Transfinite Mean Value interpolation in $\mathbb{R}^n$

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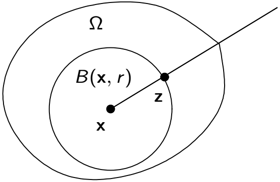
In this talk:

1. Transfinite Lagrange Mean Value interpolation in $\mathbb{R}^n$.
2. Hermite interpolation.
3. Surface deformation.

Mean Value property

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and convex domain and $\partial\Omega$ its boundary. For a fixed $\mathbf{x} \in \Omega$, let $B = B(\mathbf{x}, r)$ be a ball with radius r, centered at $\mathbf{x}$.

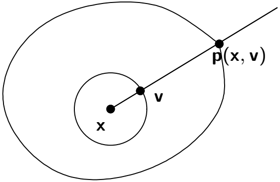


A function F satisfies the mean value property if for each $\mathbf{x}$,

$$F(\mathbf{x}) = \frac{1}{\mathcal{A}(\partial B)} \int_{\partial B} F(\mathbf{z}) d\mathbf{z}. \qquad (1)$$

Mean Value interpolation

Given $f : \partial\Omega \to \mathbb{R}$, we want to find an interpolant $g : \Omega \to \mathbb{R}$.



The mean value interpolant is

$$g(\mathbf{x}) = \int_S \frac{f(\mathbf{p}(\mathbf{x},\mathbf{v}))}{\rho(\mathbf{x},\mathbf{v})}\, d\mathbf{v} \Big/ \phi(\mathbf{x}), \qquad \phi(\mathbf{x}) = \int_S \frac{1}{\rho(\mathbf{x},\mathbf{v})}\, d\mathbf{v},$$

where S is the unit sphere in $\mathbb{R}^n$, $\mathbf{v} \in S$ and $\rho(\mathbf{x},\mathbf{v}) = \|\mathbf{p}(\mathbf{x},\mathbf{v}) - \mathbf{x}\|$.

Parametric representation

The function g can also be written as

$$g(\mathbf{x}) = \int_D w(\mathbf{x},\mathbf{t}) f(\mathbf{s}(\mathbf{t}))\, d\mathbf{t} \Big/ \phi(\mathbf{x}), \qquad \phi(\mathbf{x}) = \int_D w(\mathbf{x},\mathbf{t})\, d\mathbf{t},$$

where

$$w(\mathbf{x},\mathbf{t}) = \frac{\det(\mathbf{s}(\mathbf{t}) - \mathbf{x}, D_1\mathbf{s}(\mathbf{t}), \ldots, D_{n-1}\mathbf{s}(\mathbf{t}))}{\|\mathbf{s}(\mathbf{t}) - \mathbf{x}\|^{n+1}},$$

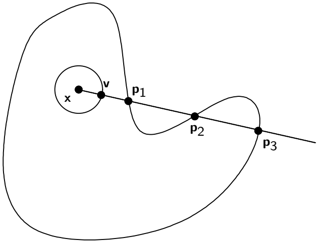
and $\mathbf{s}(\mathbf{t})$ is a parametrization $\mathbf{s} : D \to \partial\Omega$, with $\mathbf{t} \in D \subset \mathbb{R}^n$.

Non-convex domains

If Ω is non-convex, the expression for g is

$$g(\mathbf{x}) = \int_S \sum_{j=1}^{n(\mathbf{x},\mathbf{v})} \frac{(-1)^{j-1}}{\rho_j(\mathbf{x},\mathbf{v})} f(\mathbf{p}_j(\mathbf{x},\mathbf{v}))\, d\mathbf{v} \Big/ \phi(\mathbf{x}),$$

$$\phi(\mathbf{x}) = \int_S \sum_{j=1}^{n(\mathbf{x},\mathbf{v})} \frac{(-1)^{j-1}}{\rho_j(\mathbf{x},\mathbf{v})}\, d\mathbf{v}.$$



$\mathbf{n}(\mathbf{x},\mathbf{v})$ denotes the number of intersection points $\mathbf{p}_j$ for each pair $(\mathbf{x},\mathbf{v})$. If Ω is convex, $\mathbf{n}(\mathbf{x},\mathbf{v}) = 1$ for all $\mathbf{x}$, $\mathbf{v}$.

Volume and area of the unit sphere in $\mathbb{R}^k$

The volume of the unit sphere in $\mathbb{R}^k$ is $V_k = \frac{\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2}+1)}$,

where Γ is the gamma function.

For even k, $\Gamma(\frac{k}{2}+1) = \left(\frac{k}{2}\right)!$

and for odd k, $\Gamma(\frac{k}{2}+1) = \sqrt{\pi}\frac{1\cdot 3\cdot 5\cdots k}{2^{(k+1)/2}}$.

$$V_1 = 2, \quad V_2 = \pi, \quad V_3 = 4\pi/3, \quad V_4 = \pi^2/2, \quad V_5 = 8\pi^2/15, \quad \ldots.$$

We find the area by $A_k = kV_k$

$$A_1 = 2, \quad A_2 = 2\pi, \quad A_3 = 4\pi, \quad A_4 = 2\pi^2, \quad A_5 = 8\pi^2/3, \quad \ldots.$$

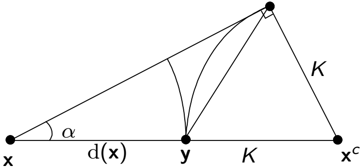
Bounds on ϕ

For $\mathbf{x} \in \Omega$, the upper bound on ϕ in $\mathbb{R}^n$ is

$$\phi(\mathbf{x}) \le \mathcal{A}_n / \mathrm{d}(\mathbf{x}).$$

If $K = \mathrm{d}(M_E, \partial\Omega) > 0$, then

$$\phi(\mathbf{x}) \ge \frac{\mathcal{A}_{n-1}}{\mathrm{d}(\mathbf{x})} G_n(\alpha(\mathbf{x})),$$



where

$$G_n(\alpha) = \int_0^\alpha \sin(\alpha - \beta) \sin^{n-2} \beta \, d\beta.$$

If Ω is convex, we have $\alpha = \pi/2$ and

$$\phi(\mathbf{x}) \ge \frac{\mathcal{A}_{n-1}}{\mathrm{d}(\mathbf{x})} \frac{1}{n-1} = \frac{V_{n-1}}{\mathrm{d}(\mathbf{x})}.$$

Proof of interpolation

g interpolates f if

- f is continuous on $\partial\Omega$
- $\mathbf{n}(\mathbf{x}, \mathbf{v})$ is finite for all $\mathbf{x}$ and $\mathbf{v}$
- $dist(M_E, \partial\Omega) > 0$.

This proof is analogous to the proof in 2D, once we have established the lower bound on ϕ in $\mathbb{R}^n$.

Hermite interpolation

We define the weight function

$$\psi(\mathbf{x}) = \frac{1}{\phi(\mathbf{x})}, \quad \mathbf{x} \in \Omega.$$

If $\mathrm{d}(M_E, \partial\Omega) > 0$ and $\mathrm{d}(M_I, \partial\Omega) > 0$ and $\mathbf{y} \in \partial\Omega$ then

$$\frac{\partial \psi}{\partial \mathbf{n}}(\mathbf{y}) = \frac{1}{V_{n-1}}.$$

and

$$\frac{\partial g}{\partial \mathbf{n}}(\mathbf{y}) = \frac{1}{V_{n-1}} \int_D w(\mathbf{y}, \mathbf{t}) \big(f(\mathbf{s}(\mathbf{t})) - f(\mathbf{y}) \big) \, d\mathbf{t}.$$

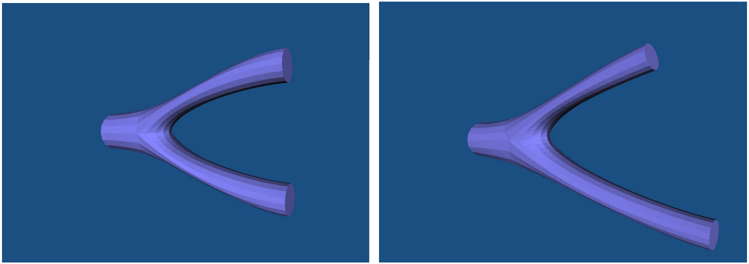
With the normal derivatives we can construct the Hermite interpolant in $\mathbb{R}^n$:

$$p(\mathbf{x}) = g(\mathbf{x}) + \psi(\mathbf{x})\hat{g}(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$

where $\hat{g}$ is a Lagrange interpolant satisfying

$$\hat{g}(\mathbf{y}) = \left(\frac{\partial f}{\partial \mathbf{n}}(\mathbf{y}) - \frac{\partial g}{\partial \mathbf{n}}(\mathbf{y}) \right) \Big/ \frac{\partial \psi}{\partial \mathbf{n}}(\mathbf{y}), \quad \mathbf{y} \in \partial\Omega.$$

Surface deformation



An application of mean value interpolation in 3D.

For triangle meshes the interpolant g can be written

$$g(\mathbf{x}) = \sum_{i=0}^{n} w_i(\mathbf{x}) f(\mathbf{p}_i) \Big/ \phi(\mathbf{x}), \qquad \phi(\mathbf{x}) = \sum_{i=0}^{n} w_i(\mathbf{x}).$$

We can now express $\mathbf{x}$ as a convex combination of the vertices $\mathbf{p}_i$:

$$\mathbf{x} = \sum_{i=0}^{n} \lambda_i(\mathbf{x}) \mathbf{p}_i, \qquad \lambda_i(\mathbf{x}) := w_i(\mathbf{x}) \Big/ \sum_{j=0}^{n} w_j(\mathbf{x}).$$

(See Ju, Schaefer, and Warren (2005) and Floater, Kos, and Reimers (2005).)

Summary

- The Mean Value Lagrange interpolant in $\mathbb{R}^n$.
  - Parametric representation.
  - Non-convex domains.
- The Mean Value interpolant is in fact an interpolant.
- Hermite interpolant.
  - Weight function.
  - Normal derivatives.
- Application in $\mathbb{R}^3$: Surface deformation

References

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Thank you for your attention!