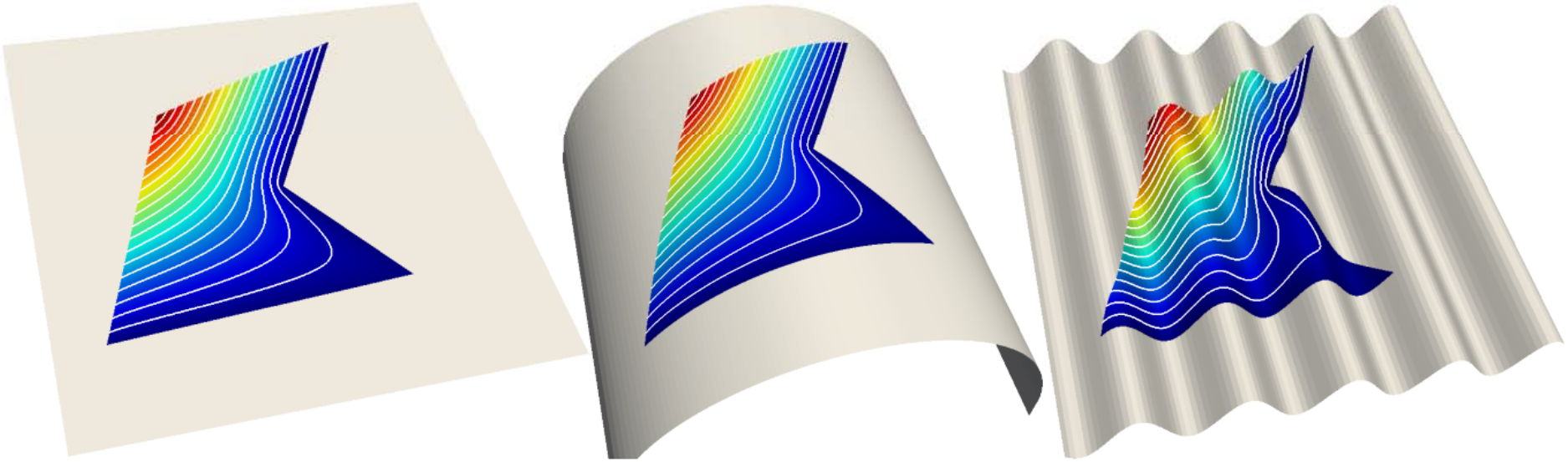
Properties

- Edge linearity



Properties

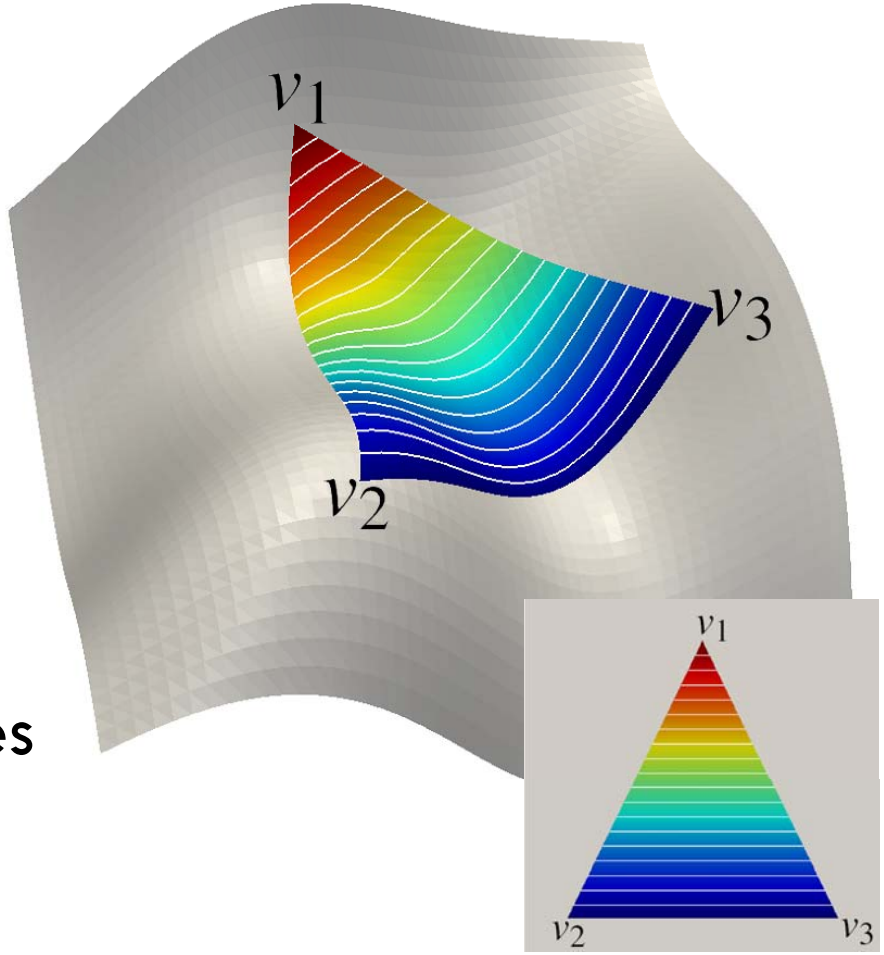
- Isometry invariance



- isometry invariance + unique reconstruction =
  = isometry map can be reconstructed

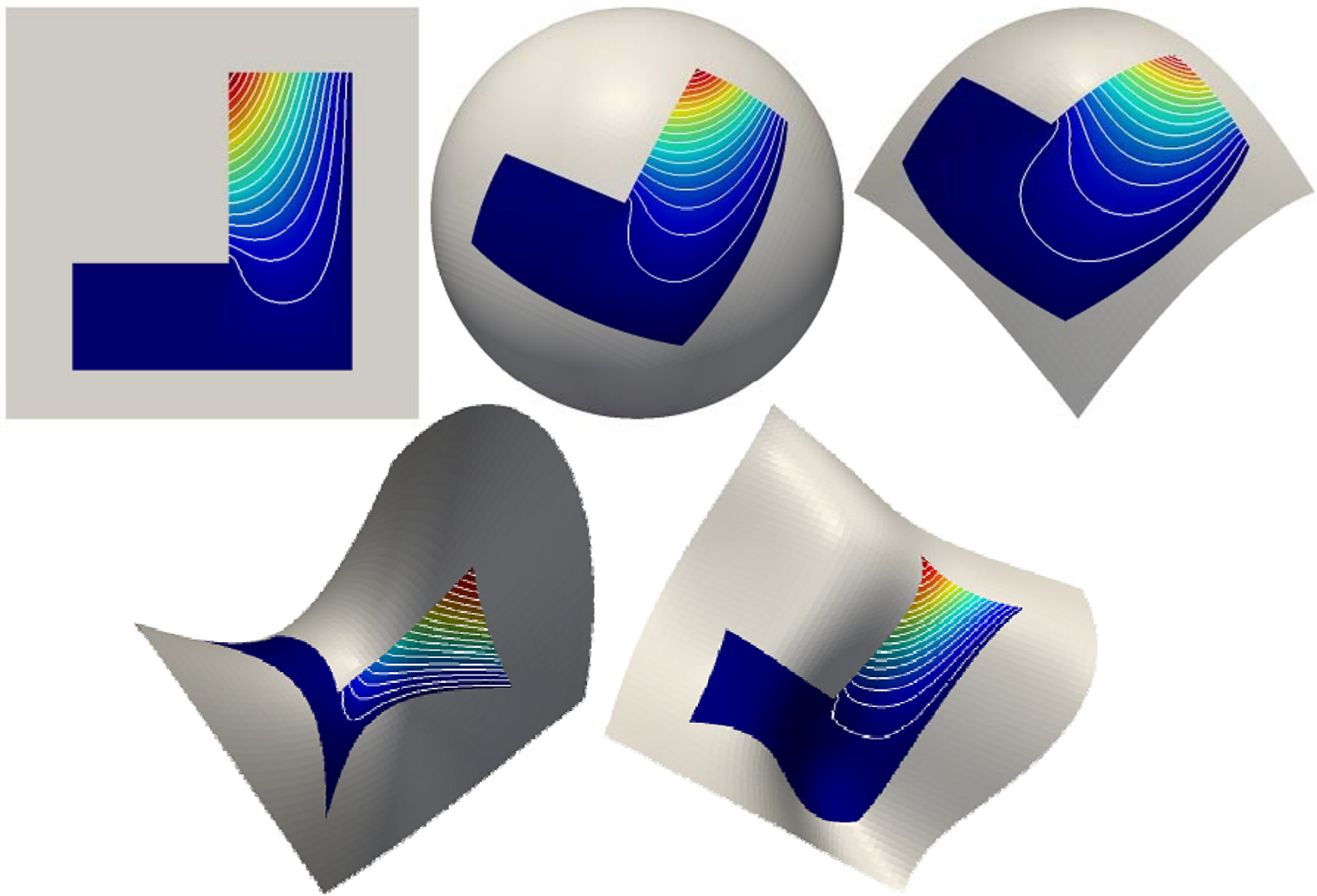
Results

- Darkest red point – vertex wrt which coords are computed
- Dark blue – small value
- Dark red – large value
- Equally spaced isolines

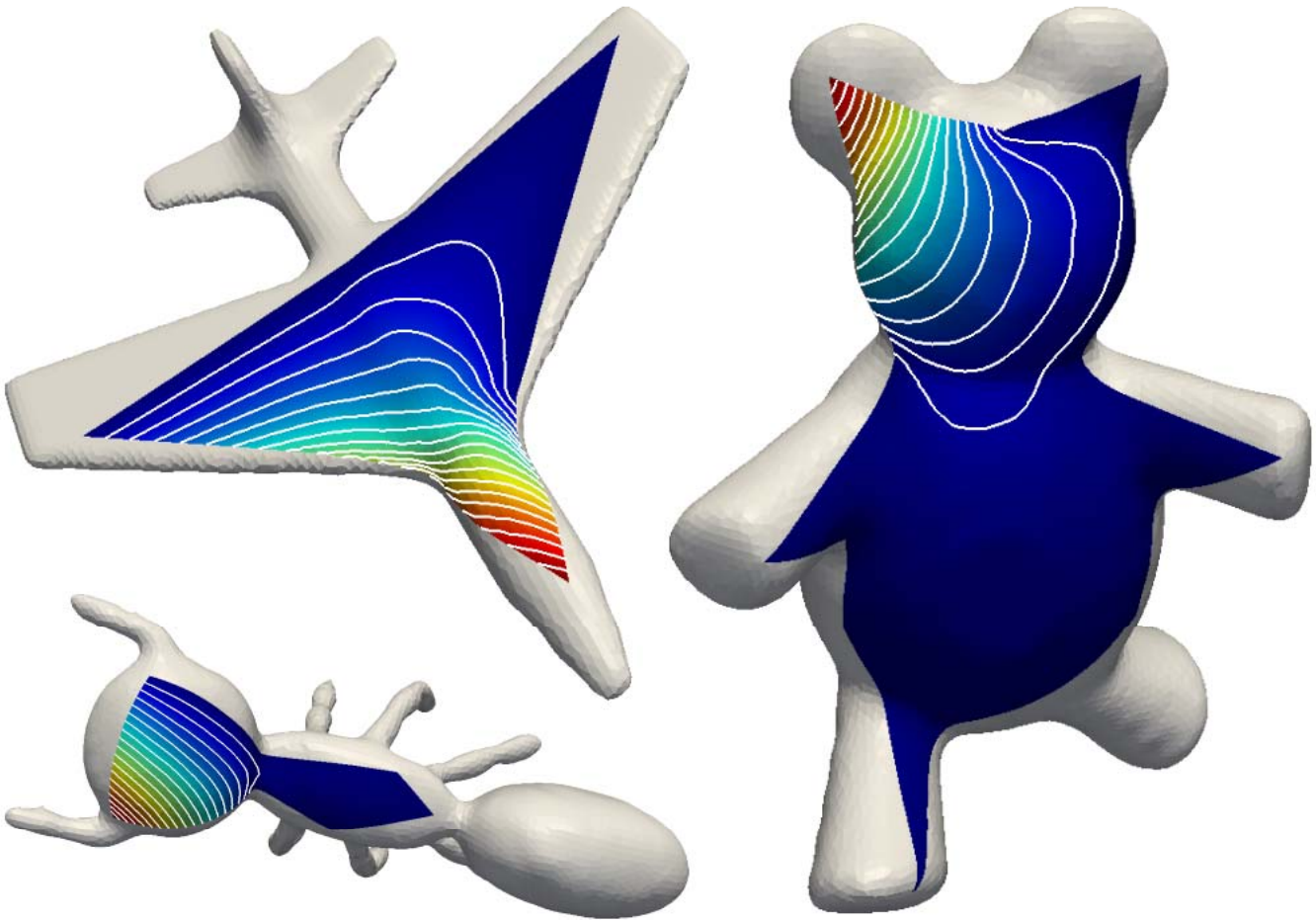


- Planar baryc coords =
  =Mean Value Coordinates

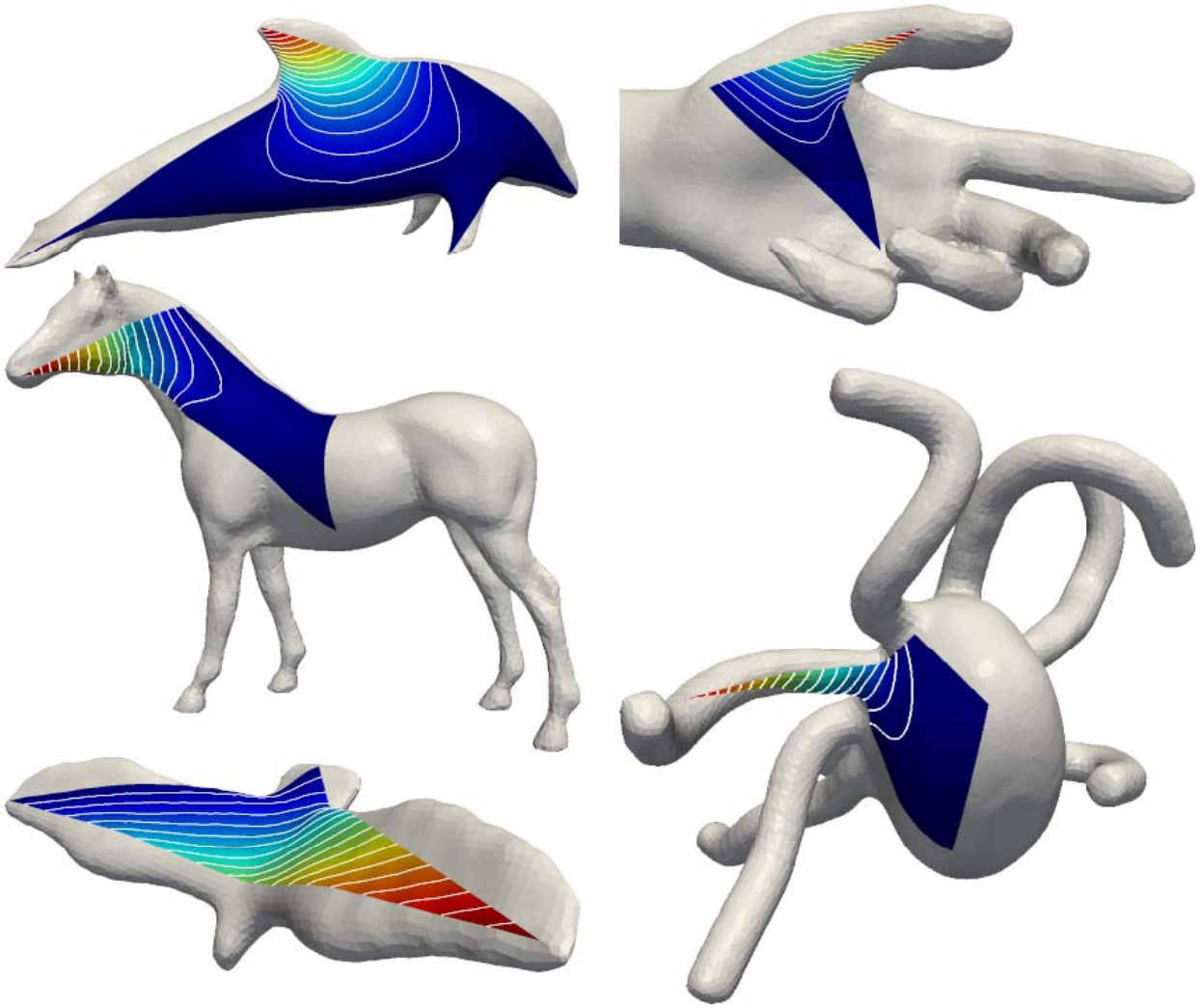
Effect of surface shape



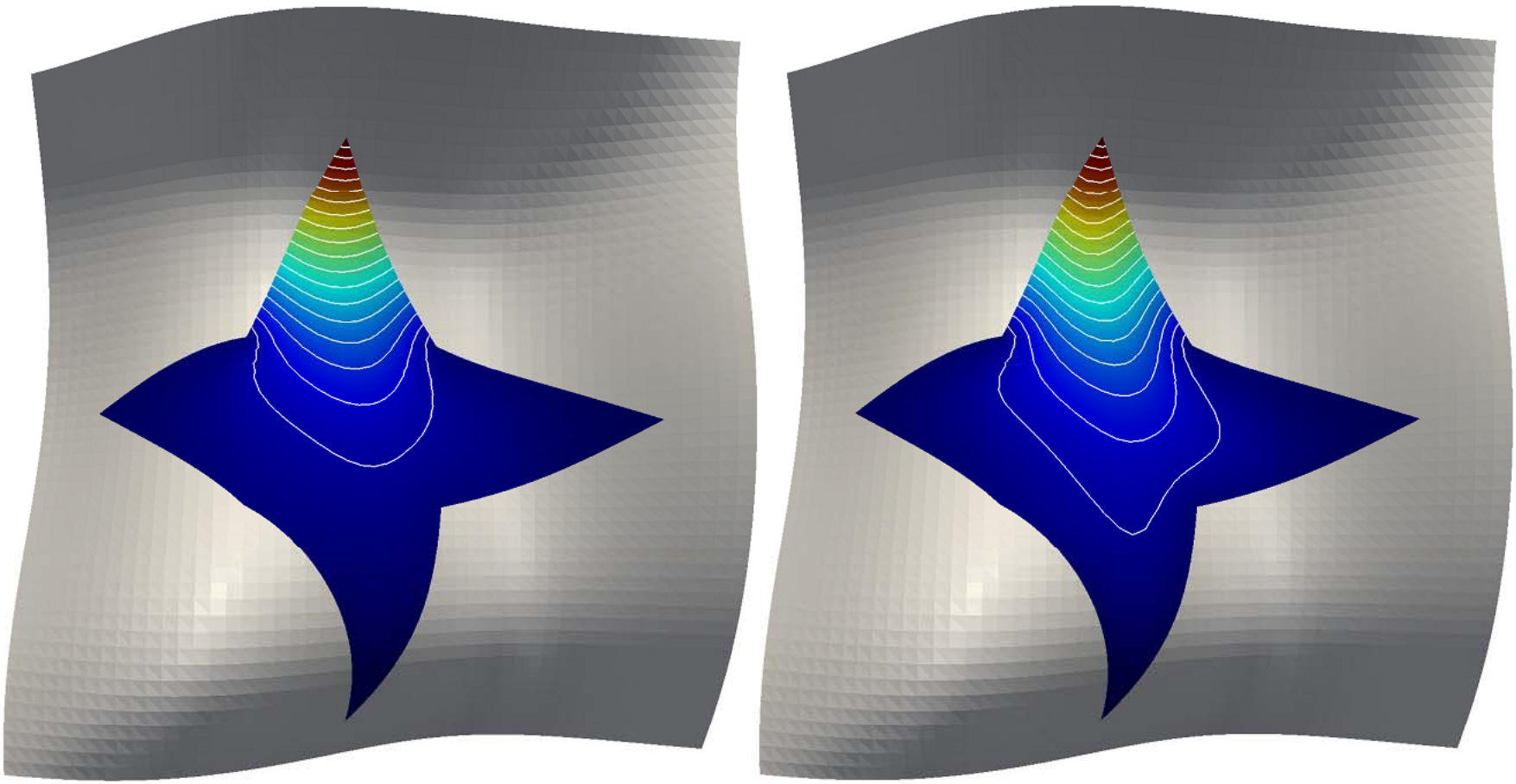
Variety



Variety



Effect of planar coordinate



Mean Value Coordinates

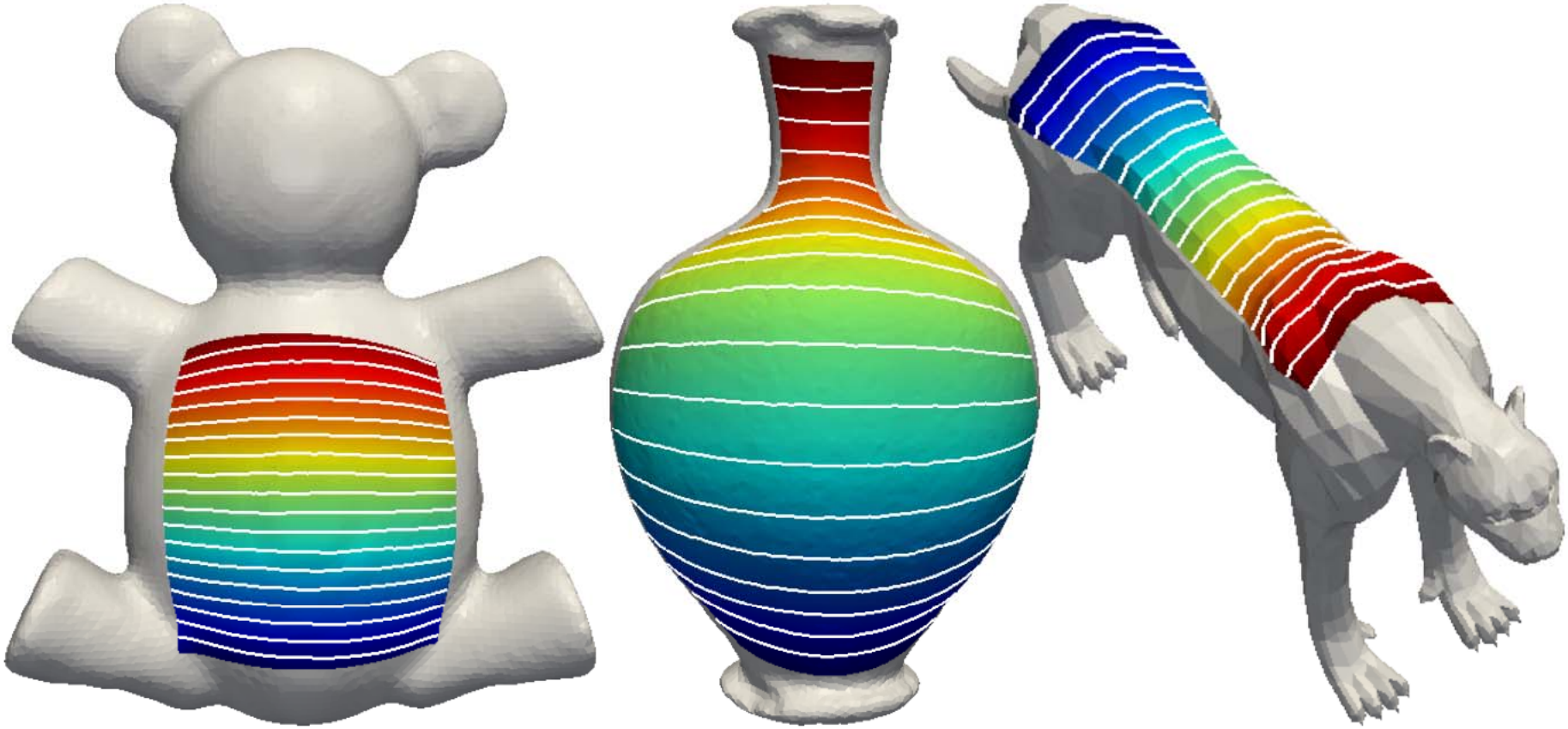
Maximum Entropy Coordinates

Timing (seconds)

|V|	|F|	Preprocess	Per-point
2K	4K	0.81	0.0018
4K	7K	1.25	0.0018
8K	16K	5.46	0.0019
17K	35K	7.12	0.0021
27K	56K	12.71	0.0024
53K	105K	28.04	0.0027

Application: Interpolation

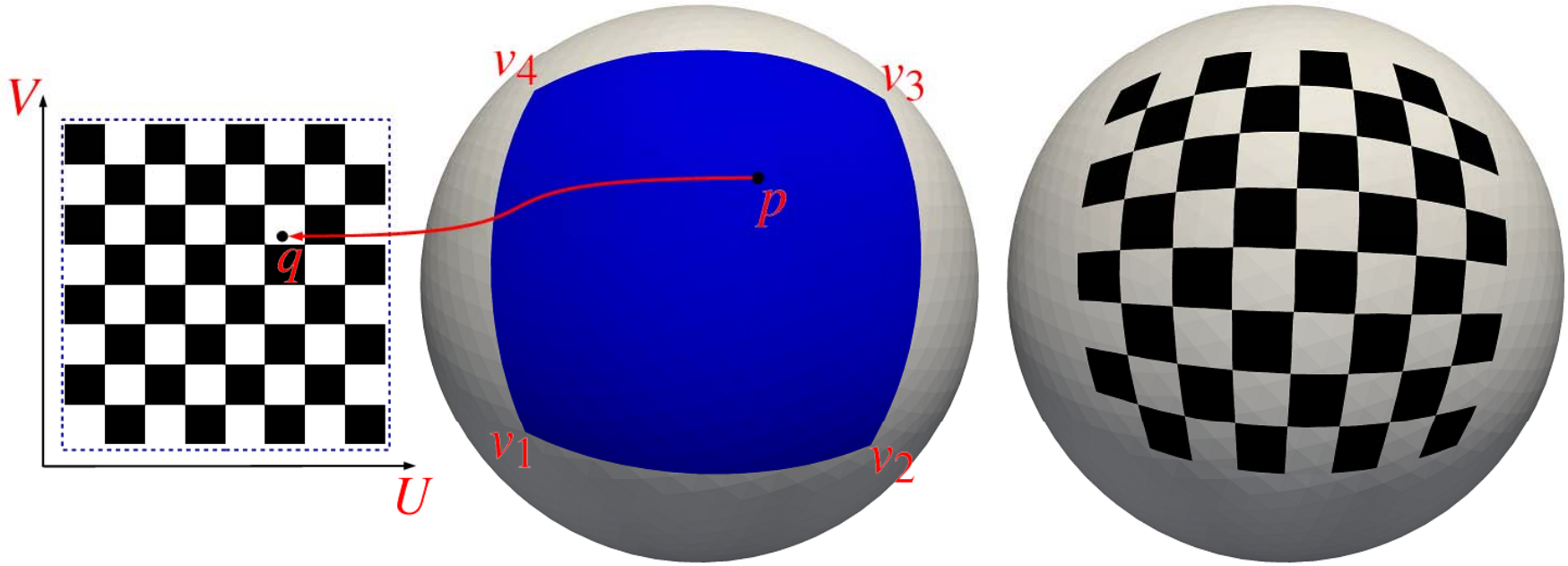
values $\{f_i\}$ on vertices $\{v_i\}$, define $f(p) = \sum f_i b_i(p)$



Application: Decal mapping

Local parameterizations – used for texturing

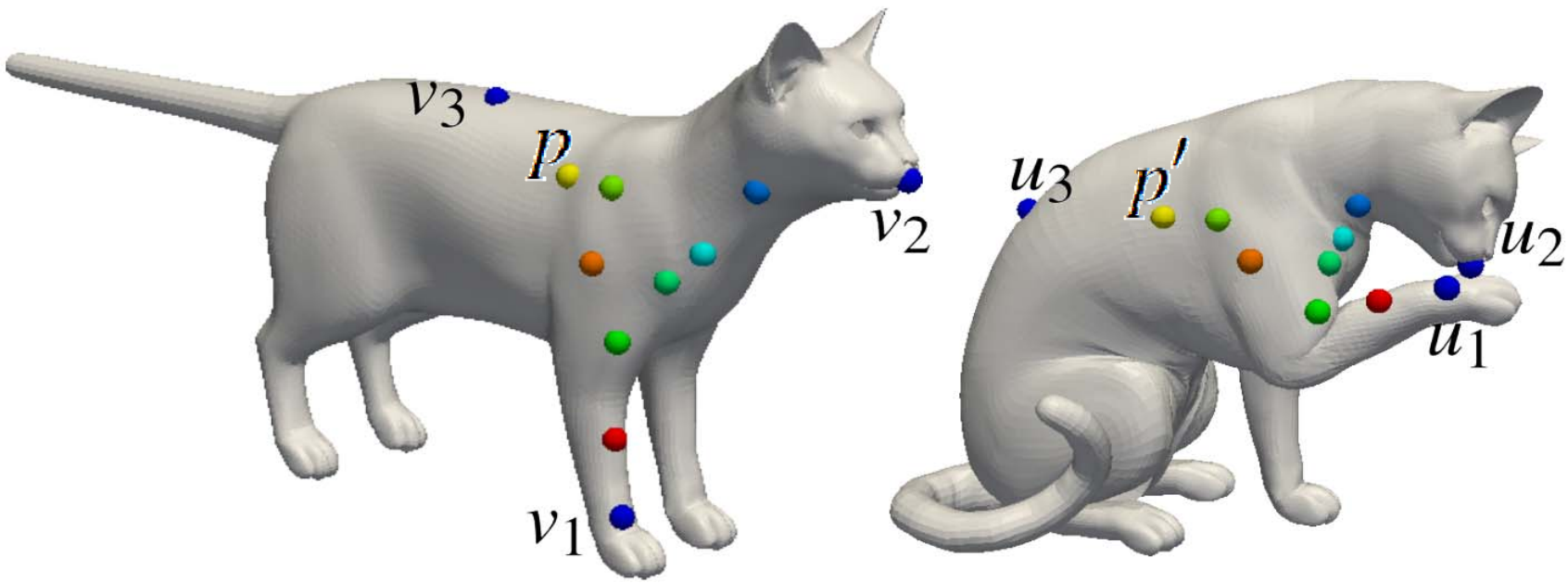
We use the same idea as in image warping



$$q = \begin{bmatrix} U \\ V \end{bmatrix} = b_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Application: Decal mapping

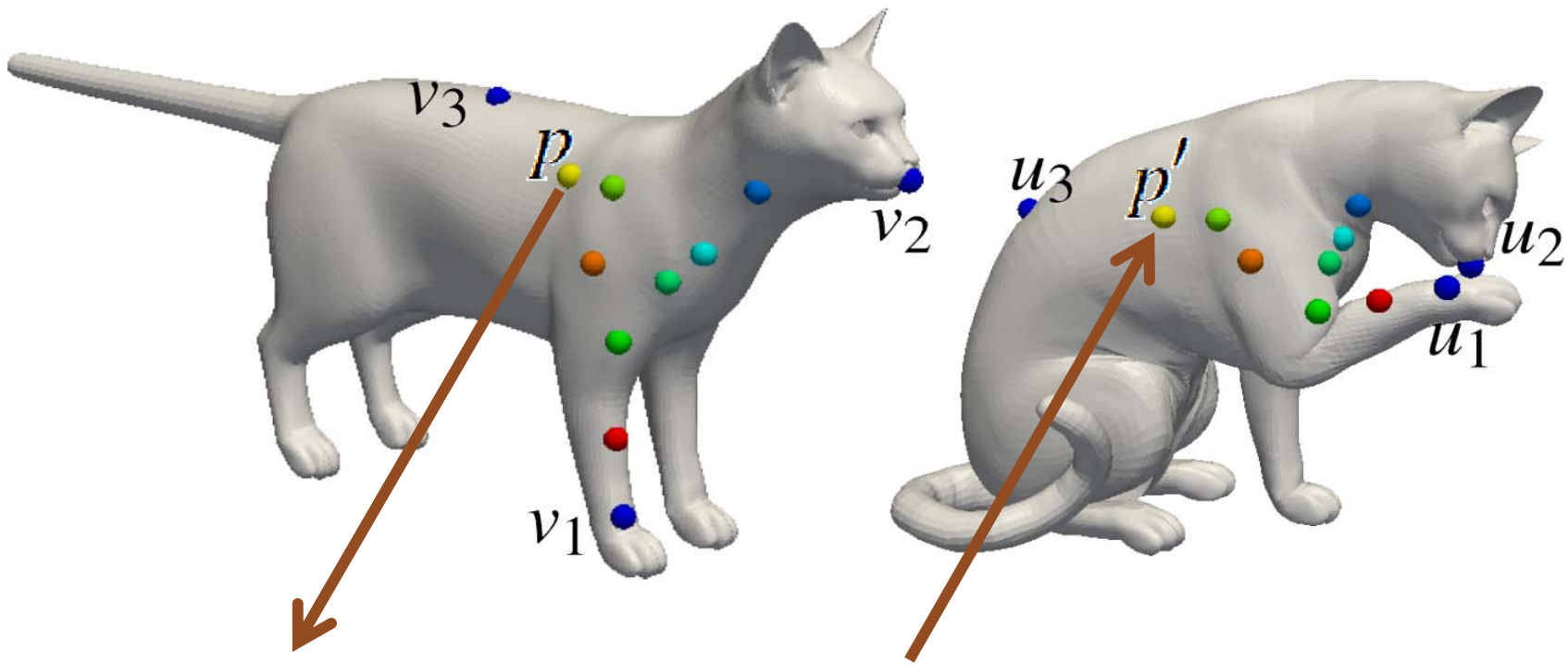
App: Correspondence Refinement



Based on isometry reconstruction property

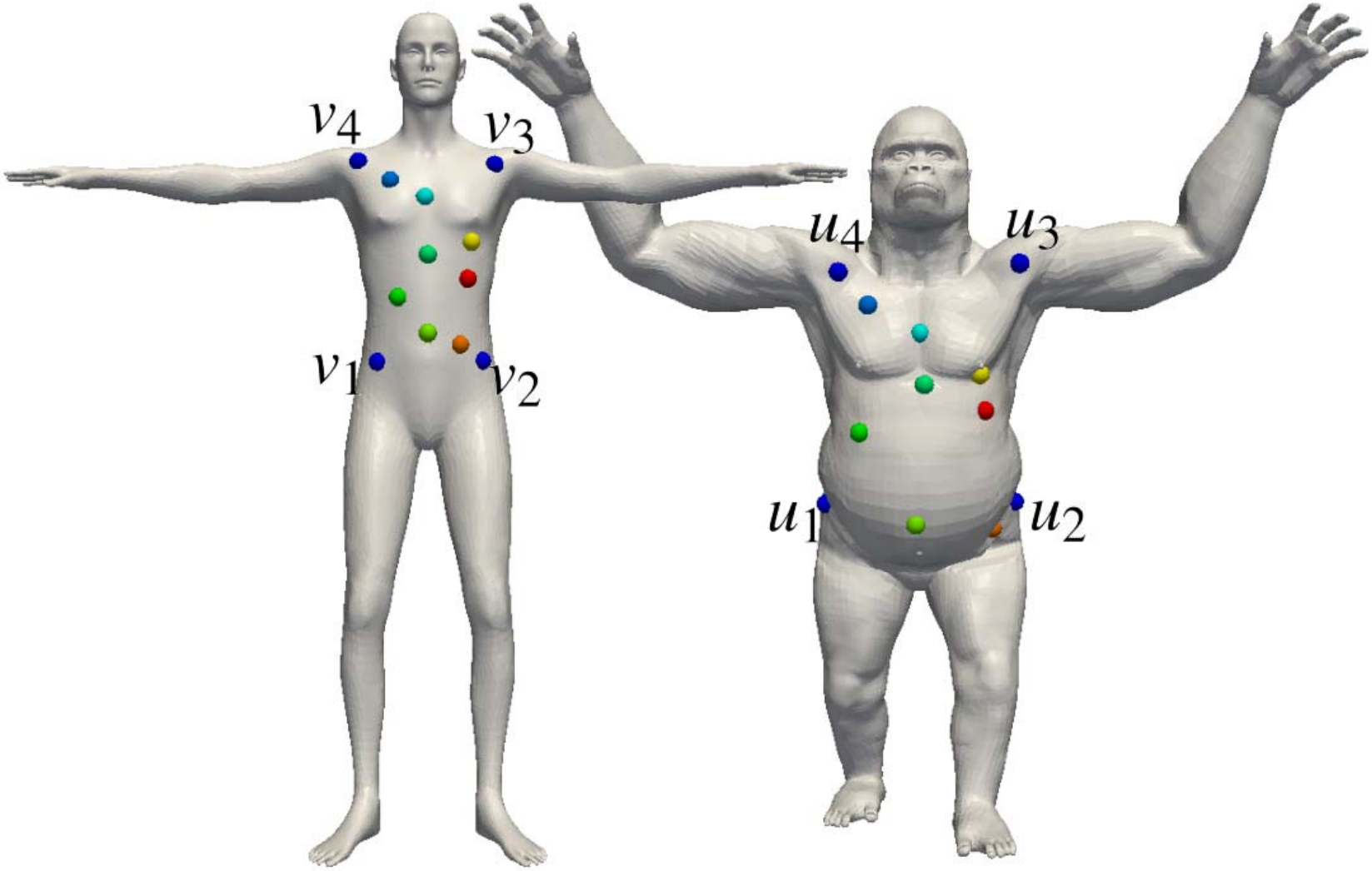
Correspondence is exact, if true isometry

App: Correspondence Refinement



$$\{b_i\} \text{ wrt } \{v_i\} \longrightarrow p' = \arg\min_x \sum b_i d^2(x, u_i)$$

App: Correspondence Refinement



Summary

Definition and construction of barycentric coordinates on surfaces:

- properly generalize existing planar coordinates
- insensitive to isometric deformations
- easy to implement, and fast to compute

Future work

- Better Karcher's theorem
  - Other distances instead of geodesic in $U(x) = \sum m_i d^2(x, v_i)$
  - Uniqueness of c.m. for larger polygons?
- More empirical studies of various choices of
  - Distance
  - Planar barycentric coordinates
- Further applications of surface coordinates

Thank you

Software

- Szymon Rusinkiewicz for Trimesh2
- Danil Kirsanov for Exact Geodesic

3D models

- Daniela Giorgi
- AIM@SHAPE

Raison d'être

- Remy of Ratatouille whose posing for [Joshi et al 2007] got me interested in barycentric coordinates

Thank you

Thank You!