# New algebraic and geometric characterizations of planar quintic Pythagorean-hodograph curves

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**Abstract**

The aim of this work is to provide new characterizations of planar quintic Pythagorean-hodograph curves. The first two are algebraic and consist of two and three equations, respectively, in terms of the edges of the Bézier control polygon as complex numbers. These equations are symmetric with respect to the edge indices and cover curves with generic as well as degenerate control polygons. The last two characterizations are geometric and rely both on just two auxiliary points outside the control polygon. One requires two (possibly degenerate) quadrilaterals to be similar, and the other highlights two families of three similar triangles. All characterizations are a step forward with respect to the state of the art, and they can be linked to the well-established counterparts for planar cubic Pythagorean-hodograph curves. The key ingredient for proving the aforementioned results is a novel general expression for the hodograph of the curve.

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## 1 Introduction

A *Pythagorean-hodograph* (PH) curve is a parametric polynomial curve for which the norm of the hodograph, that is, its first derivative with respect to the parameter, is also a polynomial [10]. For a planar polynomial curve this means that the two derivative components and the parametric speed form a Pythagorean triple [12]. As a consequence, the arc length of a PH curve can be computed by simply evaluating a polynomial and fixed-distance offset curves are rational, so that they can be represented exactly in common CAD systems. Another class of planar polynomial curves with rational offsets are *indirect Pythagorean-hodograph* (iPH) curves. The two derivative components and the parametric speed of an iPH curve form a *rational* Pythagorean triple after applying a suitable rational quadratic reparameterization to the curve [13, 14].

Due to their established practical value in a variety of applications ranging from CNC machining to motion control and railway design, low degree PH and iPH curves have been the subject of investigation of several works over the last decades [5]. In particular, the derivation of algebraic and geometric characterizations related to their Bézier control polygons [3, 4, 9, 11, 16, 17] as well as the interpolation of first-order Hermite data (end points and derivatives) have been of primary interest [1, 7, 8, 15].

Our work focusses on quintic PH curves since they are considered the lowest-degree PH curves suitable for free-form design. More precisely, after setting the notation and recalling some preliminaries about planar PH curves (Section 2), we introduce two new algebraic characterizations (Section 3) and two new geometric characterizations (Section 4) of planar quintic PH curves. The proposed algebraic characterizations are more compact than the one established by Farouki [4] since they cover the generic as well as the degenerate case with one common system of symmetric equations. Compared to the geometric characterization by Fang and Wang [3], ours are entirely geometric and simpler since both involve only two instead of four auxiliary points. Moreover, one of the geometric characterizations leads to a simple construction of a quintic PH curve from two cubic PH companion curves.

## 2 Preliminaries and notation

In the plane, a generic polynomial curve $\boldsymbol{r}$ of *degree* $n \in \mathbb{N}$ can be expressed in complex *Bézier form* as

$$\boldsymbol{r}: \mathbb{R} \to \mathbb{C}, \qquad t \mapsto \sum_{k=0}^{n} \boldsymbol{p}_k B_k^n(t), \tag{1}$$

for some *control points* $\boldsymbol{p}_0, \dots, \boldsymbol{p}_n \in \mathbb{C}$, where

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k$$

denotes the $k$-th *Bernstein polynomial* of degree $n$. The PH property is then equivalent to the requirement that $|\boldsymbol{r}'|$ is a real polynomial. As pointed out by Wang and Fang [17], a very general description of the hodograph of such curves follows from [4, 10, 12].

**Theorem 1 [17, Theorem 3].** A regular polynomial curve $\boldsymbol{r}$ is a PH curve if and only if

$$\boldsymbol{r}'(t) = p(t)\boldsymbol{w}(t)^2, \tag{2}$$

for some real polynomial $p$ and some complex polynomial $\boldsymbol{w}$ with $p(t), \boldsymbol{w}(t) \neq 0$ for $t \in \mathbb{R}$.

Note that this result can also be seen as a special case of a more general characterization of polynomial curves with rational offsets [13, Theorem 1]. Since we are interested in quintic PH curves, we consider $\deg(\boldsymbol{r}') = 4$. Following Theorem 1, there are three cases to distinguish:

- $\deg(p) = 4$ and $\deg(\boldsymbol{w}) = 0$
  In this case, $\boldsymbol{r}(t)$ is just a line, parameterized by a strictly monotonic quintic polynomial.
- $\deg(p) = 2$ and $\deg(\boldsymbol{w}) = 1$
  Fang and Wang [3] were the first to analyse this case in depth, and they derive a geometric characterization in terms of the Bézier control edges of such a so-called *class II quintic PH curve*. They further show that there are usually four curves of this type that solve a given set of Hermite interpolation conditions.
- $\deg(p) = 0$ and $\deg(\boldsymbol{w}) = 2$
  In this case, $\boldsymbol{r}'$ is called a *primitive* Pythagorean hodograph [5], and the curve $\boldsymbol{r}$ may be referred to as a *class I quintic PH curve* [3]. This is by far the best-studied case in the literature and the most relevant for applications, and we discuss only curves of this type in the following.

If $\boldsymbol{r}$ is a regular class I quintic PH curve, then there exist $\boldsymbol{w}_0, \boldsymbol{w}_1, \boldsymbol{w}_2 \in \mathbb{C}$ with $\boldsymbol{w}_0, \boldsymbol{w}_2 \neq 0$, such that

$$\begin{aligned} \boldsymbol{r}'(t) &= \left(\boldsymbol{w}_0(1-t)^2 + 2\boldsymbol{w}_1(1-t)t + \boldsymbol{w}_2 t^2\right)^2 \\ &= \boldsymbol{w}_0^2 B_0^4(t) + \boldsymbol{w}_0\boldsymbol{w}_1 B_1^4(t) + \frac{2\boldsymbol{w}_1^2 + \boldsymbol{w}_0\boldsymbol{w}_2}{3} B_2^4(t) + \boldsymbol{w}_1\boldsymbol{w}_2 B_3^4(t) + \boldsymbol{w}_2^2 B_4^4(t). \end{aligned} \tag{3}$$

Denoting the $k$-th edge of the Bézier control polygon of $\boldsymbol{r}$ by $\boldsymbol{e}_k = \boldsymbol{p}_{k+1} - \boldsymbol{p}_k$, differentiating (1) for $n = 5$, and using the properties of Bernstein polynomials, the hodograph $\boldsymbol{r}'$ can also be expressed as

$$\boldsymbol{r}'(t) = 5\boldsymbol{e}_0 B_0^4(t) + 5\boldsymbol{e}_1 B_1^4(t) + 5\boldsymbol{e}_2 B_2^4(t) + 5\boldsymbol{e}_3 B_3^4(t) + 5\boldsymbol{e}_4 B_4^4(t). \tag{4}$$

Comparing the coefficients of the Bernstein polynomials in (3) and (4) yields the well-known relations [4] between the control edges $\boldsymbol{e}_k$ of a class I quintic PH curve in Bézier form and the parameters $\boldsymbol{w}_0, \boldsymbol{w}_1, \boldsymbol{w}_2$,

$$5\boldsymbol{e}_0 = \boldsymbol{w}_0^2, \qquad 5\boldsymbol{e}_1 = \boldsymbol{w}_0\boldsymbol{w}_1, \qquad 5\boldsymbol{e}_2 = \frac{2\boldsymbol{w}_1^2 + \boldsymbol{w}_0\boldsymbol{w}_2}{3}, \qquad 5\boldsymbol{e}_3 = \boldsymbol{w}_1\boldsymbol{w}_2, \qquad 5\boldsymbol{e}_4 = \boldsymbol{w}_2^2. \tag{5}$$

## 3 Algebraic Characterization

Based on the relations in (5), Farouki [4] shows that a regular quintic curve $\boldsymbol{r}$ is a PH curve if and only if its control edges satisfy

$$\boldsymbol{e}_0\boldsymbol{e}_3^2 = \boldsymbol{e}_1^2\boldsymbol{e}_4 \tag{6a}$$

and are consistent with the six constraints

$$\begin{aligned} 3\boldsymbol{e}_0\boldsymbol{e}_1\boldsymbol{e}_2 - \boldsymbol{e}_0^2\boldsymbol{e}_3 - 2\boldsymbol{e}_1^3 &= 0, \\ 3\boldsymbol{e}_4\boldsymbol{e}_3\boldsymbol{e}_2 - \boldsymbol{e}_4^2\boldsymbol{e}_1 - 2\boldsymbol{e}_3^3 &= 0, \\ 3\boldsymbol{e}_0\boldsymbol{e}_3\boldsymbol{e}_2 - \boldsymbol{e}_4\boldsymbol{e}_0\boldsymbol{e}_1 - 2\boldsymbol{e}_1^2\boldsymbol{e}_3 &= 0, \\ 3\boldsymbol{e}_4\boldsymbol{e}_1\boldsymbol{e}_2 - \boldsymbol{e}_0\boldsymbol{e}_4\boldsymbol{e}_3 - 2\boldsymbol{e}_3^2\boldsymbol{e}_1 &= 0, \\ 9\boldsymbol{e}_0\boldsymbol{e}_2^2 - 6\boldsymbol{e}_1^2\boldsymbol{e}_2 - 2\boldsymbol{e}_0\boldsymbol{e}_1\boldsymbol{e}_3 - \boldsymbol{e}_0^2\boldsymbol{e}_4 &= 0, \\ 9\boldsymbol{e}_4\boldsymbol{e}_2^2 - 6\boldsymbol{e}_3^2\boldsymbol{e}_2 - 2\boldsymbol{e}_4\boldsymbol{e}_3\boldsymbol{e}_1 - \boldsymbol{e}_4^2\boldsymbol{e}_0 &= 0. \end{aligned} \tag{6b}$$

In the generic case with $\boldsymbol{e}_1$, $\boldsymbol{e}_3$ both non-zero, the condition in (6a) and any one of the first four conditions in (6b) suffice to characterize a PH quintic. In the degenerate case with $\boldsymbol{e}_1$, $\boldsymbol{e}_3$ both zero, the condition in (6a) and the first four conditions in (6b) are automatically satisfied, and either of the last two conditions in (6b) suffices to characterize a PH quintic. Our aim is to unify these conditions by coming up with a set of constraints that does not have to distinguish between the different cases. To this end, let us first express the hodograph in (3) in an alternative way.

**Corollary 1.** *A regular quintic curve* $\boldsymbol{r}$ *is a PH curve, if and only if there exist* $\boldsymbol{a},\boldsymbol{u},\boldsymbol{v}\in\mathbb{C}$ *with* $\boldsymbol{a},\boldsymbol{u}\neq 0$*, such that*

$$\boldsymbol{r}'(t)=5\boldsymbol{a}\left(\boldsymbol{u}(1-t)^2+2\boldsymbol{v}(1-t)t+\frac{1}{\boldsymbol{u}}t^2\right)^2. \tag{7}$$

*Proof.* Expanding (7), we get

$$\boldsymbol{r}'(t)=5\boldsymbol{a}\boldsymbol{u}^2B_0^4(t)+5\boldsymbol{a}\boldsymbol{u}\boldsymbol{v}B_1^4(t)+\frac{5\boldsymbol{a}(2\boldsymbol{v}^2+1)}{3}B_2^4(t)+\frac{5\boldsymbol{a}\boldsymbol{v}}{\boldsymbol{u}}B_3^4(t)+\frac{5\boldsymbol{a}}{\boldsymbol{u}^2}B_4^4(t). \tag{8}$$

To derive the coefficients in (8) from those in (3), we first fix $\boldsymbol{g}=\sqrt{\boldsymbol{w}_0\boldsymbol{w}_2}$ (using any of the two possible square roots), and then set $\boldsymbol{a}=\boldsymbol{g}^2/5=\boldsymbol{w}_0\boldsymbol{w}_2/5$, $\boldsymbol{u}=\boldsymbol{w}_0/\boldsymbol{g}=\boldsymbol{g}/\boldsymbol{w}_2$, and $\boldsymbol{v}=\boldsymbol{w}_1/\boldsymbol{g}$. Vice versa, starting with the coefficients in (8), we get those in (3) by first fixing $\boldsymbol{g}=\sqrt{5\boldsymbol{a}}$ (again, either of the two square roots can be taken), and then setting $\boldsymbol{w}_0=\boldsymbol{g}\boldsymbol{u}$, $\boldsymbol{w}_1=\boldsymbol{g}\boldsymbol{v}$, and $\boldsymbol{w}_2=\boldsymbol{g}/\boldsymbol{u}$. □

Note that the representations in (3) and (8) are not unique: the triplet $(-\boldsymbol{w}_0,-\boldsymbol{w}_1,-\boldsymbol{w}_2)$ leads to the same hodograph as $(\boldsymbol{w}_0,\boldsymbol{w}_1,\boldsymbol{w}_2)$ in (3), and the same holds for the triplets $(\boldsymbol{a},\boldsymbol{u},\boldsymbol{v})$ and $(\boldsymbol{a},-\boldsymbol{u},-\boldsymbol{v})$ in (8).

Comparing, as above, the coefficients in (8) and (4), we find the relations

$$\boldsymbol{e}_0=\boldsymbol{a}\boldsymbol{u}^2,\qquad \boldsymbol{e}_1=\boldsymbol{a}\boldsymbol{u}\boldsymbol{v},\qquad \boldsymbol{e}_2=\boldsymbol{a}\frac{2\boldsymbol{v}^2+1}{3},\qquad \boldsymbol{e}_3=\boldsymbol{a}\frac{\boldsymbol{v}}{\boldsymbol{u}},\qquad \boldsymbol{e}_4=\boldsymbol{a}\frac{1}{\boldsymbol{u}^2} \tag{9}$$

between the control edges $\boldsymbol{e}_k$ of $\boldsymbol{r}$ and the parameters $\boldsymbol{a},\boldsymbol{u},\boldsymbol{v}$, which imply four ways to express $\boldsymbol{a}$ in terms of the control edges,

$$\boldsymbol{a}^2=\boldsymbol{e}_0\boldsymbol{e}_4,\qquad \frac{\boldsymbol{e}_0\boldsymbol{e}_3}{\boldsymbol{e}_1}=\boldsymbol{a}=\frac{\boldsymbol{e}_1\boldsymbol{e}_4}{\boldsymbol{e}_3},\qquad \boldsymbol{a}=3\boldsymbol{e}_2-\left(\frac{\boldsymbol{e}_1^2}{\boldsymbol{e}_0}+\frac{\boldsymbol{e}_3^2}{\boldsymbol{e}_4}\right). \tag{10}$$

While the first expression is ambiguous with respect to the sign of $\boldsymbol{a}$ and the next two are not well-defined if $\boldsymbol{e}_1=0$ or $\boldsymbol{e}_3=0$, respectively, the last expression can always be used to derive the parameter $\boldsymbol{a}$ from the control edges of a regular quintic PH curve.

**Definition 1.** For a regular quintic Bézier curve $\boldsymbol{r}$ with control edges $\boldsymbol{e}_0,\dots,\boldsymbol{e}_4$, where $\boldsymbol{e}_0,\boldsymbol{e}_4\neq 0$, we call

$$\boldsymbol{k}=3\boldsymbol{e}_2-\left(\frac{\boldsymbol{e}_1^2}{\boldsymbol{e}_0}+\frac{\boldsymbol{e}_3^2}{\boldsymbol{e}_4}\right) \tag{11}$$

the *kern*[1] of $\boldsymbol{r}$. If $\boldsymbol{k}=1$, then the curve is said to be in *normal form*.

By Definition 1, every quintic Bézier curve $\boldsymbol{r}$ with a non-vanishing kern and in particular, every regular quintic PH curve, has an associated curve $\tilde{\boldsymbol{r}}$ in normal form,

$$\tilde{\boldsymbol{r}}(t)=\frac{\boldsymbol{r}(t)}{\boldsymbol{k}},\quad t\in\mathbb{R}$$

with control points and control edges given by

$$\tilde{\boldsymbol{p}}_k=\frac{\boldsymbol{p}_k}{\boldsymbol{k}},\quad k=0,\dots,5\qquad\text{and}\qquad \tilde{\boldsymbol{e}}_k=\frac{\boldsymbol{e}_k}{\boldsymbol{k}},\quad k=0,\dots,4. \tag{12}$$

We are now ready to establish our novel algebraic characterization of planar PH quintics. Since the multiplication by a non-zero complex number $\boldsymbol{c}$ applies a rotation by $\arg(\boldsymbol{c})$ and a uniform scaling by $|\boldsymbol{c}|$, transformations that preserve the Pythagorean-hodograph property, we start by considering curves in normal form.

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[1]The term *kern* is inspired by the German phrase “des Pudels Kern” (literally, “the poodle’s core”), referring to the crux or the heart of the matter, because $\boldsymbol{k}$ will turn out to play an essential role in the subsequent analysis. This phrase was used by Goethe in “Faust” when reflecting on the episode in which a black poodle transforms into Mephistopheles, an agent of the devil in disguise. This context suggests to say that a curve with $\boldsymbol{k}=0$ is in *divine form*. However, we shall not pursue such curves any further, since (10) ensures that $\boldsymbol{k}\neq 0$ for regular PH quintics.

**Theorem 2.** *A regular quintic Bézier curve $\tilde{\boldsymbol{r}}$ in normal form is a PH curve, if and only if its control edges satisfy the conditions*

$$\tilde{\boldsymbol{e}}_0\tilde{\boldsymbol{e}}_4 = 1, \tag{13a}$$
$$\tilde{\boldsymbol{e}}_0\tilde{\boldsymbol{e}}_4 = 3\tilde{\boldsymbol{e}}_2 - 2\tilde{\boldsymbol{e}}_1\tilde{\boldsymbol{e}}_3. \tag{13b}$$

*Proof.* Since $\tilde{\boldsymbol{r}}$ is in normal form, we have

$$3\tilde{\boldsymbol{e}}_2 - \left(\frac{\tilde{\boldsymbol{e}}_1^2}{\tilde{\boldsymbol{e}}_0} + \frac{\tilde{\boldsymbol{e}}_3^2}{\tilde{\boldsymbol{e}}_4}\right) = 1, \tag{14}$$

and it follows from Corollary 1 and Equations (9) and (12), that $\tilde{\boldsymbol{r}}$ is a PH curve, if and only if there exist $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{C}$ with $\boldsymbol{u} \neq 0$, such that

$$\tilde{\boldsymbol{e}}_0 = \boldsymbol{u}^2, \qquad \tilde{\boldsymbol{e}}_1 = \boldsymbol{u}\boldsymbol{v}, \qquad \tilde{\boldsymbol{e}}_2 = \frac{2\boldsymbol{v}^2+1}{3}, \qquad \tilde{\boldsymbol{e}}_3 = \frac{\boldsymbol{v}}{\boldsymbol{u}}, \qquad \tilde{\boldsymbol{e}}_4 = \frac{1}{\boldsymbol{u}^2}. \tag{15}$$

To show the sufficiency of the conditions, assume that $\tilde{\boldsymbol{r}}$ is a regular quintic PH curve. Then, by (15), we have

$$\tilde{\boldsymbol{e}}_0\tilde{\boldsymbol{e}}_4 = 1 \quad \text{and} \quad \tilde{\boldsymbol{e}}_1\tilde{\boldsymbol{e}}_3 = \boldsymbol{v}^2 \quad \Longrightarrow \quad 3\tilde{\boldsymbol{e}}_2 = 2\boldsymbol{v}^2 + 1 = 2\tilde{\boldsymbol{e}}_1\tilde{\boldsymbol{e}}_3 + \tilde{\boldsymbol{e}}_0\tilde{\boldsymbol{e}}_4,$$

confirming that both conditions in (13) are satisfied.

Now suppose that the conditions in (13) hold. Condition (13a) implies

$$\tilde{\boldsymbol{e}}_4 = \frac{1}{\tilde{\boldsymbol{e}}_0} \tag{16}$$

and can be used to rewrite (13b) as

$$3\tilde{\boldsymbol{e}}_2 = 2\tilde{\boldsymbol{e}}_1\tilde{\boldsymbol{e}}_3 + 1. \tag{17}$$

Plugging this into (14), we get

$$\frac{\tilde{\boldsymbol{e}}_1^2}{\tilde{\boldsymbol{e}}_0} + \frac{\tilde{\boldsymbol{e}}_3^2}{\tilde{\boldsymbol{e}}_4} = 2\tilde{\boldsymbol{e}}_1\tilde{\boldsymbol{e}}_3. \tag{18}$$

Setting $\boldsymbol{v} = \sqrt{\tilde{\boldsymbol{e}}_1\tilde{\boldsymbol{e}}_3}$ (as above, it does not matter which square root is used), it follows from (17) that

$$\tilde{\boldsymbol{e}}_2 = \frac{2\tilde{\boldsymbol{e}}_1\tilde{\boldsymbol{e}}_3 + 1}{3} = \frac{2\boldsymbol{v}^2+1}{3}.$$

To obtain the other identities in (15), we distinguish two cases. On the one hand, if $\tilde{\boldsymbol{e}}_1 = 0$ or $\tilde{\boldsymbol{e}}_3 = 0$, then (18) implies that both edges must vanish, and since $\boldsymbol{v} = 0$ in this case, we get the identities for $\tilde{\boldsymbol{e}}_1$ and $\tilde{\boldsymbol{e}}_3$ in (15). By (16), the remaining identities for $\tilde{\boldsymbol{e}}_0$ and $\tilde{\boldsymbol{e}}_4$ are then satisfied for $\boldsymbol{u} = \sqrt{\tilde{\boldsymbol{e}}_0}$ (taking either of the two square roots). On the other hand, if $\tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_3 \neq 0$, then $\boldsymbol{v} \neq 0$ and setting $\boldsymbol{u} = \tilde{\boldsymbol{e}}_1/\boldsymbol{v} = \boldsymbol{v}/\tilde{\boldsymbol{e}}_3$ gives

$$\tilde{\boldsymbol{e}}_1 = \boldsymbol{u}\boldsymbol{v} \quad \text{and} \quad \tilde{\boldsymbol{e}}_3 = \frac{\boldsymbol{v}}{\boldsymbol{u}}.$$

It then follows from (18) and (16) that

$$2\boldsymbol{v}^2 = 2\tilde{\boldsymbol{e}}_1\tilde{\boldsymbol{e}}_3 = \frac{\tilde{\boldsymbol{e}}_0\boldsymbol{v}^2}{\boldsymbol{u}^2} + \boldsymbol{u}^2\boldsymbol{v}^2\tilde{\boldsymbol{e}}_4 \quad \Longrightarrow \quad 2 = \frac{\tilde{\boldsymbol{e}}_0}{\boldsymbol{u}^2} + \frac{\boldsymbol{u}^2}{\tilde{\boldsymbol{e}}_0} \quad \Longrightarrow \quad \tilde{\boldsymbol{e}}_0 = \boldsymbol{u}^2 \quad \text{and} \quad \tilde{\boldsymbol{e}}_4 = \frac{1}{\boldsymbol{u}^2},$$

which completes the proof. □

The two conditions in Theorem 2 are symmetric and compact, but we still need to extend them to curves that are not in normal form.

**Corollary 2.** *A regular quintic Bézier curve $\boldsymbol{r}$ is a PH curve, if and only if its control edges satisfy the conditions*

$$\boldsymbol{e}_0\boldsymbol{e}_4 = \boldsymbol{k}^2, \tag{19a}$$
$$\boldsymbol{e}_0\boldsymbol{e}_4 = 3\boldsymbol{k}\boldsymbol{e}_2 - 2\boldsymbol{e}_1\boldsymbol{e}_3, \tag{19b}$$

*where $\boldsymbol{k}$ is the kern of $\boldsymbol{r}$.*

*Proof.* On the one hand, if $\boldsymbol{r}$ is a PH curve, then the control edges $\tilde{\boldsymbol{e}}_k = \boldsymbol{e}_k/\boldsymbol{k}$ of the associated curve $\tilde{\boldsymbol{r}}$ satisfy the conditions in (13) by Theorem 2, and the conditions in (19) follow by substituting $\tilde{\boldsymbol{e}}_k$ with $\boldsymbol{e}_k/\boldsymbol{k}$ and multiplying with $\boldsymbol{k}^2$. On the other hand, if the control edges satisfy (19), then $\boldsymbol{k} \neq 0$, because $\boldsymbol{r}$ is regular with $\boldsymbol{e}_0\boldsymbol{e}_4 \neq 0$. Hence, the control edges $\tilde{\boldsymbol{e}}_k = \boldsymbol{e}_k/\boldsymbol{k}$ of the associated curve $\tilde{\boldsymbol{r}}$ are well-defined and satisfy (13). Therefore, $\tilde{\boldsymbol{r}}$ is a PH quintic by Theorem 2, and so is $\boldsymbol{r} = \boldsymbol{k}\tilde{\boldsymbol{r}}$, because it is similar to $\tilde{\boldsymbol{r}}$. In fact, $\boldsymbol{r}$ is just $\tilde{\boldsymbol{r}}$, rotated and scaled by $\boldsymbol{k}$. □

Note that after using (11) and multiplying (19a) by $\boldsymbol{e}_0^2\boldsymbol{e}_4^2$ and (19b) by $\boldsymbol{e}_0\boldsymbol{e}_4$, we can express the conditions in (19) in terms of polynomials of the edges $\boldsymbol{e}_k$,

$$\boldsymbol{e}_0^3\boldsymbol{e}_4^3 = \left(\boldsymbol{e}_0\boldsymbol{e}_3^2 - 3\boldsymbol{e}_0\boldsymbol{e}_2\boldsymbol{e}_4 + \boldsymbol{e}_1^2\boldsymbol{e}_4\right)^2, \tag{20a}$$

$$\boldsymbol{e}_0^2\boldsymbol{e}_4^2 = -3\boldsymbol{e}_2\left(\boldsymbol{e}_0\boldsymbol{e}_3^2 - 3\boldsymbol{e}_0\boldsymbol{e}_2\boldsymbol{e}_4 + \boldsymbol{e}_1^2\boldsymbol{e}_4\right) - 2\boldsymbol{e}_0\boldsymbol{e}_1\boldsymbol{e}_3\boldsymbol{e}_4, \tag{20b}$$

which reveals that they are of degrees 6 and 4, respectively, compared to the conditions of degree 3 in (6). This increase in degree seems to be the price to pay for combining the conditions for the generic and the degenerate case into one common system of symmetric equations.

We now provide an equivalent algebraic characterization with one more equation that presents more structure, which makes it worthy in its own right.

**Corollary 3.** *A regular quintic Bézier curve* $\boldsymbol{r}$ *is a PH curve, if and only if its control edges satisfy the conditions*

$$4\boldsymbol{e}_0\boldsymbol{e}_1^2\boldsymbol{e}_3^2\boldsymbol{e}_4 = \left(\boldsymbol{e}_0\boldsymbol{e}_3^2 + \boldsymbol{e}_1^2\boldsymbol{e}_4\right)^2, \tag{21a}$$

$$6\boldsymbol{e}_0\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\boldsymbol{e}_4 = (\boldsymbol{e}_0\boldsymbol{e}_4 + 2\boldsymbol{e}_1\boldsymbol{e}_3)(\boldsymbol{e}_0\boldsymbol{e}_3^2 + \boldsymbol{e}_1^2\boldsymbol{e}_4), \tag{21b}$$

$$9\boldsymbol{e}_0\boldsymbol{e}_2^2\boldsymbol{e}_4 = (\boldsymbol{e}_0\boldsymbol{e}_4 + 2\boldsymbol{e}_1\boldsymbol{e}_3)^2. \tag{21c}$$

*Proof.* We start by observing that (19) is equivalent to

$$2\boldsymbol{e}_0\boldsymbol{e}_1\boldsymbol{e}_3\boldsymbol{e}_4 = \boldsymbol{k}(\boldsymbol{e}_0\boldsymbol{e}_3^2 + \boldsymbol{e}_1^2\boldsymbol{e}_4), \tag{22a}$$

$$3\boldsymbol{k}\boldsymbol{e}_2 = \boldsymbol{e}_0\boldsymbol{e}_4 + 2\boldsymbol{e}_1\boldsymbol{e}_3. \tag{22b}$$

Indeed, (22b) is exactly (19b), while, using (19) and (11), we get

$$2\boldsymbol{e}_1\boldsymbol{e}_3 = 3\boldsymbol{k}\boldsymbol{e}_2 - \boldsymbol{e}_0\boldsymbol{e}_4 = \boldsymbol{k}(3\boldsymbol{e}_2 - \boldsymbol{k}) = \boldsymbol{k}\left(\frac{\boldsymbol{e}_1^2}{\boldsymbol{e}_0} + \frac{\boldsymbol{e}_3^2}{\boldsymbol{e}_4}\right),$$

which leads to (22a). The conditions in (21) are then equivalent to those in (22). On the one hand, we get (21a) and (21c) by squaring both sides of (22a) and (22b) and recalling that $\boldsymbol{k}^2 = \boldsymbol{e}_0\boldsymbol{e}_4$. Moreover, condition (21b) is obtained by multiplying the left-hand sides and the right-hand sides of (22a) and (22b). On the other hand, (22a) and (22b) can be derived from (21a) and (21c) by taking the square roots on both sides and (21b) guarantees the correct signs of the square roots. □

### 3.1 Comparison to the state of the art

Comparing the conditions (19) in Corollary 2 and (21) in Corollary 3 with those in (6), it is clear that our algebraic characterizations are symmetric and more compact, in the sense that they consist of only two or three equations that cover all cases instead of different subsets of the seven equations in (6). So let us inspect the conditions a bit closer to comprehend the underlying reason.

We first observe that condition (6a) is not part of the conditions in (19). Indeed, although this condition is necessary, it is too “weak” to be sufficient. On the one hand, it is trivially satisfied in the case $\boldsymbol{e}_1 = \boldsymbol{e}_3 = 0$ and on the other hand, it does not involve $\boldsymbol{e}_2$. Moreover, while (6a) is related to the second and third expression for $\boldsymbol{a}$ in (10), which are not well-defined if $\boldsymbol{e}_1 = \boldsymbol{e}_3 = 0$, the key for deriving our set of conditions is the last expression in (10), which identifies $\boldsymbol{a}$ as the kern of the curve and is valid for *any* regular quintic PH curve.

We further note that (6a) is equivalent to (21a), because $ab = \left(\frac{a+b}{2}\right)^2$ is equivalent to $a = b$, and that Farouki [4, Eq. (51)] identifies (21c) as a necessary condition. But he also shows that it is not sufficient, even in conjunction with (6a), and the missing piece turns out to be (21b). Indeed, expanding (21c), we have

$$9\boldsymbol{e}_0\boldsymbol{e}_2^2\boldsymbol{e}_4 = \boldsymbol{e}_0^2\boldsymbol{e}_4^2 + 4\boldsymbol{e}_0\boldsymbol{e}_1\boldsymbol{e}_3\boldsymbol{e}_4 + 4\boldsymbol{e}_1^2\boldsymbol{e}_3^2 \tag{23}$$

and, after multiplying both sides by $\boldsymbol{e}_1^2$ and using (6a),

$$(3\boldsymbol{e}_0\boldsymbol{e}_2\boldsymbol{e}_3)^2 = (\boldsymbol{e}_0\boldsymbol{e}_1\boldsymbol{e}_4 + 2\boldsymbol{e}_1^2\boldsymbol{e}_3)^2. \tag{24}$$

At this point, we would like to take square roots on both sides to get

$$3\boldsymbol{e}_0\boldsymbol{e}_2\boldsymbol{e}_3 = \boldsymbol{e}_0\boldsymbol{e}_1\boldsymbol{e}_4 + 2\boldsymbol{e}_1^2\boldsymbol{e}_3, \tag{25}$$

that is, the third condition in (6b), but it is not clear why we can rule out

$$-3\boldsymbol{e}_0\boldsymbol{e}_2\boldsymbol{e}_3 = \boldsymbol{e}_0\boldsymbol{e}_1\boldsymbol{e}_4 + 2\boldsymbol{e}_1^2\boldsymbol{e}_3, \tag{26}$$

which is also consistent with (24). This is where (21b) comes into play. In fact, we get (21b) after multiplying both sides of (25) by $\boldsymbol{e}_1\boldsymbol{e}_4$ and using (6a), but (21b) is incompatible with (26). One could say that condition (21b) forces $\boldsymbol{e}_2$ to have the correct sign with respect to the signs of the other edges, while this sign remains ambiguous if we consider only conditions (21a) and (21c). Once the sign is fixed correctly, we can multiply both sides of (25) by $\boldsymbol{e}_3$ and use (6a) to get

$$4\boldsymbol{e}_1^2\boldsymbol{e}_3^2 = 6\boldsymbol{e}_1^2\boldsymbol{e}_2\boldsymbol{e}_4 - 2\boldsymbol{e}_0\boldsymbol{e}_1\boldsymbol{e}_3\boldsymbol{e}_4, \tag{27}$$

and the fifth condition in (6b) then follows from plugging (27) into (23) and dividing both sides by $\boldsymbol{e}_4 \neq 0$. Moreover, we get the first condition in (6b) from (25) if we multiply both sides by $\boldsymbol{e}_1/\boldsymbol{e}_3$ and use (6a). Note that the latter works only if $\boldsymbol{e}_3 \neq 0$, but if $\boldsymbol{e}_3 = 0$, then (6a) implies $\boldsymbol{e}_1 = 0$, because $\boldsymbol{e}_0, \boldsymbol{e}_4 \neq 0$, and the first condition in (6b) is then trivially satisfied. The second, fourth, and sixth condition in (6b) can be derived from (21) with the same arguments after multiplying both sides of (23) by $\boldsymbol{e}_3^2$ and using (6a) to get

$$(3\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_4)^2 = (\boldsymbol{e}_0\boldsymbol{e}_3\boldsymbol{e}_4 + 2\boldsymbol{e}_1\boldsymbol{e}_3^2)^2.$$

### 3.2 Observations

Farouki [4] notes that (6a) stands out from the conditions in (6), because it is invariant under the substitution

$$(\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4) \mapsto (-\boldsymbol{e}_4, -\boldsymbol{e}_3, -\boldsymbol{e}_2, -\boldsymbol{e}_1, -\boldsymbol{e}_0),$$

which corresponds to a reparameterization $t \mapsto 1-t$. The same is true for all conditions in (19), (20), and (21), and they are also invariant under the substitutions

$$(\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4) \mapsto (\boldsymbol{e}_0, -\boldsymbol{e}_1, \boldsymbol{e}_2, -\boldsymbol{e}_3, \boldsymbol{e}_4),$$

and

$$(\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4) \mapsto (-\boldsymbol{e}_0, \boldsymbol{e}_1, -\boldsymbol{e}_2, \boldsymbol{e}_3, -\boldsymbol{e}_4),$$

and any composition of these three substitutions.

An interesting observation is that the conditions in (21) can be expressed compactly in matrix from as

$$\boldsymbol{E}_0\boldsymbol{E}_2^\top = \boldsymbol{E}_1\boldsymbol{E}_1^\top,$$

where

$$\boldsymbol{E}_0 = \begin{pmatrix} 2\boldsymbol{e}_0\boldsymbol{e}_1\boldsymbol{e}_3 \\ 3\boldsymbol{e}_0\boldsymbol{e}_2 \end{pmatrix}, \qquad \boldsymbol{E}_1 = \begin{pmatrix} \boldsymbol{e}_0\boldsymbol{e}_3^2 + \boldsymbol{e}_1^2\boldsymbol{e}_4 \\ 2\boldsymbol{e}_1\boldsymbol{e}_3 + \boldsymbol{e}_0\boldsymbol{e}_4 \end{pmatrix}, \qquad \boldsymbol{E}_2 = \begin{pmatrix} 2\boldsymbol{e}_1\boldsymbol{e}_3\boldsymbol{e}_4 \\ 3\boldsymbol{e}_2\boldsymbol{e}_4 \end{pmatrix},$$

which resembles the algebraic characterization of cubic PH curves with control edges $\boldsymbol{d}_0, \boldsymbol{d}_1, \boldsymbol{d}_2$ [4], namely $\boldsymbol{d}_0\boldsymbol{d}_2 = \boldsymbol{d}_1^2$. Moreover, it is striking that the conditions in (19) and (20) both involve a power of the product between the first and the last control edge, akin to the cubic case, but with more complex right-hand sides.

Note that the conditions in (21) are also satisfied by irregular PH quintics, that is, if the quadratic polynomial $\boldsymbol{w}$ in (2) has one or two real roots. If $\boldsymbol{w}(t)$ does not vanish for $t = 0$ and $t = 1$, so that $\boldsymbol{w}_0\boldsymbol{w}_2 \neq 0$, then this follows from the proof of Theorem 2, since it relies solely on the non-zero condition of $\boldsymbol{w}_0$ and $\boldsymbol{w}_2$. Otherwise, if $\boldsymbol{w}_0 = 0$ or $\boldsymbol{w}_2 = 0$, then (5) implies $\boldsymbol{e}_0 = \boldsymbol{e}_1 = 0$ or $\boldsymbol{e}_3 = \boldsymbol{e}_4 = 0$, and in both cases all three conditions are trivially satisfied. However, the opposite is not true. For example, the edges

$$\boldsymbol{e}_0 = 0, \qquad \boldsymbol{e}_1 = 0, \qquad \boldsymbol{e}_2 = 1, \qquad \boldsymbol{e}_3 = \mathbf{i}, \qquad \boldsymbol{e}_4 = 1$$

satisfy (21), but they do not describe a quintic PH curve.

By construction, the three conditions in (21) are in geometric progression, that is, the ratios between the left-hand sides and the right-hand sides of (21a) and (21b) are the same as those of (21b) and (21c), namely

$$\frac{2\boldsymbol{e}_1\boldsymbol{e}_3}{3\boldsymbol{e}_2} \qquad \text{and} \qquad \frac{\boldsymbol{e}_0\boldsymbol{e}_3^2+\boldsymbol{e}_1^2\boldsymbol{e}_4}{2\boldsymbol{e}_1\boldsymbol{e}_3+\boldsymbol{e}_0\boldsymbol{e}_4},$$

respectively. This shows the dependency among these conditions, at least if $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 \neq 0$. Indeed, in this case (21c) follows from (21a) and (21b), and (21a) follows from (21b) and (21c). Nonetheless, (21b) is not redundant, because it implies constraints on the signs of the square roots of (21a) and (21c). For example, the choice $\boldsymbol{e}_0 = \boldsymbol{e}_1 = \boldsymbol{e}_3 = \boldsymbol{e}_4 = 1$ and $\boldsymbol{e}_2 = -1$ satisfies the first and the third condition, but not the second.

### 3.3 Special cases

Let us now take a look at some special cases of PH quintics and how they affect the algebraic conditions in (19) and (21). We specify these cases in terms of the parameter $\boldsymbol{v}$ from Corollary 1:

(i) $\boldsymbol{v} = 0$

By (9), this case is equivalent to $\boldsymbol{e}_1 = \boldsymbol{e}_3 = 0$, hence $\boldsymbol{k} = 3\boldsymbol{e}_2$, and we conclude that both (19) and (21) reduce to the single condition $9\boldsymbol{e}_2^2 = \boldsymbol{e}_0\boldsymbol{e}_4$, stemming from (19a), (19b), and (21c), while the other conditions in (21) are trivially satisfied. By Corollary 1, this case corresponds to $\boldsymbol{w}_1 = 0$ in (3).

(ii) $\boldsymbol{v} = \pm\mathbf{i}/\sqrt{2}$

By (9), this happens if and only if $\boldsymbol{e}_2 = 0$. In this case, (19b) and (21c) reduce to $2\boldsymbol{e}_1\boldsymbol{e}_3 + \boldsymbol{e}_0\boldsymbol{e}_4 = 0$, implying that (21b) is trivially satisfied, while condition (21a) becomes equivalent to (20a) and thus to (19a). For example, if $\boldsymbol{e}_0, \boldsymbol{e}_4 \neq 0$ are given, then the curve is a PH quintic, if and only if $\boldsymbol{e}_1 = \mathbf{i}^k\boldsymbol{e}_0\sqrt{\boldsymbol{g}/2}$ and $\boldsymbol{e}_3 = (-1)^{k+1}\boldsymbol{e}_1\boldsymbol{g}$ for any $k = 0,1,2,3$ and $\boldsymbol{g} = \sqrt{\boldsymbol{e}_4/\boldsymbol{e}_0}$. Similarly, if $\boldsymbol{e}_1, \boldsymbol{e}_3 \neq 0$ are given, then the conditions are satisfied if $\boldsymbol{e}_0 = (-1)^k\boldsymbol{e}_1\sqrt{-2\boldsymbol{g}}$ and $\boldsymbol{e}_4 = \boldsymbol{e}_0/\boldsymbol{g}^2$ for any $k = 0,1$ and $\boldsymbol{g} = \boldsymbol{e}_1/\boldsymbol{e}_3$. By Corollary 1, this case corresponds to $\boldsymbol{w}_1 = \pm\mathbf{i}\sqrt{\boldsymbol{w}_0\boldsymbol{w}_2/2}$ in (3).

(iii) $\boldsymbol{v} = \pm 1$

By (9), this case corresponds to a sequence of control edges in geometric progression, that is, $\boldsymbol{e}_k = \boldsymbol{e}_{k-1}/\boldsymbol{u} = \boldsymbol{e}_0/\boldsymbol{u}^k$ for $k = 1,2,3,4$ for some $\boldsymbol{u} \in \mathbb{C}\setminus\{0\}$, and it is straightforward to verify that the conditions in (19) and (21) are always satisfied in this case. Hence, while having control edges in geometric progression is a defining property for cubic PH curves, it is only a sufficient condition for quintic PH curves, characterizing a specific subfamily of PH quintics. By Corollary 1, this case corresponds to $\boldsymbol{w}_1 = \pm\sqrt{\boldsymbol{w}_0\boldsymbol{w}_2}$ in (3).

(iv) $\boldsymbol{v} = (\boldsymbol{u} + 1/\boldsymbol{u})/2$ for some $\boldsymbol{u} \in \mathbb{C}\setminus\{0\}$

By (7), this captures the case in which the hodograph simplifies to the square of a linear polynomial, namely $\boldsymbol{r}'(t) = \boldsymbol{a}\big(\boldsymbol{u}(1-t) + t/\boldsymbol{u}\big)^2$. In fact, the curve is a degree-raised cubic PH curve with control edges $\boldsymbol{d}_0 = \frac{5}{3}\boldsymbol{e}_0$, $\boldsymbol{d}_1 = \boldsymbol{d}_0/\boldsymbol{u}^2$, $\boldsymbol{d}_2 = \boldsymbol{d}_1/\boldsymbol{u}^2$. By Corollary 1, this case corresponds to $\boldsymbol{w}_1 = (\boldsymbol{w}_0 + \boldsymbol{w}_2)/2$ in (3).

Note that these special cases, apart from case (iii), have already been identified and described by Farouki [4].

## 4 Geometric Characterization

Farouki [4] notes that condition (6a) has a simple geometric interpretation in terms of the edge lengths $E_k = |\boldsymbol{e}_k|$, $k = 0,\dots,4$ and the interior signed angles $\theta_k = \angle\boldsymbol{p}_{k-1}\boldsymbol{p}_k\boldsymbol{p}_{k+1} = \pi - \arg(\boldsymbol{e}_{k-1}/\boldsymbol{e}_k)$, $k = 1,\dots,4$ of the control polygon, namely

$$\frac{E_1}{E_3} = \sqrt{\frac{E_0}{E_4}} \qquad \text{and} \qquad \theta_1 + \theta_4 = \theta_2 + \theta_3, \tag{28}$$

but the conditions in (6b) do not admit similarly intuitive geometric interpretations.

A more geometric characterization of planar quintic PH curves was later found by Fang and Wang [3, Theorem 1]. Given the control points of a quintic Bézier curve, they first construct the auxiliary points $\boldsymbol{q}_1$ and $\boldsymbol{q}_4$ on the lines $\overline{\boldsymbol{p}_0\boldsymbol{p}_1}$ and $\overline{\boldsymbol{p}_4\boldsymbol{p}_5}$, respectively, such that the lines $\overline{\boldsymbol{q}_1\boldsymbol{p}_2}$ and $\overline{\boldsymbol{p}_3\boldsymbol{q}_4}$ are parallel and the angles $\angle\boldsymbol{p}_0\boldsymbol{q}_1\boldsymbol{p}_2$ and $\angle\boldsymbol{p}_3\boldsymbol{q}_4\boldsymbol{p}_5$ are equal, as well as the two auxiliary points $\boldsymbol{q}_2$ and $\boldsymbol{q}_3$ on the lines $\overline{\boldsymbol{q}_1\boldsymbol{p}_2}$ and $\overline{\boldsymbol{p}_3\boldsymbol{q}_4}$, respectively, such that $\angle\boldsymbol{p}_0\boldsymbol{p}_1\boldsymbol{p}_2 = \angle\boldsymbol{p}_1\boldsymbol{p}_2\boldsymbol{q}_3$ and $\angle\boldsymbol{q}_2\boldsymbol{p}_3\boldsymbol{p}_4 = \angle\boldsymbol{p}_3\boldsymbol{p}_4\boldsymbol{p}_5$ (see Figure 2). Using these four auxiliary points, they then show that the curve is PH, if and only if the quadrilateral $\square(\boldsymbol{p}_2,\boldsymbol{q}_2,\boldsymbol{p}_3,\boldsymbol{q}_3)$ is a parallelogram, the triangles $\triangle(\boldsymbol{p}_1,\boldsymbol{q}_1,\boldsymbol{p}_2)$ and $\triangle(\boldsymbol{p}_3,\boldsymbol{q}_4,\boldsymbol{p}_4)$ are similar, and the edge lengths satisfy

$$2E_1^2=3E_0F, \qquad 2E_3^2=3E_4F, \qquad E_0E_4=9G^2, \tag{29}$$

where $F=|\boldsymbol{f}|$ and $G=|\boldsymbol{g}|$ are the lengths of the parallelogram's edges $\boldsymbol{f}=\boldsymbol{q}_3-\boldsymbol{p}_2=\boldsymbol{p}_3-\boldsymbol{q}_2$ and $\boldsymbol{g}=\boldsymbol{q}_2-\boldsymbol{p}_2=\boldsymbol{p}_3-\boldsymbol{q}_3$.

Our aim is to provide two simpler and entirely geometric characterizations. The first is optimal, in the sense of involving just a single similarity condition for two quadrilaterals. The second is more in the spirit of the well-known geometric characterization for PH cubics and the one proposed by Fang and Wang [3, Theorem 1], in the sense that it is expressed in terms of similarities of triangles, and it leads to a simple way to construct PH quintics starting from two PH cubics. However, this second characterization does not cover the case $\boldsymbol{e}_1=\boldsymbol{e}_3=0$.

The first characterization requires the two auxiliary points

$$\boldsymbol{t}_2=\boldsymbol{p}_3-\boldsymbol{k} \qquad \text{and} \qquad \boldsymbol{t}_3=\boldsymbol{p}_2+\boldsymbol{k}, \tag{30}$$

where $\boldsymbol{k}$ is the kern of the given curve. They can be constructed geometrically in two steps (see Figure 1).

1. Let $\boldsymbol{r}_2$, $\boldsymbol{r}_3$ be two points, such that $\triangle(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{r}_3)$ is similar to $\triangle(\boldsymbol{p}_0,\boldsymbol{p}_1,\boldsymbol{p}_2)$ and $\triangle(\boldsymbol{r}_2,\boldsymbol{p}_3,\boldsymbol{p}_4)$ is similar to $\triangle(\boldsymbol{p}_3,\boldsymbol{p}_4,\boldsymbol{p}_5)$. If $\boldsymbol{e}_1=0$, so that $\triangle(\boldsymbol{p}_0,\boldsymbol{p}_1,\boldsymbol{p}_2)$ is degenerate, then let $\boldsymbol{r}_3=\boldsymbol{p}_2$, and likewise $\boldsymbol{r}_2=\boldsymbol{p}_3$ if $\boldsymbol{e}_3=0$. In any case, the supporting points $\boldsymbol{r}_2,\boldsymbol{r}_3$ can be expressed algebraically as

$$\boldsymbol{r}_2=\boldsymbol{p}_3-\frac{\boldsymbol{e}_3^2}{\boldsymbol{e}_4} \qquad \text{and} \qquad \boldsymbol{r}_3=\boldsymbol{p}_2+\frac{\boldsymbol{e}_1^2}{\boldsymbol{e}_0}. \tag{31}$$

2. Add the vector $\boldsymbol{d}$ from $\boldsymbol{r}_2$ to $\boldsymbol{r}_3$ and $-\boldsymbol{e}_2$ to $\boldsymbol{p}_2$ to get $\boldsymbol{t}_2$. Likewise, add $-\boldsymbol{d}$ and $\boldsymbol{e}_2$ to $\boldsymbol{p}_3$ to get $\boldsymbol{t}_3$, that is,

$$\boldsymbol{t}_2=\boldsymbol{p}_2+(\boldsymbol{r}_3-\boldsymbol{r}_2)-(\boldsymbol{p}_3-\boldsymbol{p}_2) \qquad \text{and} \qquad \boldsymbol{t}_3=\boldsymbol{p}_3-(\boldsymbol{r}_3-\boldsymbol{r}_2)+(\boldsymbol{p}_3-\boldsymbol{p}_2).$$

This construction gives the correct points, because

$$\boldsymbol{t}_2=\boldsymbol{p}_3-3(\boldsymbol{p}_3-\boldsymbol{p}_2)+(\boldsymbol{r}_3-\boldsymbol{p}_2)+(\boldsymbol{p}_3-\boldsymbol{r}_2)=\boldsymbol{p}_3-3\boldsymbol{e}_2+\frac{\boldsymbol{e}_1^2}{\boldsymbol{e}_0}+\frac{\boldsymbol{e}_3^2}{\boldsymbol{e}_4}=\boldsymbol{p}_3-\boldsymbol{k}$$

and similarly for $\boldsymbol{t}_3$. Given $\boldsymbol{t}_2$ and $\boldsymbol{t}_3$, we can now state our first geometric characterization of planar quintic PH curves.

**Theorem 3.** *Let $\boldsymbol{r}$ be a regular quintic curve and let $\boldsymbol{t}_2$, $\boldsymbol{t}_3$ be defined as in (30). Then, $\boldsymbol{r}$ is a PH curve, if and only if*

$$\square(\boldsymbol{p}_0,\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{t}_3) \quad \textit{is similar to} \quad \square(\boldsymbol{t}_2,\boldsymbol{p}_3,\boldsymbol{p}_4,\boldsymbol{p}_5). \tag{32}$$

*Proof.* Assume that $\boldsymbol{r}$ is a PH curve. Then, by (30), (9), and the fact that $\boldsymbol{a}=\boldsymbol{k}\neq 0$, we have

$$\begin{aligned}
\frac{\boldsymbol{a}}{\boldsymbol{e}_0}(\boldsymbol{p}_1-\boldsymbol{p}_0)&=\boldsymbol{a}\frac{\boldsymbol{e}_0}{\boldsymbol{e}_0}=\boldsymbol{a}=\boldsymbol{p}_3-\boldsymbol{t}_2,\\
\frac{\boldsymbol{a}}{\boldsymbol{e}_0}(\boldsymbol{p}_2-\boldsymbol{p}_1)&=\boldsymbol{a}\frac{\boldsymbol{e}_1}{\boldsymbol{e}_0}=\boldsymbol{a}\frac{\boldsymbol{v}}{\boldsymbol{u}}=\boldsymbol{e}_3=\boldsymbol{p}_4-\boldsymbol{p}_3,\\
\frac{\boldsymbol{a}}{\boldsymbol{e}_0}(\boldsymbol{t}_3-\boldsymbol{p}_2)&=\frac{\boldsymbol{a}^2}{\boldsymbol{e}_0}=\boldsymbol{e}_4=\boldsymbol{p}_5-\boldsymbol{p}_4,
\end{aligned}$$

which implies (32), with $\boldsymbol{a}/\boldsymbol{e}_0$ representing the rotation and uniform scaling that maps $\square(\boldsymbol{p}_0,\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{t}_3)$ to $\square(\boldsymbol{t}_2,\boldsymbol{p}_3,\boldsymbol{p}_4,\boldsymbol{p}_5)$.

Conversely, suppose that (32) holds. Then there exists some $\boldsymbol{z}\neq 0$ which represents the similarity transformation, so that

$$\boldsymbol{z}\,\boldsymbol{e}_0=\boldsymbol{z}(\boldsymbol{p}_1-\boldsymbol{p}_0)=\boldsymbol{p}_3-\boldsymbol{t}_2=\boldsymbol{k}, \tag{33a}$$

$$\boldsymbol{z}\,\boldsymbol{e}_1=\boldsymbol{z}(\boldsymbol{p}_2-\boldsymbol{p}_1)=\boldsymbol{p}_4-\boldsymbol{p}_3=\boldsymbol{e}_3, \tag{33b}$$

$$\boldsymbol{z}\,\boldsymbol{k}=\boldsymbol{z}(\boldsymbol{t}_3-\boldsymbol{p}_2)=\boldsymbol{p}_5-\boldsymbol{p}_4=\boldsymbol{e}_4, \tag{33c}$$

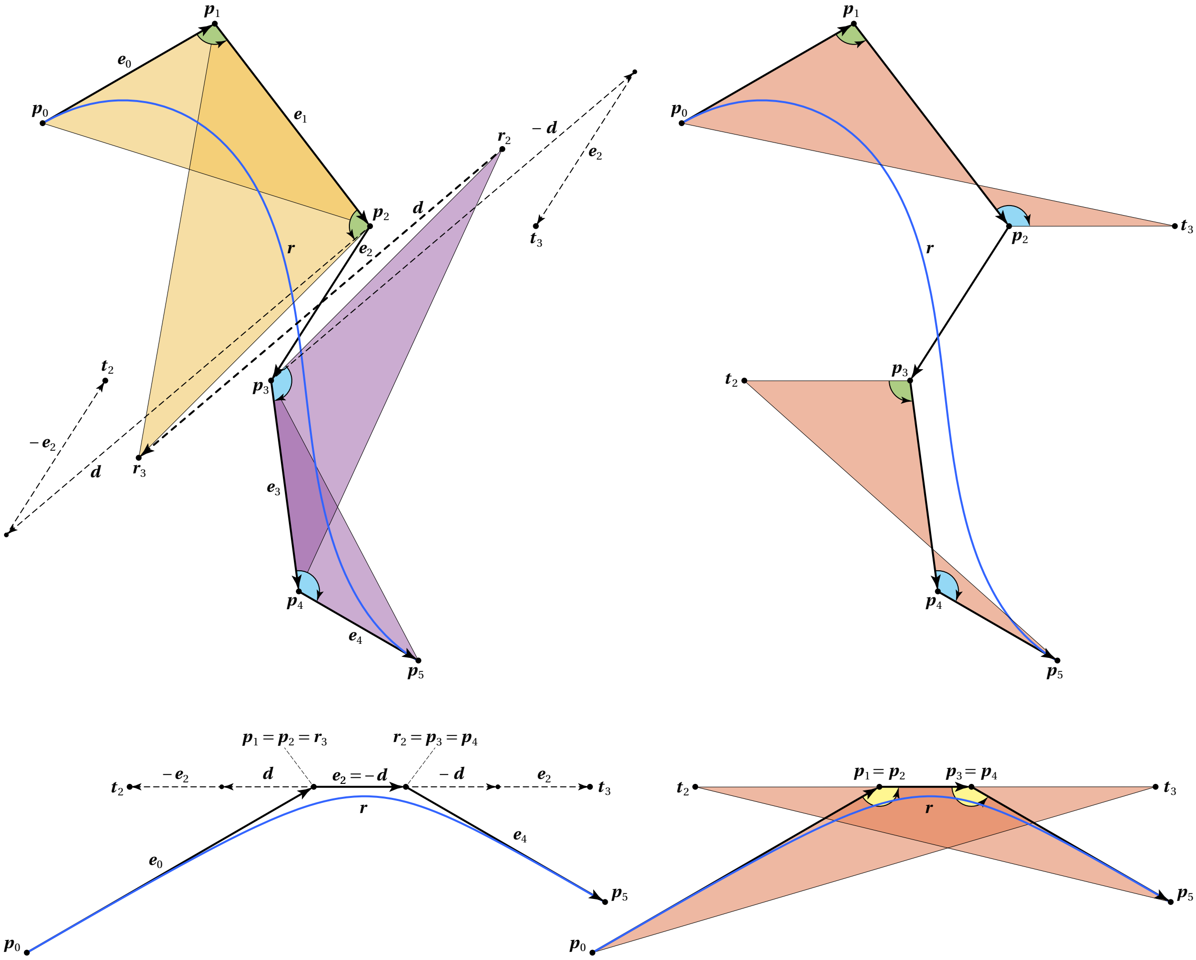


**Figure 1:** Construction of the supporting points $\boldsymbol{r}_2, \boldsymbol{r}_3$ and the auxiliary points $\boldsymbol{t}_2, \boldsymbol{t}_3$ (left) and similar quadrilaterals $\square(\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{t}_3)$ and $\square(\boldsymbol{t}_2, \boldsymbol{p}_3, \boldsymbol{p}_4, \boldsymbol{p}_5)$ used in Theorem 3 (right) for a general quintic PH curve $\boldsymbol{r}$ (top) and a quintic PH curve with $\boldsymbol{e}_1 = \boldsymbol{e}_3 = 0$ (bottom). In the latter case, the quadrilaterals degenerate to similar triangles.

which implies $\boldsymbol{k} \neq 0$, because $\boldsymbol{r}$ is a regular curve with $\boldsymbol{e}_0, \boldsymbol{e}_4 \neq 0$. From (33a) and (33c), we then have

$$\boldsymbol{z} = \frac{\boldsymbol{k}}{\boldsymbol{e}_0} = \frac{\boldsymbol{e}_4}{\boldsymbol{k}} \qquad \Longrightarrow \qquad \boldsymbol{e}_0 \boldsymbol{e}_4 = \boldsymbol{k}^2,$$

and using (33b), we get

$$\frac{\boldsymbol{k}}{\boldsymbol{e}_0}\boldsymbol{e}_1 = \boldsymbol{e}_3 \qquad \text{and} \qquad \frac{\boldsymbol{e}_4}{\boldsymbol{k}}\boldsymbol{e}_1 = \boldsymbol{e}_3 \qquad \Longrightarrow \qquad \boldsymbol{e}_0 \boldsymbol{e}_3 = \boldsymbol{k}\boldsymbol{e}_1 \qquad \text{and} \qquad \boldsymbol{e}_1 \boldsymbol{e}_4 = \boldsymbol{k}\boldsymbol{e}_3.$$

Therefore,

$$\boldsymbol{k}(\boldsymbol{e}_0 \boldsymbol{e}_3^2 + \boldsymbol{e}_1^2 \boldsymbol{e}_4) = 2\boldsymbol{k}^2 \boldsymbol{e}_1 \boldsymbol{e}_3 = 2\boldsymbol{e}_0 \boldsymbol{e}_1 \boldsymbol{e}_3 \boldsymbol{e}_4$$

and, by (11),

$$3\boldsymbol{e}_2 \boldsymbol{k} = \boldsymbol{k}^2 + \boldsymbol{k}\left(\frac{\boldsymbol{e}_1^2}{\boldsymbol{e}_0} + \frac{\boldsymbol{e}_3^2}{\boldsymbol{e}_4}\right) = \boldsymbol{k}^2 + \frac{\boldsymbol{k}(\boldsymbol{e}_0 \boldsymbol{e}_3^2 + \boldsymbol{e}_1^2 \boldsymbol{e}_4)}{\boldsymbol{e}_0 \boldsymbol{e}_4} = \boldsymbol{e}_0 \boldsymbol{e}_4 + 2\boldsymbol{e}_1 \boldsymbol{e}_3,$$

which we recognize as the two conditions in (22). Following the proof of Corollary 2, we conclude that $\boldsymbol{r}$ is a PH curve. □

Figure 1 visualizes the result of Theorem 3. Note that the theorem holds even if the two involved quadrilaterals self-intersect or degenerate to triangles.

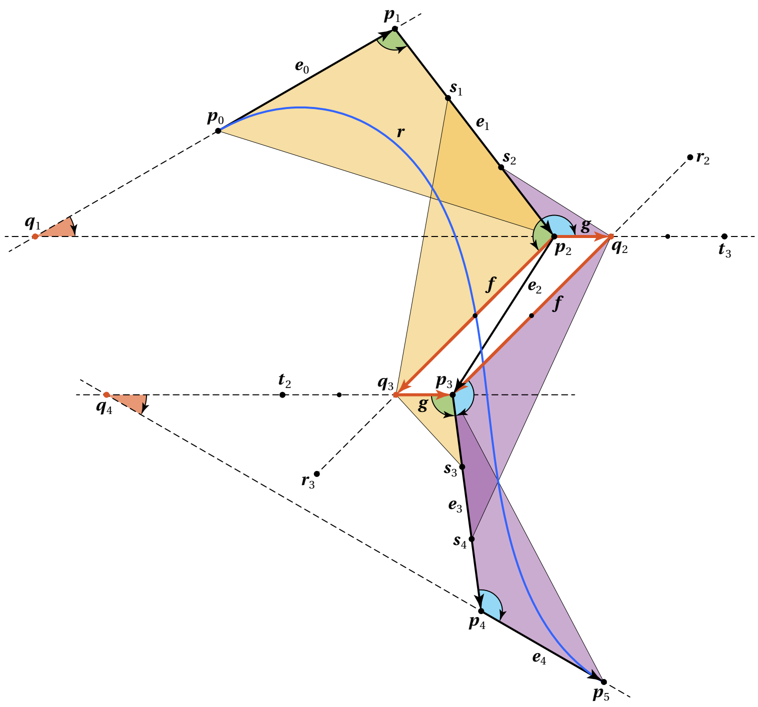


**Figure 2:** Control points $\boldsymbol{p}_k$, control edges $\boldsymbol{e}_k$, supporting points $\boldsymbol{r}_2, \boldsymbol{r}_3$, auxiliary points $\boldsymbol{q}_k$, split points $\boldsymbol{s}_k$, parallelogram (red) with edges $\boldsymbol{f}, \boldsymbol{g}$, and families of similar triangles (yellow, purple) used by Fang and Wang [3, Theorem 1] and in Corollary 4, for the quintic PH curve $\boldsymbol{r}$ defined by (3) with $\boldsymbol{w}_0 = \sqrt{6}e^{\pi\mathbf{i}/12}$, $\boldsymbol{w}_1 = \pi e^{-3\pi\mathbf{i}/8}$, and $\boldsymbol{w}_2 = 5e^{-\pi\mathbf{i}/12}/\sqrt{6}$. The signed interior angles $\theta_1$ and $\theta_4$ of the control polygon are marked in green and blue, respectively.

For the second characterization, let us split the edges $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$ into three equal parts by introducing the points

$$\boldsymbol{s}_1 = \frac{2\boldsymbol{p}_1 + \boldsymbol{p}_2}{3}, \qquad \boldsymbol{s}_2 = \frac{\boldsymbol{p}_1 + 2\boldsymbol{p}_2}{3}, \qquad \boldsymbol{s}_3 = \frac{2\boldsymbol{p}_3 + \boldsymbol{p}_4}{3}, \qquad \boldsymbol{s}_4 = \frac{\boldsymbol{p}_3 + 2\boldsymbol{p}_4}{3}. \tag{34}$$

It then follows from (29) that, in case of the curve being PH, the two triangles $\triangle(\boldsymbol{s}_1, \boldsymbol{p}_2, \boldsymbol{q}_3)$ and $\triangle(\boldsymbol{q}_3, \boldsymbol{p}_3, \boldsymbol{s}_3)$ are similar to $\triangle(\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2)$, and the two triangles $\triangle(\boldsymbol{q}_2, \boldsymbol{p}_3, \boldsymbol{s}_4)$ and $\triangle(\boldsymbol{s}_2, \boldsymbol{p}_2, \boldsymbol{q}_2)$ are similar to $\triangle(\boldsymbol{p}_3, \boldsymbol{p}_4, \boldsymbol{p}_5)$, where $\boldsymbol{q}_2, \boldsymbol{q}_3$ are the auxiliary points from Fang and Wang [3, Theorem 1]. Moreover, our construction of the supporting points $\boldsymbol{r}_2, \boldsymbol{r}_3$ implies that

$$\boldsymbol{q}_2 = \frac{2\boldsymbol{r}_2 + \boldsymbol{p}_3}{3} \qquad \text{and} \qquad \boldsymbol{q}_3 = \frac{\boldsymbol{p}_2 + 2\boldsymbol{r}_3}{3}, \tag{35}$$

where $\boldsymbol{r}_2$ and $\boldsymbol{r}_3$ are defined as in (31). This observation gives rise to our second geometric characterization of planar quintic PH curves (see Figure 2).

**Corollary 4.** *Let $\boldsymbol{r}$ be a regular quintic curve with $\boldsymbol{e}_1, \boldsymbol{e}_3 \neq 0$, let $\boldsymbol{s}_2, \boldsymbol{s}_3$ be defined as in* (34)*, and let $\boldsymbol{q}_2, \boldsymbol{q}_3$ be defined as in* (35)*. Then, $\boldsymbol{r}$ is a PH curve, if and only if*

$$\triangle(\boldsymbol{q}_3, \boldsymbol{p}_3, \boldsymbol{s}_3) \quad \textit{is similar to} \quad \triangle(\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2), \tag{36a}$$

$$\triangle(\boldsymbol{s}_2, \boldsymbol{p}_2, \boldsymbol{q}_2) \quad \textit{is similar to} \quad \triangle(\boldsymbol{p}_3, \boldsymbol{p}_4, \boldsymbol{p}_5). \tag{36b}$$

*Proof.* By construction, the auxiliary points $\boldsymbol{q}_2, \boldsymbol{q}_3$ can be expressed as

$$\boldsymbol{q}_2 = \boldsymbol{p}_3 - \frac{2\boldsymbol{e}_3^2}{3\boldsymbol{e}_4} \qquad \text{and} \qquad \boldsymbol{q}_3 = \boldsymbol{p}_2 + \frac{2\boldsymbol{e}_1^2}{3\boldsymbol{e}_0}. \tag{37}$$

If $\boldsymbol{r}$ is a PH curve, then it follows from (6a) that $\boldsymbol{e}_1^2/\boldsymbol{e}_0 = \boldsymbol{e}_3^2/\boldsymbol{e}_4$, hence, by (11) and (37),

$$\boldsymbol{k} = 3\boldsymbol{e}_2 - 2\frac{\boldsymbol{e}_1^2}{\boldsymbol{e}_0} = 3\big((\boldsymbol{p}_3 - \boldsymbol{p}_2) + (\boldsymbol{p}_2 - \boldsymbol{q}_3)\big) = 3(\boldsymbol{p}_3 - \boldsymbol{q}_3)$$

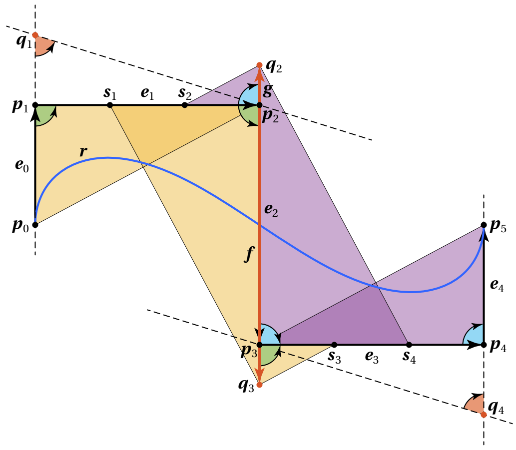


**Figure 3:** Control points, control edges, auxiliary points, split points, degenerate parallelogram, and families of similar triangles (cf. Figure 2) for the quintic PH curve $\boldsymbol{r}$ defined by (3) with $\boldsymbol{w}_0 = \boldsymbol{w}_2 = \sqrt{15}(1+\mathbf{i})$ and $\boldsymbol{w}_1 = \sqrt{105/2}(1-\mathbf{i})$. The construction by Fang and Wang [3, Theorem 1] fails for this example.

and, using (30),

$$\boldsymbol{q}_3 = \frac{3\boldsymbol{p}_3 - \boldsymbol{k}}{3} = \frac{2\boldsymbol{p}_3 + \boldsymbol{t}_2}{3}.$$

Therefore, $\triangle(\boldsymbol{q}_3, \boldsymbol{p}_3, \boldsymbol{s}_3)$ is similar to $\triangle(\boldsymbol{t}_2, \boldsymbol{p}_3, \boldsymbol{p}_4)$, which in turn is similar to $\triangle(\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2)$ by Theorem 3. This confirms (36a), and (36b) can be derived analogously.

Now suppose that the conditions in (36) hold. Then, since $\triangle(\boldsymbol{s}_1, \boldsymbol{p}_2, \boldsymbol{q}_3)$ is similar to $\triangle(\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2)$ by construction and therefore similar to $\triangle(\boldsymbol{q}_3, \boldsymbol{p}_3, \boldsymbol{s}_3)$ by (36a), and likewise $\triangle(\boldsymbol{q}_2, \boldsymbol{p}_3, \boldsymbol{s}_4)$ is similar to $\triangle(\boldsymbol{s}_2, \boldsymbol{p}_2, \boldsymbol{q}_2)$, we have, since $\boldsymbol{e}_1, \boldsymbol{e}_3 \neq 0$,

$$\frac{\boldsymbol{q}_3 - \boldsymbol{p}_2}{2\boldsymbol{e}_1/3} = \frac{\boldsymbol{q}_3 - \boldsymbol{p}_2}{\boldsymbol{p}_2 - \boldsymbol{s}_1} = \frac{\boldsymbol{s}_3 - \boldsymbol{p}_3}{\boldsymbol{p}_3 - \boldsymbol{q}_3} = \frac{\boldsymbol{e}_3/3}{\boldsymbol{p}_3 - \boldsymbol{q}_3} \qquad \text{and} \qquad \frac{2\boldsymbol{e}_3/3}{\boldsymbol{p}_3 - \boldsymbol{q}_2} = \frac{\boldsymbol{s}_4 - \boldsymbol{p}_3}{\boldsymbol{p}_3 - \boldsymbol{q}_2} = \frac{\boldsymbol{q}_2 - \boldsymbol{p}_2}{\boldsymbol{p}_2 - \boldsymbol{s}_2} = \frac{\boldsymbol{q}_2 - \boldsymbol{p}_2}{\boldsymbol{e}_1/3}.$$

Consequently,

$$(\boldsymbol{q}_3 - \boldsymbol{p}_2)(\boldsymbol{p}_3 - \boldsymbol{q}_3) = \frac{2\boldsymbol{e}_1\boldsymbol{e}_3}{9} = (\boldsymbol{q}_2 - \boldsymbol{p}_2)(\boldsymbol{p}_3 - \boldsymbol{q}_2) \qquad \Longrightarrow \qquad \frac{\boldsymbol{q}_3 - \boldsymbol{p}_2}{\boldsymbol{p}_2 - \boldsymbol{q}_2} = \frac{\boldsymbol{q}_2 - \boldsymbol{p}_3}{\boldsymbol{p}_3 - \boldsymbol{q}_3},$$

and so the triangles $\triangle(\boldsymbol{q}_2, \boldsymbol{p}_2, \boldsymbol{q}_3)$ and $\triangle(\boldsymbol{q}_3, \boldsymbol{p}_3, \boldsymbol{q}_2)$ are congruent, because they share the edge $\overline{\boldsymbol{q}_2\boldsymbol{q}_3}$. Therefore, $\boldsymbol{q}_3 = \boldsymbol{p}_2 + (\boldsymbol{p}_3 - \boldsymbol{q}_2)$, and, using (37), (11), and (30), we get

$$\boldsymbol{q}_3 = \frac{1}{2}\boldsymbol{q}_3 + \frac{1}{2}\big(\boldsymbol{p}_2 + (\boldsymbol{p}_3 - \boldsymbol{q}_2)\big) = \frac{1}{2}\boldsymbol{p}_2 + \frac{\boldsymbol{e}_1^2}{3\boldsymbol{e}_0} + \frac{1}{2}\boldsymbol{p}_2 + \frac{\boldsymbol{e}_3^2}{3\boldsymbol{e}_4} = \boldsymbol{p}_3 - \frac{1}{3}\left(3\boldsymbol{e}_2 - \frac{\boldsymbol{e}_1^2}{\boldsymbol{e}_0} - \frac{\boldsymbol{e}_3^2}{\boldsymbol{e}_4}\right) = \frac{3\boldsymbol{p}_3 - \boldsymbol{k}}{3} = \frac{2\boldsymbol{p}_3 + \boldsymbol{t}_2}{3}.$$

This implies that $\triangle(\boldsymbol{t}_2, \boldsymbol{p}_3, \boldsymbol{p}_4)$ is similar to $\triangle(\boldsymbol{q}_3, \boldsymbol{p}_3, \boldsymbol{s}_3)$, which in turn is similar to $\triangle(\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2)$ by (36a). Analogously, we can show that $\triangle(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{t}_3)$ is similar to $\triangle(\boldsymbol{p}_3, \boldsymbol{p}_4, \boldsymbol{p}_5)$. Therefore, $\square(\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{t}_3)$ is similar to $\square(\boldsymbol{t}_2, \boldsymbol{p}_3, \boldsymbol{p}_4, \boldsymbol{p}_5)$ and $\boldsymbol{r}$ is a PH curve by Theorem 3. □

### 4.1 Comparison to the state of the art

Comparing our conditions (32) in Theorem 3 and (36) in Corollary 4 with the ones by Fang and Wang [3, Theorem 1], it is clear that ours are simpler and more compact. On the one hand, we need only two instead of four auxiliary points for Theorem 3, and the construction of $\boldsymbol{s}_2, \boldsymbol{s}_3, \boldsymbol{q}_2, \boldsymbol{q}_3$ for Corollary 4 is simpler than the construction of $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_4$ described by Fang and Wang. On the other hand, our characterizations are entirely geometric and do not involve conditions on edge lengths. Moreover, Theorem 3 works for all regular quintic PH curves and Corollary 4 for curves with $\boldsymbol{e}_1, \boldsymbol{e}_3 \neq 0$, while the construction by Fang and Wang can fail even in the latter case.

For example, if we consider the quintic PH curve in Figure 3 with Bézier control points

$$\boldsymbol{p}_0 = -\sqrt{126}, \qquad \boldsymbol{p}_1 = -\sqrt{126} + 6\mathbf{i}, \qquad \boldsymbol{p}_2 = 6\mathbf{i}, \qquad \boldsymbol{p}_3 = -6\mathbf{i}, \qquad \boldsymbol{p}_4 = \sqrt{126} - 6\mathbf{i}, \qquad \boldsymbol{p}_5 = \sqrt{126},$$

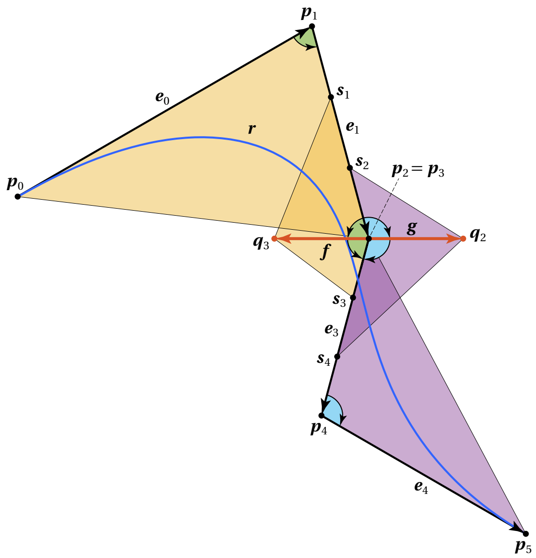


**Figure 4:** Control points, control edges, auxiliary points, split points, degenerate parallelogram, and families of similar triangles (cf. Figure 2) for the quintic PH curve $\boldsymbol{r}$ with $\boldsymbol{e}_2 = 0$ defined by (3) with $\boldsymbol{w}_0 = \sqrt{6}e^{\pi\mathbf{i}/12}$, $\boldsymbol{w}_1 = -\mathbf{i}\sqrt{5/2}$, and $\boldsymbol{w}_2 = 5e^{-\pi\mathbf{i}/12}/\sqrt{6}$.

then $\boldsymbol{q}_1 = \boldsymbol{p}_0 + \lambda\mathbf{i}$ and $\boldsymbol{q}_4 = \boldsymbol{p}_5 - \lambda\mathbf{i}$ satisfy their conditions for these two auxiliary points for any $\lambda \in \mathbb{R}$, and their subsequent construction leads to $\boldsymbol{q}_2 = \boldsymbol{p}_2$ and $\boldsymbol{q}_3 = \boldsymbol{p}_3$, independently of $\lambda$. However, the edge length conditions in (29) are not satisfied for this choice of $\boldsymbol{q}_2$ and $\boldsymbol{q}_3$. Instead, our construction delivers the correct auxiliary points $\boldsymbol{q}_2 = 8\mathbf{i}$ and $\boldsymbol{q}_3 = -8\mathbf{i}$.

For those cases that are covered by the approach of Fang and Wang, their conditions follow easily from Corollary 4 (see Figure 2). We first obtain $\boldsymbol{q}_2$ and $\boldsymbol{q}_3$ as in (35) and then construct $\boldsymbol{q}_1$ by intersecting the lines $\overline{\boldsymbol{p}_0\boldsymbol{p}_1}$ and $\overline{\boldsymbol{p}_2\boldsymbol{q}_2}$ and $\boldsymbol{q}_4$ by intersecting $\overline{\boldsymbol{p}_4\boldsymbol{p}_5}$ and $\overline{\boldsymbol{p}_3\boldsymbol{q}_3}$. The resulting triangles $\triangle(\boldsymbol{p}_1, \boldsymbol{q}_1, \boldsymbol{p}_2)$ and $\triangle(\boldsymbol{p}_3, \boldsymbol{q}_4, \boldsymbol{p}_4)$ are similar, because they share the interior angle $\theta_1$ at $\boldsymbol{p}_1$ and $\boldsymbol{p}_3$ and the exterior angle $\theta_4$ at $\boldsymbol{p}_2$ and $\boldsymbol{p}_4$. Moreover, as shown in the proof of Corollary 4, the similarity conditions (36a) and (36b) imply the congruence of the triangles $\triangle(\boldsymbol{q}_2, \boldsymbol{p}_2, \boldsymbol{q}_3)$ and $\triangle(\boldsymbol{q}_3, \boldsymbol{p}_3, \boldsymbol{q}_2)$, which means that $\square(\boldsymbol{p}_2, \boldsymbol{q}_2, \boldsymbol{p}_3, \boldsymbol{q}_3)$ is a parallelogram, and it further follows from these conditions that the lengths of the edges $\boldsymbol{f} = \boldsymbol{q}_3 - \boldsymbol{p}_2 = \boldsymbol{p}_3 - \boldsymbol{q}_2$ and $\boldsymbol{g} = \boldsymbol{q}_2 - \boldsymbol{p}_2 = \boldsymbol{p}_3 - \boldsymbol{q}_3$ of this parallelogram are $F = |\boldsymbol{f}| = \frac{2}{3}E_1^2/E_0 = \frac{2}{3}E_3^2/E_4$ and $G = |\boldsymbol{g}| = \frac{1}{3}E_1E_4/E_3 = \frac{1}{3}E_3E_0/E_1$, thus giving (29). Note that our $\boldsymbol{q}_2$ and $\boldsymbol{q}_3$ in (37) can be different from the auxiliary points given by the geometric construction in [3] for a generic, non-PH quintic Bézier curve.

Farouki's geometric interpretation of the algebraic condition (6a) can be derived from (36), too. The length condition in (28) follows directly from the previously mentioned identities for $F$, and the angle condition holds, because $\theta_2 = \theta_1 + \alpha$ and $\theta_3 = \theta_4 - \alpha$, where $\alpha = \angle\boldsymbol{q}_3\boldsymbol{p}_2\boldsymbol{p}_3 = \angle\boldsymbol{q}_2\boldsymbol{p}_3\boldsymbol{p}_2$ (see Figure 2).

### 4.2 Observations

Corollary 4 also holds if $\boldsymbol{e}_2 = 0$, so that the parallelogram $\square(\boldsymbol{p}_2, \boldsymbol{q}_2, \boldsymbol{p}_3, \boldsymbol{q}_3)$ degenerates to a segment (see Figure 4), but it cannot be applied if $\boldsymbol{e}_1 = \boldsymbol{e}_3 = 0$. In the latter case (see Figure 1, bottom), the similarity condition (32) in Theorem 3 degenerates to the condition

$$\triangle(\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{t}_3) \quad \text{is similar to} \quad \triangle(\boldsymbol{t}_2, \boldsymbol{p}_4, \boldsymbol{p}_5),$$

where

$$\boldsymbol{t}_2 = \boldsymbol{p}_4 - 3\boldsymbol{e}_2 \qquad \text{and} \qquad \boldsymbol{t}_3 = \boldsymbol{p}_1 + 3\boldsymbol{e}_2,$$

which is equivalent to the algebraic condition $9\boldsymbol{e}_2^2 = \boldsymbol{e}_0\boldsymbol{e}_4$ that we encountered in case (i) in Section 3.3.

This actually provides a recipe for generating a quintic PH curve from a cubic PH curve: all we need to do is to stretch the first and the last control edge by a factor of three. Indeed, if a cubic PH curve with control edges $\boldsymbol{d}_0, \boldsymbol{d}_1, \boldsymbol{d}_2$ is given, then it follows from the observations above that the quintic curve with control edges $\boldsymbol{e}_0 = 3\boldsymbol{d}_0$, $\boldsymbol{e}_1 = 0$, $\boldsymbol{e}_2 = \boldsymbol{d}_1$, $\boldsymbol{e}_3 = 0$, $\boldsymbol{e}_4 = 3\boldsymbol{d}_2$ is PH (see Figure 5).

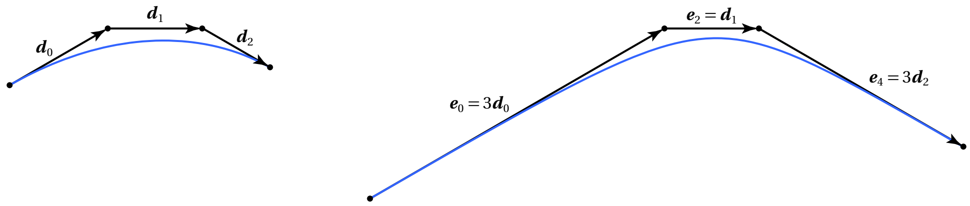


**Figure 5:** A cubic PH curve (left) can be turned into a degenerate quintic PH curve (right) with $\boldsymbol{e}_1 = \boldsymbol{e}_3 = 0$ by stretching the first and the last leg by a factor of three.

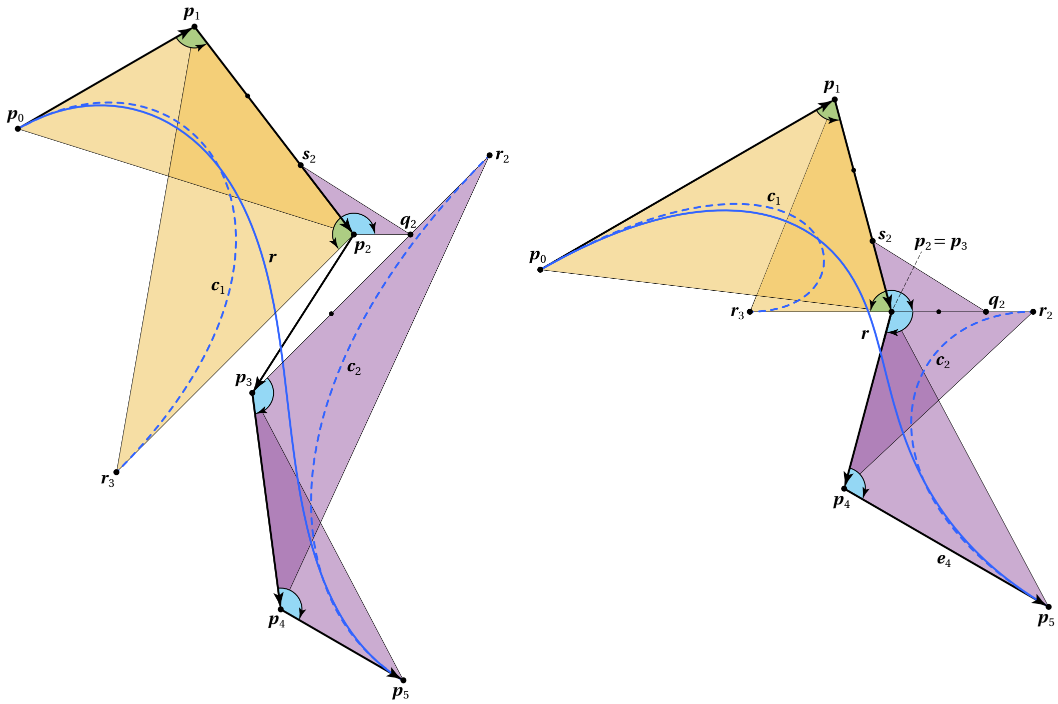

**Figure 6:** Quintic PH curves $\boldsymbol{r}$ from Figure 2 (left) and Figure 4 (right), together with their cubic PH companion curves $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$.

More interestingly, it follows from the construction of the supporting points $\boldsymbol{r}_2$, $\boldsymbol{r}_3$ in (31) that a regular quintic PH curve with $\boldsymbol{e}_1, \boldsymbol{e}_3 \neq 0$ is intimately related to *two* cubic PH curves. Indeed, it is clear that $\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{r}_3$ and $\boldsymbol{r}_2, \boldsymbol{p}_3, \boldsymbol{p}_4, \boldsymbol{p}_5$ are the control points of two cubic PH *companion curves* $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$, where the last control edge of $\boldsymbol{c}_1$ and the first control edge of $\boldsymbol{c}_2$ are translates of each other. Vice versa, given two cubic PH curves $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ with control points $\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{r}_3$ and $\boldsymbol{r}_2, \boldsymbol{p}_3, \boldsymbol{p}_4, \boldsymbol{p}_5$, we can generate a quintic PH curve by first rotating and scaling $\boldsymbol{c}_2$ such that its first control edge is parallel to and has the same length as the last control edge of $\boldsymbol{c}_1$, that is, $\boldsymbol{r}_3 - \boldsymbol{p}_2 = \boldsymbol{p}_3 - \boldsymbol{r}_2$, and then translating $\boldsymbol{c}_2$ such that the triangles $\triangle(\boldsymbol{s}_2, \boldsymbol{p}_2, \boldsymbol{q}_2)$ and $\triangle(\boldsymbol{p}_3, \boldsymbol{p}_4, \boldsymbol{p}_5)$ are similar, where $\boldsymbol{s}_2 = (\boldsymbol{p}_1 + 2\boldsymbol{p}_2)/3$ and $\boldsymbol{q}_2 = (2\boldsymbol{r}_2 + \boldsymbol{p}_3)/3$ (see Figure 6). Alternatively, the correct translation can also be determined by forcing the triangles $\triangle(\boldsymbol{q}_3, \boldsymbol{p}_3, \boldsymbol{s}_3)$ and $\triangle(\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2)$ to be similar, where $\boldsymbol{s}_3 = (2\boldsymbol{p}_3 + \boldsymbol{p}_4)/3$ and $\boldsymbol{q}_3 = (2\boldsymbol{r}_3 + \boldsymbol{p}_2)/3$.

## 5 Conclusions

One potential application of the algebraic characterization in Corollary 2 is for deciding whether a given planar quintic curve, specified by its Bézier control points, is a PH curve or not. We first convert the control edges of the curve into complex numbers. We then use complex arithmetic to compute the kern of the curve as in (11) and to check if the two conditions (19) are satisfied up to some tolerance. It remains future work to analyse this approach in detail and to compare it to the methods developed by Farouki et al. [6], which are based on real arithmetic and can detect planar as well as spatial PH curves. It would further be interesting to see if there is any numerical advantage in deriving the coefficients $\boldsymbol{w}_0, \boldsymbol{w}_1, \boldsymbol{w}_2$ of the pre-image polynomial $\boldsymbol{w}$

that generates the hodograph of the curve from the conditions in (9) and Corollary 1 instead of using the conditions in (5), as suggested by Farouki et al. [6].

In addition to verifying (19) for a given curve, these constraints can also be used for constructing quintic PH curves. For example, the first-order Hermite interpolation problem fixes the edges $\boldsymbol{e}_0$ and $\boldsymbol{e}_4$ and the sum of all edges, and the conditions in (19) can be used to derive formulas for the remaining edges $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$. However, the resulting expressions are not simpler than those derived by Farouki and Neff [8] and Dong and Farouki [2], and the same holds for the other variants of this problem, where different pairs of edges are fixed [9].


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