On Higher Order Voronoi Diagrams of Line Segments*

Evanthia Papadopoulou and Maksym Zavershynskyi

Faculty of Informatics, Università della Svizzera Italiana, Lugano, Switzerland
{evanthia.papadopoulou,maksym.zavershynskyi}@usi.ch

Abstract. We analyze structural properties of the order-k Voronoi diagram of line segments, which surprisingly has not received any attention in the computational geometry literature. We show that order-k Voronoi regions of line segments may be disconnected; in fact a single order-k Voronoi region may consist of $\Omega(n)$ disjoint faces. Nevertheless, the structural complexity of the order-k Voronoi diagram of non-intersecting segments remains $O(k(n-k))$ similarly to points. For intersecting line segments the structural complexity remains $O(k(n-k))$ for $k \geq n/2$.

Keywords: computational geometry, Voronoi diagrams, line segments, higher order Voronoi diagrams.

1 Introduction

Given a set of n simple geometric objects in the plane, called sites, the order-k Voronoi diagram of S is a partitioning of the plane into regions, such that every point within a fixed order-k region has the same set of k nearest sites. For $k = 1$ this is the *nearest-neighbor Voronoi diagram*, and for $k = n - 1$ the *farthest-site Voronoi diagram*. For n point sites in the plane, the order-k Voronoi diagram has been well studied, see e.g [9,1,4,6]. Its structural complexity has been shown to be $O(k(n-k))$ [9]. Surprisingly, order-k Voronoi diagrams of more general sites, including simple line segments, have been largely ignored. The farthest line segment Voronoi diagram was only recently considered in [3], showing properties surprisingly different than its counterpart for points. The nearest neighbor Voronoi diagram of line segments has received extensive attention, see e.g. [10,14,8] or [4] for a survey.

In this paper, we analyze the structural properties of the order-k Voronoi diagram of line segments. We first consider disjoint line segments and then extend our results to line segments that may share endpoints, such as line segments forming simple polygons or line segments forming a planar straight-line graph, and intersecting line segments. Unlike points, order-k Voronoi regions of line segments may be disconnected; in fact a single order-k Voronoi region may disconnect to $\Omega(n)$ disjoint faces. However, the structural complexity of the order-k

* Supported in part by the Swiss National Science Foundation grant 200021-127137 and the ESF EUROCORES program EuroGIGA/VORONOI, SNF 20GG21-134355.

K.-M. Chao, T.-s. Hsu, and D.-T. Lee (Eds.): ISAAC 2012, LNCS 7676, pp. 177–186, 2012.

line segment Voronoi diagram remains $O(k(n-k))$, assuming non-intersecting line segments, similarly to points, despite the disconnected regions. For intersecting line segments the dependency of the structural complexity on the number of intersections reduces as k increases and it remains $O(k(n-k))$ for $k \geq n/2$. The case of line segments involving polygonal objects is important for applications such as [11] that motivated our study.

For points, the derivation of the $O(k(n-k))$ bound relies on three facts: 1. an exact formula in [9] that relates F_k, the total number of faces on the order-k Voronoi diagram, with n, k and the number of unbounded faces in previous diagrams, 2. a symmetry property stating that $S_k = S_{n-k}$, where S_k denotes the number of unbounded faces in the order-k Voronoi diagram, and 3. an upper bound result from k-set theory [2,6]. In the case of line segments we first show that the formula of [9] remains valid, despite the presence of disconnected regions. However, the symmetry property no longer holds and results available from k-set theory are not directly applicable. Thus, a different approach has to be derived.

2 Preliminaries

Let $S = \{s_1, s_2, \ldots, s_n\}$ be a set of n line segments in $\mathbb{R}^2$. Each segment consists of three *elementary sites*: two endpoints and an open line segment. We make a *general position assumption* that no more than three elementary sites can touch the same circle.

The Euclidean distance between two points p, q is denoted as $d(p,q)$. The distance between point p and a line segment s is the minimum Euclidean distance $d(p,s) = \min_{q \in s} d(p,q)$. The bisector of two segments s_i and s_j is the locus of points equidistant from both, i.e., $b(s_i, s_j) = \{x \mid d(x, s_i) = d(x, s_j)\}$. If s_i and s_j are disjoint their bisector is a curve that consists of a constant number of line segments, rays and parabolic arcs. If segments intersect at point p the bisector consists of two such curves intersecting at point p. If segments share a common endpoint the bisector contains a two-dimensional region. In the following, we assume that segments are disjoint. We deal with segments that share endpoints in Section 5 and segments that intersect in Section 6.

Let $H \subset S$. The generalized Voronoi region of H, $\mathcal{V}(H,S)$ is the locus of points that are closer to all segments in H than to any segment not in H.

$$\mathcal{V}(H,S) = \{x \mid \forall s \in H, \forall t \in S \setminus H \ d(x,s) < d(x,t)\} \tag{1}$$

For $|H| = k$, $\mathcal{V}(H,S)$ is the order-k Voronoi region of H, denoted $\mathcal{V}_k(H,S)$.

$$\mathcal{V}_k(H,S) = \mathcal{V}(H,S) \text{ for } |H| = k \tag{2}$$

The order-k Voronoi diagram of S, $V_k(S)$, is the partitioning of the plane into order-k Voronoi regions. A maximal interior-connected subset of a region is called a face. The farthest Voronoi diagram of S is denoted as $V_f(S)$ ($V_f(S) = V_{n-1}(S)$) and a farthest Voronoi region as $\mathcal{V}_f(s,S)$ ($\mathcal{V}_f(s,S) = \mathcal{V}_{n-1}(S \setminus \{s\}, S)$).

An order-k Voronoi region $\mathcal{V}_k(H,S)$ can be interpreted as the locus of points closer to H than to any other subset of S of size k, where the distance between a point x and a set H is measured as the farthest distance $d(x,H) = \max_{s\in H} d(x,s)$.

The following lemma is a simple generalization of [3] for $1 \le k \le n-1$.

Lemma 1. *Consider a face F of region $\mathcal{V}_k(H,S)$. F is unbounded (in the direction r) iff there exists an open halfplane (normal to r) that intersects all segments in H but no segment in $S \setminus H$.*

Corollary 1. *There is an unbounded Voronoi edge separating regions $\mathcal{V}_k(H \cup \{s_1\}, S)$ and $\mathcal{V}_k(H \cup \{s_2\}, S)$ iff a line through the endpoints of s_1 and s_2 induces an open halfplane $r(s_1,s_2)$ such that $r(s_1,s_2)$ intersects all segments in H but no segment in $S \setminus H$.*

3 Disconnected Regions

The order-k line segment Voronoi diagram may have disconnected regions, unlike its counterpart of points, see e.g., Fig. 1. This phenomenon was first pointed out in [3] for the farthest line segment Voronoi diagram, where a single region was shown possible to be disconnected in $\Theta(n)$ faces.

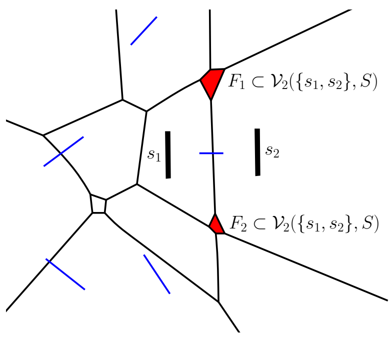

Fig. 1. $V_2(S)$ with two disconnected faces, induced by the same pair of sites

Lemma 2. *An order-k region of $V_k(S)$ can have $\Omega(n)$ disconnected faces, in the worst case, for $k > 1$.*

Proof. We describe an example where an order-k Voronoi region is disconnected in $\Omega(n-k)$ bounded faces. Consider k almost parallel long segments H. These segments induce a region $\mathcal{V}_k(H,S)$. Consider a minimum disk, that intersects all segments in H, and moves along their length. We place the remaining $n-k$ segments of $S \setminus H$ in a such way that they create obstacles for the disk. While the disk moves along the tree of $V_f(H)$ it intersects the segments of $S \setminus H$ one by one, and creates $\Omega(n-k)$ disconnectivities (see Fig. 2 (a)).

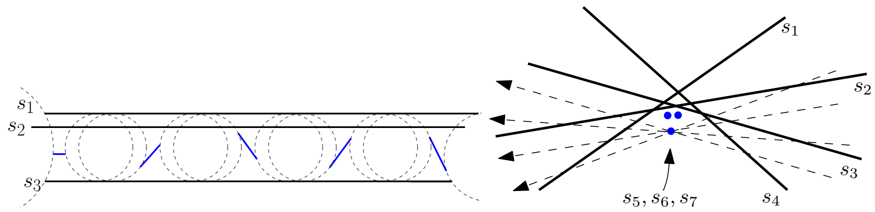

Fig. 2. (a) While the circle moves it encounters 5 obstacles, which induce $\Omega(n-k)$ disconnectivities of the region $\mathcal{V}_3(H,S)$; (b) $\mathcal{V}_4(\{s_1,s_2,s_3,s_4\},S)$ has $k=4$ disconnected unbounded faces. The dashed arrows represent the rotation of the directed line g.

We now follow [3] and describe an example where an order-k Voronoi region is disconnected in $\Omega(k)$ unbounded faces. Consider $n-k$ segments in $S \setminus H$ degenerated into points placed close to each other. The remaining k non-degenerate segments in H are organized in a cyclic fashion around them (see Fig. 2 (b)). Consider a directed line g through one of the degenerate segments s'. Rotate g around s' and consider the open halfplane to the left of g. During the rotation, the positions of g in which the halfplane intersects all k segments, alternate with the positions in which it does not. The positions in which the halfplane touches endpoints of non-degenerate segments, correspond to unbounded Voronoi edges. Each pair of consecutive unbounded Voronoi edges bounds a distinct unbounded face. Following [3], the line segments in H can be untangled into non-crossing segments while the same phenomenon remains.

Note that for small k, $1 < k \leq n/2$, $\Omega(n-k) = \Omega(n)$, while for large k, $n/2 \leq k \leq n-1$, $\Omega(k) = \Omega(n)$. □

Lemma 3. *An order-k region $\mathcal{V}_k(H,S)$ has $O(k)$ unbounded disconnected faces.*

Proof. We show that an endpoint p of a segment $s \in H$ may induce at most two unbounded Voronoi edges bordering $\mathcal{V}_k(H,S)$ (see Fig. 3). Consider two such unbounded Voronoi edges. By Corollary 1 there are open halfplanes $r(s,t_1)$, $r(s,t_2)$, for $s \in H$ and $t_1, t_2 \in S \setminus H$, that intersect all segments in H but no segments in $S \setminus H$. Thus, any halfplane $r(s,t_3)$, $t_3 \in S \setminus H$ must intersect either t_1 or t_2. Since $|H| = k$ and a segment has two endpoints, the lemma follows. □

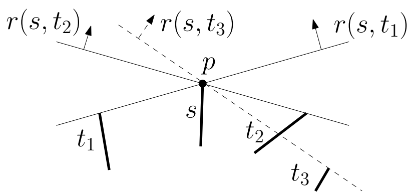

Fig. 3. Every endpoint of a segment $s \in H$ can induce at most 2 halfplanes

4 Structural Complexity

In this section we prove that the structural complexity of the order-k Voronoi diagram of n disjoint line segments is $O(k(n-k))$ despite disconnected regions. We first prove Theorem 1 which is a generalization of [9] for line segments exploiting the fact that the farthest line segment Voronoi diagram remains a tree structure [3]. Then in Lemma 7 we analyze the number of unbounded faces of the order-k Voronoi diagram in dual setting using results on arrangements of wedges [3,7] and ($\leq k$)-level in arrangements of Jordan-curves [13]. Combining Theorem 1 and Lemma 7 we derive the $O(k(n-k))$ bound.

Voronoi vertices in $V_k(S)$ are classified into *new* and *old*. A Voronoi vertex of $V_k(S)$ is called *new* (respectively *old*) if it is the center of a disk that touches 3 line segments and its interior intersects exactly $k-1$ (respectively $k-2$) segments. By the definition of the order-k Voronoi diagram we have the following properties:

1. Every Voronoi vertex of $V_k(S)$ is either *new* or *old*.
2. A *new* Voronoi vertex in $V_k(S)$ is an *old* Voronoi vertex in $V_{k+1}(S)$.
3. Under a general position assumption, an *old* Voronoi vertex in $V_k(S)$ is a *new* Voronoi vertex in $V_{k-1}(S)$.

Lemma 4. *Consider a face F of the region $\mathcal{V}_{k+1}(H,S)$. The portion of $V_k(S)$ enclosed in F is exactly the farthest Voronoi diagram $V_f(H)$ enclosed in F.*

Proof. Let x be a point in F. Suppose that among all segments in H, x is farthest from s_i. Then $x \in \mathcal{V}_f(s_i,H)$. Let $H_i = H \setminus \{s_i\}$. Since $x \in \mathcal{V}_{k+1}(H,S)$, x is farthest from s_i among all segments in H, and $|H_i| = k$, $x \in \mathcal{V}_k(H_i,S)$. □

Regions in the farthest line segment Voronoi diagram have the following *visibility property*: Let x be a point in $\mathcal{V}_f(s,H)$ of $V_f(H)$ for a set of segments H. Let $r(s,x)$ be the ray realizing the distance $d(s,x)$, emanating from point $p \in s$ such that $d(p,x) = d(s,x)$, extending to infinity (see Fig. 4). Ray $r(s,x)$ intersects the boundary of $\mathcal{V}_f(s,H)$ at a point a_x and the part of the ray beyond a_x is entirely in $\mathcal{V}_f(s,H)$. Using this property we derive the following lemma.

Lemma 5. *Let F be a face of region $\mathcal{V}_{k+1}(H,S)$ in $V_{k+1}(S)$. The graph structure of $V_k(S)$ enclosed in F is a connected tree that consists of at least one edge. Each leaf of the tree ends at a vertex of face F (see Fig. 4).*

Proof. (Sketch) Assume that the tree of $V_f(H)$ in F is disconnected. Consider a point x on the path that connects two disconnected parts and bounds $\mathcal{V}_f(s,H)$ such that $x \notin F$. Using the *visibility property* we derive a contradiction. □

Corollary 2. *Consider a face F of the Voronoi region $\mathcal{V}_{k+1}(H,S)$. Let m be the number of Voronoi vertices of $V_k(S)$ enclosed in its interior. Then F encloses $e = 2m+1$ Voronoi edges of $V_k(S)$.*

Let F_k, E_k, V_k and S_k denote respectively the number of faces, edges, vertices and unbounded faces in $V_k(S)$. By Euler's formula we derive

$$E_k = 3(F_k - 1) - S_k, V_k = 2(F_k - 1) - S_k \tag{3}$$

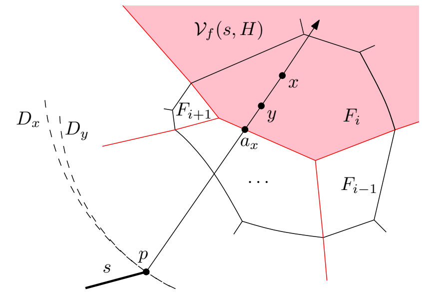

Fig. 4. The part of the ray $r(s, x)$ beyond a_x entirely belongs to $\mathcal{V}_f(s, H)$

Lemma 6. *The total number of unbounded faces in the order-k Voronoi diagram for all orders is* $\sum_{i=1}^{n-1} S_i = n(n-1)$

Theorem 1. *The number of Voronoi faces in order-k Voronoi diagram of n disjoint line segments is:*

$$F_k = 2kn - k^2 - n + 1 - \sum_{i=1}^{k-1} S_i,\ F_k = 1 - (n-k)^2 + \sum_{i=k}^{n-1} S_i \qquad (4)$$

Proof. (Sketch) Corollary 2 implies that $E_{k+1} = 2V'_k + F_{k+2}$, where V'_k is the number of new Voronoi vertices in $V_k(S)$. Combining Corollary 2 and (3) we obtain a recursive formula $F_{k+3} = 2F_{k+2} - F_{k+1} - 2 - S_{k+2} + S_{k+1}$. Using as base cases $F_1 = n$ and $F_2 = 3(n-1) - S_1$ we prove the first part of (4). Lemma 6 implies $\sum_{i=1}^{k-1} S_i + \sum_{i=k}^{n-1} S_i = \sum_{i=1}^{n-1} S_i = n(n-1)$. Combining this with the first part of (4) we derive the second part. □

Lemma 7. *For a given set of n segments,* $\sum_{i=k}^{n-1} S_i$ *is* $O(k(n-k))$*, for* $k \geq n/2$.

Proof. Following [3], we use the well-known point-line duality transformation T, which maps a point $p = (a, b)$ in the primal plane to a line $T(p) : y = ax - b$ in the dual plane, and vice versa. We call the set of points above both lines $T(p)$ and $T(q)$ the *wedge* of $s = (p, q)$. Consider a line l and a segment $s = (p, q)$. Segment s is above line l iff point $T(l)$ is strictly above lines $T(p)$ and $T(q)$ [3].

Consider the arrangement W of wedges w_i, $i = 1, \ldots, n$, as defined by the segments of $S = \{s_1, \ldots, s_n\}$. For our analysis we need the notions of r-level and $(\leq r)$-level. The r-level of W is a set of edges such that every point on it is above r wedges. The r-level shares its vertices with the $(r-1)$-level and the $(r+1)$-level. The $(\leq r)$-level of W is the set of edges such that every point on it is above at most r wedges. The complexity of the r-level and the $(\leq r)$-level is the number of their vertices, excluding the wedge apices. We denote the maximum complexity of the r-level and the $(\leq r)$-level of n wedges as $g_r(n)$ and $g_{\leq r}(n)$, respectively.

Claim: The number of unbounded Voronoi edges of $V_k(S)$, unbounded in direction $\phi \in [\pi, 2\pi]$, is exactly the number of vertices shared by the $(n-k-1)$-level and the $(n-k)$-level of W. Thus $S_k = O(g_{n-k}(n))$.

Proof of claim: Consider a vertex p (see Fig. 5) of the r-level and $(r+1)$-level. Let w_i, w_j be the wedges, that intersect at p, and let s_i, s_j be their corresponding segments. Let W_p, $|W_p| = n - r - 2$, be the set of wedges strictly above p, and let S_p be the set of the corresponding segments. Then $T(p)$ induces the unbounded Voronoi edge that separates regions $\mathcal{V}_{n-r-1}(S_p \cup \{s_i\}, S)$ and $\mathcal{V}_{n-r-1}(S_p \cup \{s_j\}, S)$ of $V_{n-r-1}(S)$. Let $r = n - k - 1$ to derive the claim.

The claim implies the following.

$$\sum_{i=k}^{n-1} S_i = O(g_{\le n-k}(n)) \tag{5}$$

Since the arrangement of wedges is a special case of arrangements of Jordan curves, we use the formula from [13] to bound the complexity of the ($\le r$)-level:

$$g_{\le r}(n) = O\left((r+1)^2 g_0\left(\left\lfloor \frac{n}{r+1} \right\rfloor\right)\right) \tag{6}$$

It is known that the complexity of the lower envelope of such wedges is $g_0(x) = O(x)$ [7,3]. (Note that [13] implies a weaker $g_0(x) = O(x \log x)$)). Therefore, $g_{\le r}(n) = O(n(r+1))$. Substituting into formula (5) we obtain $\sum_{i=k}^{n-1} S_i = O(n(n-k))$. Since $n/2 \le k \le n-1$, $\sum_{i=k}^{n-1} S_i = O(k(n-k))$. □

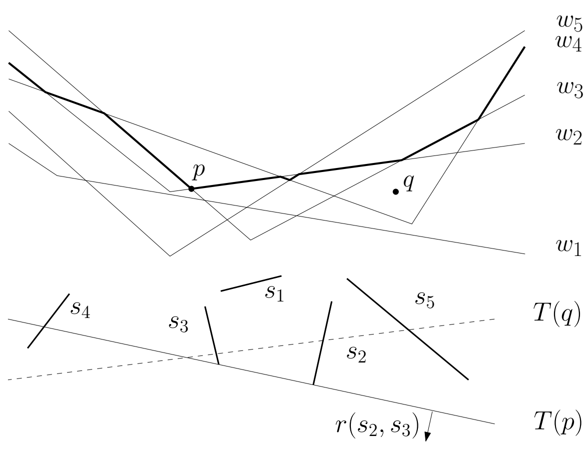

Fig. 5. (a) In the dual plane point p belongs to the 2-level and the 3-level of the arrangement W. (b) In the primal plane the halfplane $r(s_2, s_3)$ below $T(p)$ defines the unbounded Voronoi edge that separates $\mathcal{V}_2(\{s_2, s_4\}, S)$ and $\mathcal{V}_2(\{s_3, s_4\}, S)$.

Combining Lemma 7 and Theorem 1 we obtain the following theorem.

Theorem 2. *The number of Voronoi faces in order-k Voronoi diagram of n disjoint line segments is $F_k = O(k(n-k))$.*

5 Line Segments Forming a Planar Straight-Line Graph

In this section we consider line segments that may touch at endpoints, such as line segments forming a simple polygon, more generally line segments forming a planar straight-line graph. This is important for applications involving polygonal shapes such as [11].

When line segments share endpoints, bisectors may contain 2-dimensional portions. The standard approach to avoid this issue for $k = 1$ is to consider S as a set of distinct elementary sites. Our goal is to extend this notion for the order-k Voronoi diagram without altering the structure of the order-k Voronoi diagram for disjoint line segments. Note that we cannot simply consider elementary sites as distinct when defining an order-k Voronoi region as this will lead to a different type of order-k Voronoi diagram for disjoint segments that is not very interesting. We first extend the notion of a subset of S of cardinality k as follows.

Definition 1. *A set H, $H \subseteq S$, is called an order-k subset iff*

1. *$|H| = k$ (type 1) or*
2. *$H = H' \cup I(p)$, where $H' \subseteq S$, $|H'| < k$, p is a segment endpoint incident to set of segments $I(p)$, and $|H' \cup I(p)| > k$ (type 2). Set $rep(H) = \{p\} \cup \{H \setminus I(p)\}$ is called the* representative *of the order-k subset.*

An order-k Voronoi region is now defined as

$$\mathcal{V}_k(H, S) = \mathcal{V}(H, S), \text{ where } H \text{ is an order-}k \text{ subset of } S$$

Note that for disjoint segments all order-k subsets are of type 1 and the definition of $\mathcal{V}_k(H, S)$ is equivalent to (2). The following lemma clarifies Def. 1.

Lemma 8. *An order-k subset H induces a non-empty order-k Voronoi region iff there exists a disk that intersects or touches all segments in H but it does not intersect nor touch any segment in $S \setminus H$.*

The order-k Voronoi diagram defined in this way has some differences from the standard order-k Voronoi diagram of disjoint objects. In the standard case, any two neighboring order-k regions belong to two order-k subsets, H_1 and H_2, which are of type 1 and differ by exactly two elements, i.e. $|H_1 \triangle H_2| = 2$, where $\triangle$ denotes the symmetric difference. The bisector of these two elements defines exactly the Voronoi edge separating the two regions. Here, two neighboring regions of order-k subsets H_1 and H_2 that are not both type 1, may differ as $|H_1 \triangle H_2| \geq 1$, see e.g., regions $V(6, 5)$ and $V(3, 4)$ or regions $V(5, 7, 8)$ and $V(7, 8)$ in Fig. 6. However, the representatives of H_1 and H_2 may differ in exactly one or two elements. If $|rep(H_1) \triangle rep(H_2)| = 1$ then the bisector bounding the two regions is $b(p, h)$, where $H_1 = H_1' \cup I(p)$, $h \in H_2 \setminus H_1$, and $|H_2 \setminus H_1| = 1$. (For a type 1 subset, $rep(H) = H$). Voronoi edges bounding the regions of order-k subsets, which are not both of type 1, may remain in the order-k Voronoi diagram for severaf orders, while $|H_1 \cup H_2| > k$.

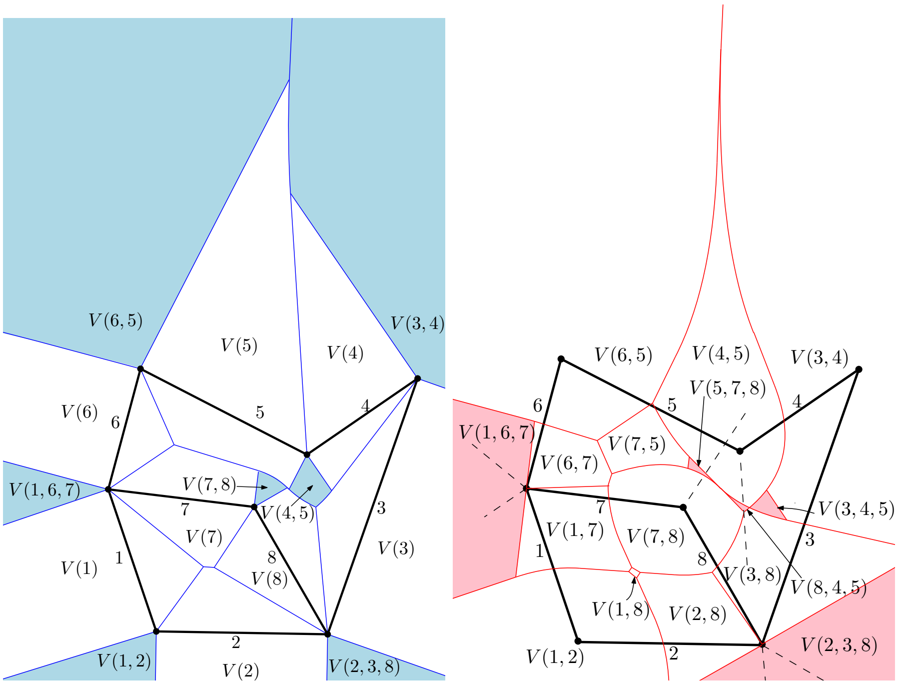

Fig. 6. (a) $V_1(S)$ of a planar straight-line graph. The bold regions are induced by order-k subsets of type-2; (b) $V_2(S)$ of a planar-straight line graph. For brevity we use $V(s_1, \ldots, s_m)$ notation instead of $\mathcal{V}_k(\{s_1, \ldots, s_m\}, S)$.

6 Intersecting Line Segments

Let S be a set of line segments that may intersect in a total I intersection points. For simplicity we assume that no two segments share an endpoint and that no more than two segments intersect at a common point. Intuitively, intersections influence Voronoi diagrams of small order and the influence grows weaker as k increases. Recall that the number of faces, edges and vertices of $\mathcal{V}_k(S)$ are denoted F_k, E_k and V_k, respectively.

Lemma 9. *The total number of unbounded faces in all orders is* $\sum_{i=1}^{n-1} S_i = n(n-1) + 2I$

Lemma 10. $F_1 = n + 2I$*,* $F_2 = 3n - 3 - S_1 + 2I$*, and* $F_3 = 5n - 8 - S_1 - S_2 + 2I$*.*

Following the induction scheme of Theorem 1 and using Lemma 10 as the base case we derive

$$F_k = 2kn - k^2 - n + 1 - \sum_{i=1}^{k-1} S_i + 2I, \; F_k = 1 - (n-k)^2 + \sum_{i=k}^{n-1} S_i$$

Lemma 7 is valid for arbitrary segments including intersecting ones.

Theorem 3. *The number of Voronoi faces in order-k Voronoi diagram of n intersecting line segments with I intersections is*

$$F_k = O(k(n-k)+I), \text{ for } 1 \leq k < n/2$$
$$F_k = O(k(n-k)), \text{ for } n/2 \leq k \leq n-1$$

7 Concluding Remarks

Any standard iterative approach can be adapted to compute the order-k Voronoi diagram of non-crossing line segments in time $O(k^2 n \log n)$. The conventions of Section 5 are important for line segments forming a planar straight-line graph. We are currently considering more efficient algorithmic techniques.

References

1. Agarwal, P., de Berg, M., Matousek, J., Schwarzkopf, O.: Constructing levels in arrangements and higher order Voronoi diagrams. SIAM J. Comput. 27(3), 654–667 (1998)
2. Alon, N., Györi, E.: The number of small semispaces of a finite set of points in the plane. J. Comb. Theory, Ser. A 41(1), 154–157 (1986)
3. Aurenhammer, F., Drysdale, R., Krasser, H.: Farthest line segment Voronoi diagrams. Inf. Process. Lett. 100(6), 220–225 (2006)
4. Aurenhammer, F., Klein, R.: Voronoi Diagrams. In: Sack, J.-R., Urrutia, J. (eds.) Handbook of Computational Geometry. North-Holland Publishing Co. (2000)
5. Boissonnat, J.-D., Devillers, O., Teillaud, M.: A Semidynamic Construction of Higher-Order Voronoi Diagrams and Its Randomized Analysis. Algorithmica 9(4), 329–356 (1993)
6. Edelsbrunner, H.: Algorithms in combinatorial geometry. EATCS Monographs on Theoretical Computer Science, ch. 13.4. Springer (1987)
7. Edelsbrunner, H., Maurer, H.A., Preparata, F.P., Rosenberg, A.L., Welzl, E., Wood, D.: Stabbing Line Segments. BIT 22(3), 274–281 (1982)
8. Karavelas, M.I.: A robust and efficient implementation for the segment Voronoi diagram. In: Proc. 1st Int. Symp. on Voronoi Diagrams in Science and Engineering, Tokyo, pp. 51–62 (2004)
9. Lee, D.T.: On k-Nearest Neighbor Voronoi Diagrams in the Plane. IEEE Trans. Computers 31(6), 478–487 (1982)
10. Lee, D.T., Drysdale, R.L.S.: Generalization of Voronoi Diagrams in the Plane. SIAM J. Comput. 10(1), 73–87 (1981)
11. Papadopoulou, E.: Net-Aware Critical Area Extraction for Opens in VLSI Circuits Via Higher-Order Voronoi Diagrams. IEEE Trans. on CAD of Integrated Circuits and Systems 30(5), 704–717 (2011)
12. Shamos, M.I., Hoey, D.: Closest-point problems. In: Proc. 16th IEEE Symp. on Foundations of Comput. Sci., pp. 151–162 (1975)
13. Sharir, M., Agarwal, P.: Davenport-Schinzel Sequences and their Geometric Applications, ch. 5.4. Cambridge University Press (1995)
14. Yap, C.-K.: An $O(n \log n)$ Algorithm for the Voronoi Diagram of a Set of Simple Curve Segments. Discrete & Computational Geometry 2, 365–393 (1987)