On the Farthest Line-Segment Voronoi Diagram*

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Abstract

The farthest line-segment Voronoi diagram shows properties surprisingly different than the farthest point Voronoi diagram: Voronoi regions may be disconnected and they are not characterized by convex-hull properties. In this paper we introduce the *farthest line-segment hull*, a cyclic structure that relates to the farthest line-segment Voronoi diagram similarly to the way an ordinary convex hull relates to the farthest-point Voronoi diagram and provide $O(n \log n)$-time algorithms for its construction. Using the farthest line-segment hull, we derive a more tight bound on the (linear) size of the farthest line-segment Voronoi diagram. We also illustrate properties of the L_∞ farthest line-segment Voronoi diagram, which finds applications in VLSI Design Automation.

1 Introduction

Let S be a set of n simple geometric objects in the plane, such as points or line segments, called sites. The *farthest-site Voronoi diagram* of S is a subdivision of the plane into regions such that the region of a site s is the locus of points farther away from s than from any other site. Surprisingly, the farthest-line-segment Voronoi diagram illustrates different properties than its counterpart for points [1]. For example, Voronoi regions are not characterized by convex-hull properties and they may be disconnected; a Voronoi region may consist of $\Theta(n)$ disconnected faces. Nevertheless, the graph structure of the diagram remains a tree and its structural complexity is $O(n)$. An abstract framework on the farthest-site Voronoi diagram (which does not include the case of intersecting line-segments) was given in [5]. Related is the farthest-polygon Voronoi diagram, recently addressed in [3].

In this paper we further study the structural properties of the farthest line-segment Voronoi diagram. We introduce the *farthest line-segment hull*, a cyclic structure that is related to the farthest line-segment Voronoi diagram similarly to the way an ordinary convex hull is related to the farthest-point Voronoi diagram. Using the farthest line-segment hull, we improve the (linear) upper bound on the structural complexity of the diagram in [1] by a constant factor. We provide $O(n \log n)$-time algorithms for the construction of the farthest line-segment hull by adapting standard approaches for the construction of an ordinary convex hull such as divide and conquer, Graham's scan, etc. We also characterize the farthest line-segment hull and the corresponding Voronoi diagram in the simpler L_∞, L_1 metrics. Once the farthest line-segment hull is available, the farthest line-segment Voronoi diagram can be constructed in additional $O(h \log h)$ time, by the simple algorithm given in [1], where h is the size of the farthest line-segment hull ($h \in O(n)$). In the L_∞ or L_1 metrics the construction simplifies to additional $O(h)$-time.

The farthest line-segment Voronoi diagram finds applications in computing the smallest disk that overlaps all given line-segments. It is necessary in defining and computing the *Hausdorff Voronoi diagram* of clusters of line segments, which finds several applications in VLSI design automation, see e.g., [6].

2 Preliminaries and Definitions

Let $S = \{s_1, \ldots, s_n\}$ be a set of n arbitrary line-segments in the plane. Line-segments may intersect or touch at single points. The distance between a point q and a line-segment s_i is $d(q, s_i) = \min\{d(q, y), \forall y \in s_i\}$, where $d(q, y)$ denotes the ordinary distance between two points, q, y, in the L_p metric, $p, 1 \leq p \leq \infty$. The farthest Voronoi region of a line-segment s_i is

$$freg(s_i) = \{x \in \mathbb{R}^2 \mid d(x, s_i) \geq d(x, s_j), 1 \leq j \leq n\}$$

The collection of all farthest Voronoi regions, together with their bounding edges and vertices, constitute the *farthest line-segment Voronoi diagram* of S, denoted as *FVD(S)* (see Fig. 1). Any maximally connected subset of a region in *FVD(S)* is called a *face*. Any Voronoi edge bounding two neighboring regions, $freg(s_i)$ and $freg(s_j)$, is portion of bisector $b(s_i, s_j)$, i.e., the locus of points equidistant from s_i and s_j. For line-segments in general position that are non-intersecting, $b(s_i, s_j)$ is an unbounded curve that consists of a constant number of simple pieces as induced by elementary bisectors between the endpoints and open portions of s_i, s_j. For more information on line-segment bisectors, see e.g., [4, 8] and references therein.

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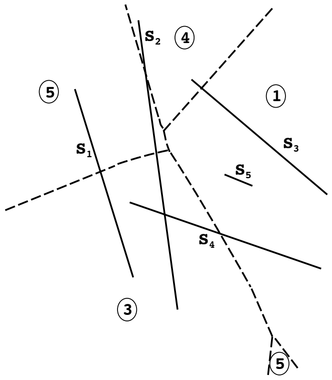

Figure 1: A farthest line-segment Voronoi diagram.

3 The farthest line-segment hull

Definition 1 *A line l through the endpoint p of a line-segment s, $s \in S$, is called a* supporting line *of S if and only if an open halfplane induced by l, denoted $H(l)$, intersects all segments in S, except s (and possibly except additional segments incident to p). Point p is said to* admit a supporting line*. The unit normal of l, pointing away from $H(l)$, is called the* unit vector *of l, denoted $\nu(l)$. A line-segment s, $s \in S$, such that the line l through s is a supporting line of S and $H(l)$ intersects all segments in $S \setminus \{s\}$, is called a* hull segment*; s is said to* admit a supporting line *and the unit vector of l is denoted as $\nu(s)$.*

For a supporting line l, through line-segment s, that intersects all segments in S, both open halfplanes of l, denoted $H(s)$ and $H'(s)$, intersect all segments in $S \setminus \{s\}$. In this case s is said to admit two supporting lines and it results in two different hull segments.

Definition 2 *The line segment $\overline{pq}$ joining the endpoints p, q, of two line segments $s_i, s_j \in S$ is called a* supporting segment *of S if and only if an open halfplane induced by the line l through $\overline{pq}$, denoted $H(\overline{pq})$, intersects all segments in S, except s_i, s_j (and possibly except additional segments incident to p, q). The unit normal of $\overline{pq}$ pointing away from $H(\overline{pq})$ is called the* unit vector*, $\nu(\overline{pq})$.*

Definition 3 *The closed polygonal curve obtained by the sequence of hull segments in S (line-segments in S that admit a supporting line), ordered according to the angular order of their unit vectors, interleaved by maximal chains of supporting segments, is called the farthest line-segment hull of S (for brevity, the farthest hull), denoted as f-hull(S).*

The vertices of the farthest hull are exactly the endpoints of S that admit a supporting line. The edges are of two types: supporting segments and hull segments. If line-segments degenerate to points, the farthest hull corresponds exactly to the convex hull of S.

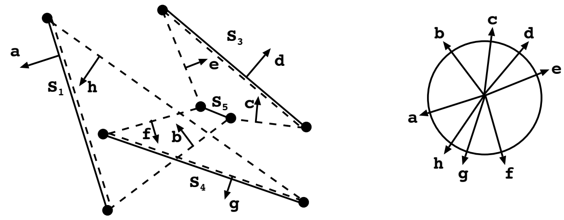

Figure 2: The farthest line-segment hull of Fig. 1

By definition of a supporting segment, any maximal chain of supporting segments between two consecutive hull segments must be convex. Fig. 2 illustrates the farthest hull of Fig. 1. Clearly, the unit vector of any supporting line of the farthest hull is unique.

Consider the circular list of the unit vectors of all edges of the farthest hull ordered angularly around the origin of the unit circle K_o. They induce a mapping of the farthest hull onto the circumference of K_o, such that every edge e is mapped to a point on the circumference of K_o as obtained by the unit vector $\nu(e)$, and every vertex is mapped to arcs as delimited by the unit vectors of the incident edges. This mapping is called the *Gaussian map* of S, for short *Gmap(S)*. (For more information on the Gmap see e.g., [2, 7]). The same notation, $\nu(e)$, is used to denote both the unit vector and the corresponding point on the circumference of K_o. *Gmap(S)* can be viewed as a cyclic sequence of arcs on the circumference of K_o, where each arc corresponds to a vertex of the farthest hull.

Lemma 1 *Consider any two consecutive unit vectors ν_i, ν_{i+1}, along Gmap(S) corresponding to edges e, e' of the farthest hull such that $\nu_i = \nu(e)$ and $\nu_{i+1} = \nu(e')$. Edges e, e' must be incident upon a common vertex v of f-hull(S), and any point along the arc on K_o from ν_i to ν_{i+1}, corresponds to the unit vector of a supporting line through v.*

Corollary 2 *FVD(S) has exactly one unbounded bisector for every supporting segment s of f-hull(S), which is unbounded in the direction opposite to $\nu(s)$. Unbounded bisectors in FVD(S) are cyclically ordered following exactly the cyclic ordering of the Gmap.*

By Corollary 2, *FVD(S)* has exactly one face for every pair of consecutive supporting segments on *f-hull(S)*. By Lemma 1, any two consecutive supporting segments are incident to the same line-segment s (either to the same endpoint or to one endpoint each), which must be the owner of the face delimited by the corresponding pair of consecutive unbounded bisectors. This face is unbounded in any direction induced by the unit vectors of points along the corresponding arc of the Gmap.

3.1 Properties of the farthest line-segment hull

Considering the Gmap as a cyclic sequence of arcs on the circumference of K_o, it forms a Davenport-Schinzel sequence of length 3 (see also [1]). Let the *upper* (resp. *lower) Gmap* denote the portion of the *Gmap* above (resp. below) the horizontal diameter of K_o. Equivalently for the *upper* and *lower f-hull*. In the following we always assume a traversal of the upper Gmap from left to right and we characterize the right and left endpoints of a segment s as a *start-vertex* and an *end-vertex* respectively.

Let an *interval* $[a_i, a_{i+1}]$ denote the portion of the upper Gmap between two consecutive (but not adjacent) occurrences of arcs for the same segment $s_a = (a', a)$, where a, a' denote the start-vertex and end-vertex of s_a respectively. Interval $[a_i, a_{i+1}]$ is assumed to be *non-trivial* i.e., it contains in its interior at least one arc other than a, a'.

Lemma 3 *Let $[a_i, a_{i+1}]$ be a (non-trivial)* interval *of segment $s_a = (a', a)$ on upper Gmap(S). We have the following properties: 1. On the upper Gmap, all occurrences of start-vertex a must precede all occurrences of end-vertex a'. 2. The vertex following a_i (resp. preceding a_{i+1}) in $[a_i, a_{i+1}]$ must be a start-vertex (resp. end-vertex). 3. If a_i is a start-vertex, i.e., $a_i = a$, (resp. a_{i+1} is end-vertex a'), no other start-vertex (resp. end-vertex) b in the interval $[a_i, a_{i+1}]$ can appear before a_i or past a_{i+1} on the upper Gmap, and no end-vertex (resp. start-vertex) e in $[a_i, a_{i+1}]$ can appear before a_i (resp. past a_{i+1}) on the upper Gmap.*

Proof. (Sketch). The proof is easier to derive using the duality transformation given in [1] that transforms a line segment to a wedge below two intersecting lines and reduces the problem of identifying the upper Gmap into the problem of computing the union of those wedges (see Fig. 3 of [1]). Here, we only point out the equivalence between the arcs of the upper Gmap and edges along the union of wedges in [1]. The proof is not hard to see. ☐

Lemma 4 *The re-appearance along the upper Gmap of any endpoint of a segment s_a can be charged to a unique vertex u of the upper f-hull such that no other re-appearance of a segment endpoint on the upper Gmap can be charged to u.*

Proof. (Sketch). For an interval $[a_i, a_{i+1}]$ such that $a_i = a$ is a start-vertex, let u be the vertex following a_i in $[a_i, a_{i+1}]$, which must be a start-vertex by Lemma 3; the appearance of a_{i+1} is charged to u. Similarly, for an interval $[a_i, a_{i+1}]$ such that $a_{i+1} = a'$ is an end-vertex, let u be the vertex preceding a_{i+1}, which must be an end-vertex; the appearance of a_i is charged to u. Using the properties of Lemma 3 it is not hard to see that no other interval $[c_j, c_{j+1}]$ of any segment s_c can be charged to u. ☐

Theorem 5 *The total number of faces of the farthest line-segment Voronoi diagram of a set S of n arbitrary line segments is at most $6n - 2$.*

Proof. (Sketch). We count the number of *maximal arcs* along the upper Gmap There are two types of maximal arcs: *composite arcs*, corresponding to hull segments, which consist of the segment unit vector and the two incident arcs of the segment endpoints, and *single-vertex* arcs, corresponding to single vertices along convex chains of the hull, which are single arcs bounded by the unit vectors of the two incident supporting segments. Consider the sequence of all occurrences of a single segment $s_a = (a', a)$ on the upper Gmap. It is a sequence of the form $\ldots a \ldots aa' \ldots a' \ldots$ or $\ldots a \ldots a \ldots a' \ldots a' \ldots$. The number of its maximal arcs is exactly one plus the number of (non-trivial) intervals. Thus, summing over all segments, the total number of maximal arcs on the upper Gmap is at most n plus the total number of vertices that may get charged for the reappearance of a single vertex arc, which is $2n$ by Lemma 4. Thus, in total $3n$. Similarly for the lower Gmap. ☐

The above theorem improves the $8n + 4$ upper bound on the number of faces of the farthest line-segment Voronoi diagram given in [1]. A known lower bound is $4n - 4$ [1]. For disjoint segments the corresponding upper bound is $2n - 2$ (see [3]) as the Gmap corresponds to a Davenport-Schinzel sequence of length 2.

4 Algorithms for the farthest line-segment hull

Most approaches to compute an ordinary convex hull can be adapted to compute the farthest line-segment hull within the same time complexity. Lemma 3 gives insight for a Graham's scan type of approach (to be given in a more complete version of this paper). A standard divide and conquer approach can be designed using the properties listed below. A two-pass divide and conquer construction in the dual space (under the standard point-line duality) is given in [1].

Let L, R be two subsets partitioning S. Given a unit vector $\nu(e)$ in *Gmap(L)*, let its *neighboring supporting vertex* in R be the vertex m in *Gmap(R)* such that $\nu(e)$ lies along an arc of m. A unit vector in *Gmap(L)* is called *valid* iff its corresponding supporting line remains a supporting line in *Gmap(L ∪ R)*. (Respectively for *Gmap(R)*).

Lemma 6 *Edge e of f-hull(L) remains valid iff the neighboring supporting vertex q of $\nu(e)$ in R lies in $H(e)$.*

Lemma 7 *Vertex $m \in$ f-hull(L) remains valid iff either an incident farthest hull edge remains valid, or m*

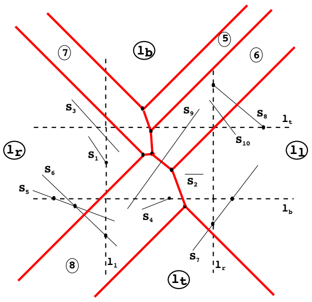

Figure 3: L_∞ *FVD(S).*

is the neighboring supporting vertex in L of an invalid unit vector in f-hull(R).

Gmap(S) can be obtained by merging *Gmap(L)* and *Gmap(R)* as follows: 1. Mark all unit vectors and vertex arcs along *Gmap(L)* and *Gmap(R)* as valid or invalid. 2. Merge *Gmap(L)* and *Gmap(R)* into *Gmap(L)*∪*Gmap(R)* following the angular order of unit vectors. 3. For any two consecutive valid vectors, one in *Gmap*(L) and one in *Gmap*(R), insert a unit vector as defined by the supporting segment joining the vertices of the two incident arcs. 4. For any valid vertex m of *Gmap*(L) between two consecutive valid vectors of *Gmap*(R), insert new unit vectors for the corresponding supporting segments incident to m. 5. Delete all invalid vectors.

5 The L_∞ farthest line segment Voronoi diagram

In L_∞, the farthest line-segment Voronoi diagram does not maintain the $O(1)$ structural complexity as its counterpart for points (see Fig. 3). All its faces are unbounded in one of the eight possible directions as implied by rays of slope $\pm1, 0, \infty$. The L_∞ farthest line-segment hull simplifies as shown in Fig. 4. Let four bounding lines be defined as follows: let l_t (resp. l_b) be the horizontal line passing through the topmost lower-endpoint (resp. the bottommost upper-endpoint) of all segments in S and let l_l (resp. l_r) be the vertical line passing through the leftmost right-endpoint (resp. the rightmost left-endpoint) of all segments in S. They partition the plane into four quadrants, labeled $1-4$ in counterclockwise order as shown in Fig. 4. Let $E_i, i = 1, 2,$ (resp. $E_j, j = 3, 4$) denote the upper (resp. lower) envelope of the line segments straddling quadrant i (resp. j). (If no segments straddle quadrant i, let E_i be the corner point O_i of quadrant i).

Definition 4 *The L_∞ farthest line-segment hull of S is a closed polygonal curve as derived by a counterclockwise traversal of $E_1, l_{t'}, E_2, l_{l'}, E_3, l_{b'}, E_4, l_{r'}$, where $l_{i'}$ denotes the portion of bounding line l_i between its two incident envelopes.*

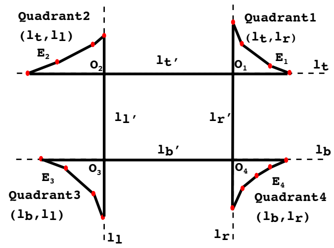

Figure 4: The L_∞ *farthest line-segment hull.*

The L_∞ farthest line-segment Voronoi diagram consists of exactly one unbounded face for each edge of the L_∞ farthest hull. A Voronoi region may consist of only a constant number of disjoint faces. Once the L_∞ farthest hull is available, the L_∞ farthest Voronoi diagram can be constructed in additional $O(h)$ time by adapting the simple algorithm in [1], where h is the size of the farthest hull which may vary from $O(1)$ to $O(n)$. The above results can be easily adapted for L_1, which is equivalent to L_∞ under rotation.

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