# Barycentric Interpolation

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**Abstract** This survey focusses on the method of barycentric interpolation, which ties up to the ideas that August Ferdinand Möbius published in his seminal work "Der barycentrische Calcul" in 1827. For univariate data, it leads to a special kind of rational interpolation which is guaranteed to have no poles and favourable approximation properties. We further discuss how to extend this idea to bivariate data, both for scattered data and for data given at the vertices of a polygon.

## 1 Introduction

Consider a system of $n+1$ particles, located at $x_0,\ldots,x_n$ and with masses $w_0,\ldots,w_n$. It is then well-known from physics that the *centre of mass* or *barycentre* of this particle system is the unique point $x$ which satisfies

$$\sum_{i=0}^{n} w_i(x-x_i) = 0,$$

that is,

$$x = \frac{\sum_{i=0}^{n} w_i x_i}{\sum_{i=0}^{n} w_i}.$$

The idea of barycentric interpolation stems from this concept, by asking the question: given a fixed set of distinct locations or *nodes* $x_0,\ldots,x_n$ and an arbitrary point $x$, do there exist some masses or *weights* $w_0,\ldots,w_n$, such that $x$ is the barycentre of the corresponding particle system? Consequently, we are interested in functions $w_0(x),\ldots,w_n(x)$, such that

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$$x = \frac{\sum_{i=0}^{n} w_i(x) x_i}{\sum_{i=0}^{n} w_i(x)}. \tag{1}$$

Möbius [24] was probably the first to answer this question in full generality. He showed that for particle systems in $\mathbb{R}^m$ such weights always exist[1] for any $x \in \mathbb{R}^m$, as long as the number of particles is greater than the dimension, that is, for $n \geq m$. Möbius called the weights $w_0(x), \dots, w_n(x)$ the *barycentric coordinates* of $x$ with respect to $x_0, \dots, x_n$.

It is clear that barycentric coordinates are *homogeneous* in the sense that they can be multiplied with a common non-zero scalar and still satisfy (1). In the context of barycentric interpolation we therefore assume without loss of generality that the barycentric coordinates sum to one for any $x$. We further demand that they are 1 at the corresponding node and 0 at all other nodes. The resulting *barycentric basis functions* $b_i \colon \mathbb{R}^m \to \mathbb{R}$, $i = 0, \dots, n$ are then characterized by the three properties

Partition of unity: $$\sum_{i=0}^{n} b_i(x) = 1, \tag{2a}$$

Barycentric property: $$\sum_{i=0}^{n} b_i(x) x_i = x, \tag{2b}$$

Lagrange property: $$b_i(x_j) = \delta_{ij}, \tag{2c}$$

where (2b) is equivalent to (1) because of (2a). Möbius observed that these barycentric basis functions are unique in the special case $n = m$, when the nodes $x_0, \dots, x_n$ can be considered the vertices of an $m$-simplex, and he gave an explicit formula for $b_i$ in this case, which reveals that $b_i$ is a linear function.

Let us now consider data $f_0, \dots, f_n$ corresponding to the nodes $x_0, \dots, x_n$ and possibly sampled from some function $f \colon \mathbb{R}^m \to \mathbb{R}$, that is, $f_i = f(x_i)$ for $i = 0, \dots, n$. The *barycentric interpolant* of this data is then given by

$$F(x) = \sum_{i=0}^{n} b_i(x) f_i. \tag{3}$$

It follows from (2c) that the function $F \colon \mathbb{R}^m \to \mathbb{R}$ interpolates the data $f_i$ at $x_i$ for $i = 0, \dots, n$, and from (2a) and (2b) that this kind of interpolation reproduces linear functions. That is, if the data $f_0, \dots, f_n$ are sampled from a linear polynomial $f \in \Pi_1$, where $\Pi_d$ denotes the space of polynomials with degree at most $d$, then $F = f$.

Vice versa, if an interpolation operator reproduces linear functions, then its cardinal basis functions clearly satisfy the three conditions in (2). Therefore, many classical interpolation methods, including interpolation with splines, radial basis functions, and subdivision schemes, just to name a few, could be called barycentric. However, we prefer to use the term *barycentric interpolation* whenever simple closed-form expressions for the barycentric basis functions $b_i$ exist, so that evaluating the interpolant (3) is efficient.

[1] Note that at least one of the $w_i(x)$ must be negative if $x$ is outside the *convex hull* of the nodes $x_0, \dots, x_n$, which is physically impossible and motivates to call the $w_i$ weights rather than masses.

In this survey, we review recent progress in the construction of such barycentric basis functions and the related interpolants. We mainly focus on the univariate setting in Section 2, but also summarize some results on scattered data interpolation in two variables in Section 3. The special case of barycentric interpolation at the vertices of a polygon in $\mathbb{R}^2$ is only briefly discussed in Section 4, as more details can be found in [12].

## 2 Univariate barycentric interpolation

Suppose we are given two distinct nodes $x_0, x_1 \in \mathbb{R}$. Then it is clear that the two functions $b_0, b_1 \colon \mathbb{R} \to \mathbb{R}$ with

$$b_0(x) = \frac{x_1 - x}{x_1 - x_0} \qquad \text{and} \qquad b_1(x) = \frac{x - x_0}{x_1 - x_0}$$

are barycentric basis functions[2] with respect to $x_0$ and $x_1$, that is, these functions satisfy the three conditions in (2). Therefore, the barycentric interpolant to the data $f_0$ and $f_1$, associated with $x_0$ and $x_1$, is the linear function

$$F_1(x) = \frac{x_1 - x}{x_1 - x_0} f_0 + \frac{x - x_0}{x_1 - x_0} f_1. \tag{4}$$

In order to generalize this approach to more than two nodes, we first rewrite $F_1(x)$ as

$$F_1(x) = \frac{(x - x_1) f_0 - (x - x_0) f_1}{-(x_1 - x_0)} = \frac{(x - x_1) f_0 - (x - x_0) f_1}{(x - x_1) \quad - (x - x_0)},$$

and then, after dividing numerator and denominator both by $(x - x_0)(x - x_1)$, as

$$F_1(x) = \frac{\dfrac{1}{x - x_0} f_0 - \dfrac{1}{x - x_1} f_1}{\dfrac{1}{x - x_0} \quad - \dfrac{1}{x - x_1}} = \frac{\sum_{i=0}^{1} \dfrac{(-1)^i}{x - x_i} f_i}{\sum_{i=0}^{1} \dfrac{(-1)^i}{x - x_i}}. \tag{5}$$

### *2.1 Berrut's interpolants*

The extension to $n+1$ distinct nodes in ascending order $x_0 < \cdots < x_n$ with associated data $f_0, \ldots, f_n$ is now as easy as changing the upper bound of summation in (5) from 1 to $n$, giving the interpolant

[2] Since $n = m = 1$, these are the unique barycentric basis functions, according to Möbius [24].

$$F_n(x) = \frac{\displaystyle\sum_{i=0}^{n} \frac{(-1)^i}{x-x_i} f_i}{\displaystyle\sum_{i=0}^{n} \frac{(-1)^i}{x-x_i}}. \tag{6}$$

To see that $F_n$ indeed interpolates the data, we multiply numerator and denominator both with

$$\ell(x) = \prod_{i=0}^{n} (x - x_i), \tag{7}$$

so that

$$F_n(x) = \frac{\displaystyle\sum_{i=0}^{n} (-1)^i \prod_{j=0, j\neq i}^{n} (x - x_j) f_i}{\displaystyle\sum_{i=0}^{n} (-1)^i \prod_{j=0, j\neq i}^{n} (x - x_j)}, \tag{8}$$

and evaluation at $x = x_k$ reveals that

$$F_n(x_k) = \frac{\displaystyle\sum_{i=0}^{n} (-1)^i \prod_{j=0, j\neq i}^{n} (x_k - x_j) f_i}{\displaystyle\sum_{i=0}^{n} (-1)^i \prod_{j=0, j\neq i}^{n} (x_k - x_j)} = \frac{\displaystyle (-1)^k \prod_{j=0, j\neq k}^{n} (x_k - x_j) f_k}{\displaystyle (-1)^k \prod_{j=0, j\neq k}^{n} (x_k - x_j)} = f_k.$$

Equation (8) shows that $F_n$ is a rational function of degree at most $n$ over $n$. This rational interpolant was discovered by Berrut [1], who also shows that $F_n$ does not have any poles in $\mathbb{R}$, because the denominator of (6) does not vanish for any $x \in \mathbb{R} \setminus \{x_0, \dots, x_n\}$. For example, if $x \in (x_0, x_1)$, then

$$\sum_{i=0}^{n} \frac{(-1)^i}{x-x_i} = \underbrace{\frac{1}{x-x_0}}_{>0} + \underbrace{\frac{1}{x_1-x} - \frac{1}{x_2-x}}_{>0} + \underbrace{\frac{1}{x_3-x} - \cdots}_{>0} > 0,$$

that is, for each negative term $-1/(x_{2i} - x)$ there is a positive term $1/(x_{2i-1} - x)$ such that their sum is positive, because $x_{2i-1} < x_{2i}$. All other cases of $x$ can be treated similarly. Another property of $F_n$ is that it is a barycentric interpolant in case $n$ is odd.

**Proposition 1.** *Berrut's first interpolant $F_n$ in* (6) *is barycentric for odd $n$.*

*Proof.* It is clear that the underlying basis functions

$$b_i(x) = \frac{\dfrac{(-1)^i}{x-x_i}}{\displaystyle\sum_{j=0}^{n} \frac{(-1)^j}{x-x_j}}, \qquad i = 0, \dots, n. \tag{9}$$

of $F_n$ satisfy conditions (2a) and (2c). Applying the construction of $F_1$ in (5) to data sampled from the identity function at $x_i$ and $x_{i+1}$ gives $F_1(x) = x$, hence

$$\left(\frac{1}{x-x_i} - \frac{1}{x-x_{i+1}}\right)x = \frac{1}{x-x_i}x_i - \frac{1}{x-x_{i+1}}x_{i+1} \tag{10}$$

for $i = 0,\dots,n-1$. Adding these equations for $i = 0,2,\dots,(n-1)/2$ gives

$$\sum_{i=0}^{n} \frac{(-1)^i}{x-x_i}x = \sum_{i=0}^{n} \frac{(-1)^i}{x-x_i}x_i,$$

which is equivalent to (2b) for the $b_i$ in (9). ☐

Unfortunately, the trick used in the proof of Proposition 1 to establish condition (2b) does not work for $n$ even, but a slight modification of $F_n$ takes care of it. We just need to weight all but the first and the last terms of the sums in (6) by a factor of 2, giving the interpolant

$$\hat{F}_n(x) = \frac{\frac{1}{x-x_0}f_0 + 2\sum_{i=1}^{n-1}\frac{(-1)^i}{x-x_i}f_i + \frac{(-1)^n}{x-x_n}f_n}{\frac{1}{x-x_0} + 2\sum_{i=1}^{n-1}\frac{(-1)^i}{x-x_i} + \frac{(-1)^n}{x-x_n}}. \tag{11}$$

This rational interpolant was also discovered by Berrut [1] and like $F_n$ it does not have any poles in $\mathbb{R}$ [13]. Its advantage, however, is that it is a barycentric interpolant for any $n$.

**Proposition 2.** *Berrut's second interpolant* $\hat{F}_n$ *in* (11) *is barycentric for any* $n$.

*Proof.* Multiplying the equations in (10) by $(-1)^i$ and adding them for $i = 0,\dots,n-1$ gives

$$\left(\frac{1}{x-x_0} + 2\sum_{i=1}^{n-1}\frac{(-1)^i}{x-x_i} + \frac{(-1)^n}{x-x_n}\right)x = \frac{1}{x-x_0}x_0 + 2\sum_{i=1}^{n-1}\frac{(-1)^i}{x-x_i}x_i + \frac{(-1)^n}{x-x_n}x_n.$$

Therefore, condition (2b) holds for the basis functions of the interpolant $\hat{F}_n$ in (11), and it is clear that these basis functions also satisfy conditions (2a) and (2c). ☐

## 2.2 General rational interpolants

Berrut's interpolants $F_n$ and $\hat{F}_n$ are special cases of the general rational function

$$F_\beta(x) = \frac{\displaystyle\sum_{i=0}^{n} \frac{\beta_i}{x - x_i} f_i}{\displaystyle\sum_{i=0}^{n} \frac{\beta_i}{x - x_i}} \tag{12}$$

with coefficients $\beta = (\beta_0, \ldots, \beta_n)$, which was introduced and studied by Schneider and Werner [28]. They show that $F_\beta$ interpolates $f_i$ at $x_i$ as long as $\beta_i \neq 0$, which can be seen immediately after multiplying numerator and denominator both with $\ell(x)$ in (7), similarly to how we did it for $F_n$ above.

Moreover, Berrut and Mittelmann [4] observe that any rational interpolant to the data $f_0, \ldots, f_n$ at $x_0, \ldots, x_n$ can be written in the form (12) for some suitable choice of $\beta$. Assume that

$$r(x) = \frac{p(x)}{q(x)}, \qquad p \in \Pi_k, \quad q \in \Pi_m$$

is a rational function of degree $k$ over $m$ with $k, m \leq n$ and $r(x_i) = f_i$ for $i = 0, \ldots, n$. Now consider the Lagrange form of $p$,

$$p(x) = \sum_{i=0}^{n} \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} p(x_i) = \ell(x) \sum_{i=0}^{n} \frac{p(x_i)}{(x - x_i)\ell'(x_i)},$$

with $\ell(x)$ as in (7), and the Lagrange form of $q$,

$$q(x) = \ell(x) \sum_{i=0}^{n} \frac{q(x_i)}{(x - x_i)\ell'(x_i)},$$

and let

$$\beta_i = \frac{q(x_i)}{\ell'(x_i)}, \qquad i = 0, \ldots, n.$$

The interpolation condition of $r$ implies $p(x_i) = q(x_i) f_i$ and substituting this in $p(x)$ as well as $\beta_i$ both in $p(x)$ and $q(x)$ then gives $r(x)$ in the form (12) after cancelling out the common factor $\ell(x)$. These coefficients $\beta$ are actually unique up to a common non-zero scaling factor.

An immediate consequence of this observation is that $F_\beta$ with

$$\beta_i = \frac{1}{\ell'(x_i)} = \prod_{j=0, j\neq i}^{n} \frac{1}{x_i - x_j}, \qquad i = 0, \ldots, n,$$

is the interpolating rational function with denominator $q(x) = 1$, that is, the interpolating polynomial of degree $n$. This special way of writing the interpolating polynomial is called the *(true) barycentric formula*[3], and it provides a fast and sta-

[3] According to Henrici [17], this terminology goes back to Rutishauser [27] and is justified because the interpolating polynomial reproduces linear functions for $n \geq 1$ and therefore is a barycentric interpolant.

ble algorithm for evaluating the interpolating polynomial, which outperforms even Newton's interpolation formula [5, 18].

Returning to the general rational interpolant $F_\beta$ in (12), a natural question arises in the context of this survey: how to choose the coefficients $\beta$ such that $F_\beta$ is a barycentric interpolant and without poles in $\mathbb{R}$? The coefficients from Berrut's second interpolant as well as those from the interpolating polynomial certainly satisfy both goals, but are there other choices? The answer is positive, but before we go into details, let us review some basic facts.

The first goal can easily be achieved by slightly constraining the coefficients $\beta$.

**Proposition 3.** *If the coefficients $\beta = (\beta_0, \ldots, \beta_n)$ satisfy*

$$\sum_{i=0}^{n} \beta_i = 0, \tag{13}$$

*then the interpolant $F_\beta$ in* (12) *is barycentric.*

*Proof.* As in the proof of Proposition 2, we consider the equations in (10). Multiplying each by $\gamma_i = \sum_{j=0}^{i} \beta_j$ and adding them for $i = 0, \ldots, n-1$ gives

$$\begin{aligned}
& \sum_{i=0}^{n-1} \gamma_i \left( \frac{1}{x - x_i} - \frac{1}{x - x_{i+1}} \right) x = \sum_{i=0}^{n-1} \gamma_i \left( \frac{1}{x - x_i} x_i - \frac{1}{x - x_{i+1}} x_{i+1} \right) \\
\iff \quad & \sum_{i=0}^{n-1} \frac{\gamma_i}{x - x_i} x - \sum_{i=1}^{n} \frac{\gamma_{i-1}}{x - x_i} x = \sum_{i=0}^{n-1} \frac{\gamma_i}{x - x_i} x_i - \sum_{i=1}^{n} \frac{\gamma_{i-1}}{x - x_i} x_i \\
\iff \quad & \frac{\gamma_0}{x - x_0} x + \sum_{i=1}^{n-1} \frac{\gamma_i - \gamma_{i-1}}{x - x_i} x - \frac{\gamma_{n-1}}{x - x_n} x = \frac{\gamma_0}{x - x_0} x_0 + \sum_{i=1}^{n-1} \frac{\gamma_i - \gamma_{i-1}}{x - x_i} x_i - \frac{\gamma_{n-1}}{x - x_n} x_n \\
\iff \quad & \sum_{i=0}^{n} \frac{\beta_i}{x - x_i} x = \sum_{i=0}^{n} \frac{\beta_i}{x - x_i} x_i,
\end{aligned}$$

where the last equivalence stems from the identities $\gamma_0 = \beta_0$, $\gamma_i - \gamma_{i-1} = \beta_i$, and $\gamma_{n-1} = -\beta_n$ by (13). This shows that condition (2b) holds for the basis functions of the interpolant $\hat{F}_\beta$ in (11), and it is clear that these basis functions also satisfy conditions (2a) and (2c). □

As for the second goal, the absence of poles, Schneider and Werner [28] derive a necessary condition: the coefficients $\beta_i$ need to have alternating sign, that is, $\beta_i \beta_{i+1} < 0$ for $i = 0, \ldots, n-1$. But as the examples in Figure 1 illustrate, this is not a sufficient condition. Schneider and Werner [28] also show that $F_\beta$ has an odd number of poles in the open interval $(x_i, x_{i+1})$ if $\beta_i$ and $\beta_{i+1}$ have the same sign. However, this is all that is known so far and deriving further conditions remains a challenging open problem.

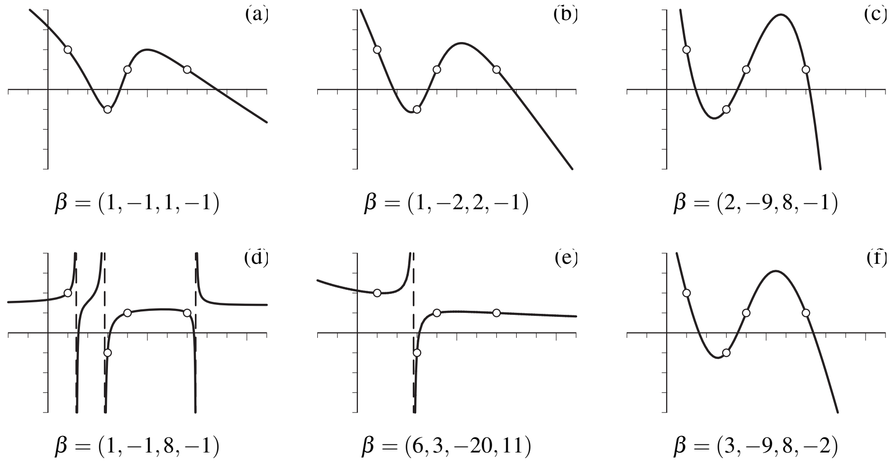


**Fig. 1** Some examples of the rational interpolant $F_\beta$ in (12) to the data $(f_0, f_1, f_2, f_3) = (2,-1,1,1)$ at the nodes $(x_0,x_1,x_2,x_3) = (1,3,4,7)$ for different choices of $\beta$: (a) Berrut's first interpolant; (b) Berrut's second interpolant; (c) interpolating cubic polynomial; (d,e) two examples of rational interpolants with poles in $\mathbb{R}$; (f) Floater–Hormann interpolant for $d = 1$.

## 2.3 Floater–Hormann interpolants

A set of coefficients $\beta$, which is different from the special cases above, is

$$\beta_i = \frac{(-1)^i}{x_{i+1} - x_i} + \frac{(-1)^i}{x_i - x_{i-1}}, \qquad i = 1,\dots,n-1,$$

and

$$\beta_0 = \frac{1}{x_1 - x_0}, \qquad \beta_n = \frac{(-1)^n}{x_n - x_{n-1}}.$$

These coefficients clearly satisfy the condition of Proposition 3, but there is another way to show that the corresponding rational interpolant $F_\beta$ is barycentric. To this end, let us rewrite the numerator of $F_\beta$ as

$$\begin{aligned}
\sum_{i=0}^{n} \frac{\beta_i}{x - x_i} f_i &= \sum_{i=0}^{n-1} \frac{(-1)^i f_i}{(x - x_i)(x_{i+1} - x_i)} + \sum_{i=1}^{n} \frac{(-1)^i f_i}{(x - x_i)(x_i - x_{i-1})} \\
&= \sum_{i=0}^{n-1} \frac{(-1)^i f_i}{(x - x_i)(x_{i+1} - x_i)} + \sum_{i=0}^{n-1} \frac{(-1)^{i+1} f_{i+1}}{(x - x_{i+1})(x_{i+1} - x_i)} \\
&= \sum_{i=0}^{n-1} \frac{(-1)^{i+1}}{(x - x_i)(x - x_{i+1})} \cdot \frac{(x_{i+1} - x) f_i + (x - x_i) f_{i+1}}{x_{i+1} - x_i}.
\end{aligned}$$

Remembering (4), we recognize the term

$$\pi_i(x) = \frac{(x_{i+1}-x)f_i + (x-x_i)f_{i+1}}{x_{i+1}-x_i}$$

as the linear interpolant to the data $f_i$ and $f_{i+1}$ at $x_i$ and $x_{i+1}$. Introducing the functions

$$\lambda_i(x) = \frac{(-1)^{i+1}}{(x-x_i)(x-x_{i+1})}, \qquad i = 0,\dots,n-1,$$

we can now write the numerator of $F_\beta$ as

$$\sum_{i=0}^{n} \frac{\beta_i}{x-x_i} f_i = \sum_{i=0}^{n-1} \lambda_i(x)\pi_i(x)$$

and the denominator as

$$\sum_{i=0}^{n} \frac{\beta_i}{x-x_i} = \sum_{i=0}^{n-1} \lambda_i(x). \tag{14}$$

It then turns out that the rational interpolant $F_\beta$ is an affine combination of the local linear interpolants $\pi_i$,

$$F_\beta(x) = \sum_{i=0}^{n-1} \mu_i(x)\pi_i(x), \tag{15}$$

with weight functions

$$\mu_i(x) = \frac{\lambda_i(x)}{\sum_{j=0}^{n-1} \lambda_j(x)}, \qquad i = 0,\dots,n-1,$$

which clearly sum to one. Now, if the data is sampled from the function $f(x) = x$, that is, $f_i = x_i$ for $i = 0,\dots,n$, then $\pi_i(x) = x$ for $i = 0,\dots,n-1$ and $F_\beta(x) = x$ by (15), which confirms $F_\beta$ to be a barycentric interpolant.

With the denominator of $F_\beta$ written as in (14), it is also easy to see that $F_\beta$ has no poles in $\mathbb{R}$. If $x \in (x_0, x_1)$, then

$$\sum_{i=0}^{n} \frac{\beta_i}{x-x_i} = \underbrace{\frac{1}{(x-x_0)(x_1-x)}}_{>0} + \underbrace{\frac{1}{(x_1-x)(x_2-x)} - \frac{1}{(x_2-x)(x_3-x)}}_{>0} + \cdots > 0,$$

similarly to our consideration for the denominator of Berrut's first interpolant $F_n$ above, and analysing the other cases shows that the denominator of $F_\beta$ does not vanish for any $x \in \mathbb{R} \setminus \{x_0,\dots,x_n\}$.

In the same way that $F_\beta$ in (15) is an affine combination of local *linear* interpolants, Berrut's first interpolant $F_n$ in (6) can be seen as an affine combination of local *constant* interpolants with the $b_i$ in (9) as weight functions, which also indicates why it does not reproduce linear functions in general.

Equipped with this new point of view, it is now straightforward to design barycentric rational interpolants which reproduce polynomials up to some general degree $d \le n$. Let us denote the unique polynomials of degree at most $d$ that interpolate the data $f_i,\dots,f_{i+d}$ at $x_i,\dots,x_{i+d}$ by $\pi_i^d \in \Pi_d$ for $i=0,\dots,n-d$ and consider their affine combination

$$F_n^d(x) = \sum_{i=0}^{n-d} \mu_i^d(x)\pi_i^d(x) \tag{16}$$

for certain weight functions $\mu_i^d(x)$. Looking at the weight functions in the constant and linear case, the obvious generalization is

$$\mu_i^d(x) = \frac{\lambda_i^d(x)}{\sum_{j=0}^{n-d}\lambda_j^d(x)}, \qquad i=0,\dots,n-d, \tag{17}$$

with

$$\lambda_i^d(x) = \frac{(-1)^{i+d}}{(x-x_i)\cdots(x-x_{i+d})}, \qquad i=0,\dots,n-d.$$

The functions $F_n^d$ in (16) were introduced by Floater and Hormann [13], who also show that they do not have any poles in $\mathbb{R}$, using similar arguments as above. Multiplying numerator and denominator of $F_n^d$ with $\ell(x)$ in (7), it is clear that this function is rational of degree $n$ over $n-d$ and that it interpolates the data $f_0,\dots,f_n$ at $x_0,\dots,x_n$. Therefore, it must be possible to convert $F_n^d$ into the general barycentric form (12) and Floater and Hormann [13] derive that

$$\beta_i = (-1)^i \sum_{j=\max(i-d,0)}^{\min(i,n-d)} \prod_{k=j,k\neq i}^{j+d} \frac{1}{|x_i-x_k|}, \qquad i=0,\dots,n. \tag{18}$$

is the correct choice of coefficients $\beta$. As $F_n^d$ clearly reproduces polynomials up to degree $d$ by construction, it is a barycentric interpolant, as long as $d \geq 1$.

This family of Floater–Hormann interpolants nicely closes the gap between Berrut's first interpolant $F_n = F_n^0$ and the interpolating polynomial $F_n^n$ and the barycentric form allows us to efficiently evaluate $F_n^d$ with $O(n)$ operations. In this regard, note that the coefficients $\beta$ in (18) do not depend on the data. Hence, they can be computed once for a specific set of nodes $x_0,\dots,x_n$ and then be used to interpolate any data $f_0,\dots,f_n$ given at these nodes. This also shows that the rational interpolant $F_n^d$ depends linearly on the data. Some examples of $F_n^d$ for different values of $d$ are shown in Figure 2.

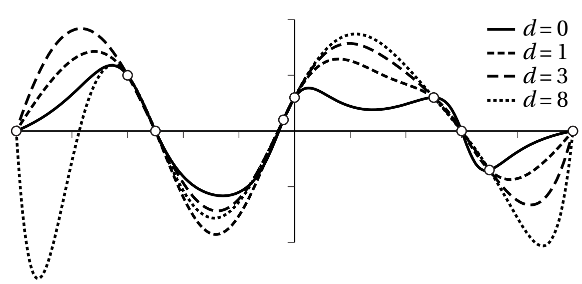


**Fig. 2** Comparison of several Floater–Hormann interpolants to data at 9 irregularly distributed nodes, including Berrut's first interpolant ($d=0$) and the interpolating polynomial ($d=8$).

In the special case of equidistant nodes $x_i = x_0 + ih$, $i = 0,\dots,n$ with spacing $h > 0$, the coefficients in (18), after multiplying them by $d!h^d$, simplify to [13]

$$\beta_i = (-1)^i \sum_{j=\max(i-d,0)}^{\min(i,n-d)} \binom{d}{i-j}, \qquad i = 0,\dots,n. \tag{19}$$

Ignoring the sign and assuming $n \geq 2d$, the first few sets of these coefficients are

$$\begin{aligned} d = 0: &\qquad 1,1,\dots,1,1,\\ d = 1: &\qquad 1,2,2,\dots,2,2,1,\\ d = 2: &\qquad 1,3,4,4,\dots,4,4,3,1,\\ d = 3: &\qquad 1,4,7,8,8,\dots,8,8,7,4,1, \end{aligned}$$

and we recognize that $F_n^1$ is identical to Berrut's second interpolant $\hat{F}_n$ in the case of equidistant nodes, but not in general, as shown in Figure 1 (b,f).

## 2.4 Approximation properties

The Floater–Hormann interpolants $F_n^d$ in (16) have some remarkable approximation properties, both with respect to the approximation order and to the Lebesgue constant. On the one hand, the approximation order of the interpolant $F_n^d$ is essentially $O(h^{d+1})$, where

$$h = \max_{0 \leq i \leq n-1} (x_{i+1} - x_i) \tag{20}$$

is the maximal distance between neighbouring nodes. On the other hand, the Lebesgue constant of $F_n^d$ grows logarithmically with $n$ for equidistant nodes, which is a setting where polynomial interpolation is known to be very ill-conditioned.

To be more precise, let $[a,b] = [x_0,x_n]$ be the interpolation interval, assume that the data $f_0,\dots,f_n$ is sampled from some function $f \in C^{d+2}[a,b]$, and denote the maximum norm on $[a,b]$ by $\|f\| = \max_{a \leq x \leq b} |f(x)|$. Floater and Hormann [13] show that for $d \geq 1$ the error between $f$ and the rational interpolant $F_n^d$ satisfies

$$\|F_n^d - f\| \leq h^{d+1}(b-a)\frac{\|f^{(d+2)}\|}{d+2}, \tag{21a}$$

if $n-d$ is odd, and if $n-d$ is even, then

$$\|F_n^d - f\| \leq h^{d+1}\left((b-a)\frac{\|f^{(d+2)}\|}{d+2} + \frac{\|f^{(d+1)}\|}{d+1}\right). \tag{21b}$$

The key idea of the proof is to note that the weighting functions $\mu_i^d$ in (17) are a partition of unity and to remember the Newton form of the error between $f$ and the interpolating polynomial $\pi_i^d$ [22]. Then,

$$\begin{aligned} f(x) - F_n^d(x) &= \sum_{i=0}^{n-d} \mu_i^d(x) \big( f(x) - \pi_i^d(x) \big) \\ &= \sum_{i=0}^{n-d} \mu_i^d(x) \prod_{j=i}^{i+d} (x - x_j) f[x_i, \dots, x_{i+d}, x] \\ &= \frac{\sum_{i=0}^{n-d} (-1)^{i+d} f[x_i, \dots, x_{i+d}, x]}{\sum_{i=0}^{n-d} \lambda_i^d(x)}, \end{aligned} \tag{22}$$

where $f[x_i, \dots, x_{i+d}, x]$ denotes the divided difference of $f$ at $x_i, \dots, x_{i+d}, x$. The error bounds in (21) then follow after bounding the numerator and the denominator in (22) suitably from above and from below, respectively.

Floater and Hormann [13] also derive similar error bounds for Berrut's first interpolant (i.e., for $d = 0$), but only if the *local mesh ratio* is bounded, that is, if a constant $R \geq 1$ exists, such that

$$\frac{1}{R} \leq \frac{x_{i+1} - x_i}{x_i - x_{i-1}} \leq R, \qquad i = 1, \dots, n-1. \tag{23}$$

For equidistant points with mesh ratio $R = 1$, these bounds show that the approximation order of $F_n$ is $O(h)$, which confirms the conjecture of Berrut [1].

Another way to bound the approximation error is by using the *Lebesgue constant* $\Lambda_n^d$ for the interpolant $F_n^d$, which is defined as the maximum of the associated *Lebesgue function*

$$\bar{\Lambda}_n^d(x) = \sum_{i=0}^{n} |b_i(x)| = \frac{\displaystyle\sum_{i=0}^{n} \frac{|\beta_i|}{|x - x_i|}}{\left| \displaystyle\sum_{i=0}^{n} \frac{\beta_i}{x - x_i} \right|} \tag{24}$$

with the coefficients $\beta$ in (18),

$$\Lambda_n^d = \max_{a \leq x \leq b} \bar{\Lambda}_n^d(x).$$

Since $F_n^d$ reproduces polynomials of degree $d$ by construction, it follows [25] that

$$\|F_n^d - f\| \leq (\Lambda_n^d + 1) \|\pi_*^d - f\|,$$

where $\pi_*^d \in \Pi_d$ is the best approximation to $f$ among all polynomials of degree at most $d$. Moroever, if $\tilde{F}_n^d$ is the Floater–Hormann interpolant to the perturbed data $\tilde{f}_i = f_i + \varepsilon_i$, $i = 0, \dots, n$ with noise $\varepsilon = \max\{|\varepsilon_0|, \dots, |\varepsilon_n|\}$, then [7]

$$\|\tilde{F}_n^d - F_n^d\| \leq \varepsilon \Lambda_n^d.$$

Hence, the interpolation process is well conditioned if the Lebesgue constant is small.

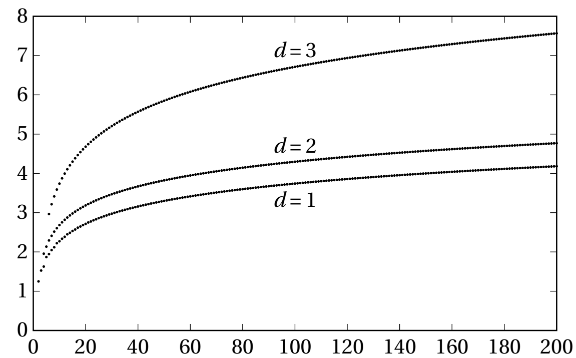


**Fig. 3** Numerically computed Lebesgue constants $\Lambda_n^d$ of the Floater–Hormann interpolants $F_n^d$ at $n+1$ equidistant nodes for $2d \le n \le 200$ and several values of $d$.

For the special case of equidistant nodes, Bos et al. [6, 7] show that the Lebesgue constant $\Lambda_n^d$ for the Floater–Hormann interpolant $F_n^d$ grows only logarithmically with $n$, as illustrated in Figure 3, while the Lebesgue constant for polynomial interpolation at such nodes is known to grow exponentially. In particular, they prove that

$$\Lambda_n^d \le \gamma_d (2 + \ln n) \tag{25}$$

with $\gamma_d = 1$ for $d = 0, 1$ and $\gamma_d = 2^{d-1}$ for $d > 1$. The key idea of the proof is to multiply both the numerator and the denominator in (24) with $(x - x_k)(x_{k+1} - x)$ for some $k \in \{0, 1, \dots, n-1\}$ and to consider $x_k < x < x_{k+1}$. It is then possible to bound the numerator from above and the denominator from below by bounds that do not depend on $k$, and (25) follows after noticing that $\bar{\Lambda}_n^d(x_i) = 1$ for $i = 0, \dots, n$.

This initial result has been improved and extended subsequently in various ways. Hormann et al. [20] tighten the upper bound on the Lebesgue constant $\Lambda_n^0$ for Berrut's first interpolant $F_n = F_n^0$ to

$$\Lambda_n^0 \le \frac{3}{4}(2 + \ln n)$$

and Zhang [32] further improves it to

$$\Lambda_n^0 \le \frac{1}{1 + \pi^2/24} \ln(n+1) + \begin{cases} 1.47, & \text{if } n \ge 10, \\ 1.00, & \text{if } n \ge 174, \\ 0.99, & \text{if } n \ge 500. \end{cases}$$

Figure 4 shows a visual comparison of these two and the initial bound in (25).

Based on extensive numerical experiments, Ibrahimoglu and Cuyt [21] predict the asymptotic growth rate of the Lebesgue constant $\Lambda_n^d$ to be

$$\Lambda_n^d \sim \gamma_d \frac{2}{\pi} \ln(n+1)$$

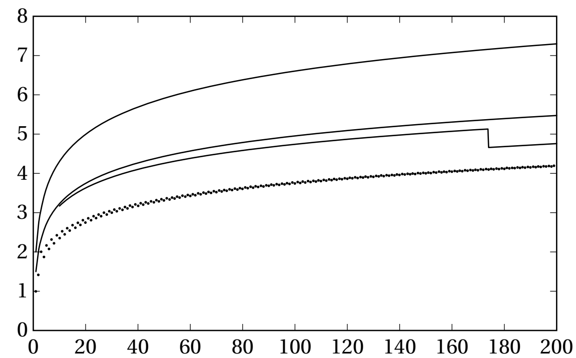


**Fig. 4** Numerically computed Lebesgue constants $\Lambda_n^0$ of Berrut's first interpolant $F_n$ at $n+1$ equidistant nodes for $1 \le n \le 200$ and the upper bounds (from top) by Bos et al. [6], Hormann et al. [20], and Zhang [32].

as $n \to \infty$. For $d = 0,1$ this is identical to the optimal growth rate of the Lebesgue constant for polynomial interpolation [30], which is obtained, for example, by sampling at the extended Chebyshev nodes.

Hormann et al. [20] generalize the upper bound in (25) to the case where the nodes are only quasi-equidistant. That is, they assume the existence of a *global mesh ratio* $M \ge 1$, independent of $n$, such that

$$\frac{h}{h_*} \le M$$

with $h$ from (20) and

$$h_* = \min_{0 \le i \le n-1} (x_{i+1} - x_i),$$

and then show

$$\Lambda_n^d \le \tilde{\gamma}_d (2 + M \ln n)$$

with $\tilde{\gamma}_0 = \frac{3}{4} M$ and $\tilde{\gamma}_d = 2^{d-1} M^d$ for $d \ge 1$.

Finally, Bos et al. [8] prove that the Lebesgue constant $\Lambda_n^0$ of Berrut's first interpolant grows logarithmically with $n$ for the very general class of well-spaced nodes. A family $X = (X_n)_{n \in \mathbb{N}}$ of sets of nodes $X_n = \{x_0, \dots, x_n\}$ is called *well-spaced* if for each $X_n$ the local mesh ratio is bounded as in (23) for some $R \ge 1$ and if

$$\frac{x_{k+1} - x_k}{x_{k+1} - x_j} \le \frac{C}{k+1-j}, \qquad j = 0, \dots, k, \quad k = 0, \dots, n-1, \tag{26}$$

$$\frac{x_{k+1} - x_k}{x_j - x_k} \le \frac{C}{j-k}, \qquad j = k+1, \dots, n, \quad k = 0, \dots, n-1, \tag{27}$$

for some $C \ge 1$, where both constants $R$ and $C$ must be independent of $n$. Under these assumptions,

$$\Lambda_n^0 \le (R+1)(1 + 2C \ln n).$$

This definition of well-spaced nodes includes equidistant nodes (with $R = C = 1$), *extended Chebyshev nodes*

$$x_i = \frac{\cos\frac{(2i+1)\pi}{2n+2}}{\cos\frac{\pi}{2n+2}}, \qquad i = 0, \dots, n$$

(with $R = 2$ and $C = \pi^2/2$), and *Chebyshev–Gauss–Lobatto* or *Clenshaw–Curtis nodes*

$$x_i = \cos\frac{k\pi}{n}, \qquad i = 0, \dots, n$$

(with $R = 9\pi/2$ and $C = 2\pi$). In general, nodes are well-spaced as long as they do not cluster too heavily, but they are allowed to cluster anywhere in the interpolation interval, not just towards its ends, and still the Lebesgue constant $\Lambda_n^0$ is guaranteed to grow only logarithmically.

## 2.5 Conclusion

We have seen in the previous sections that the rational Floater–Hormann interpolants $F_n^d$ provide a promising alternative to other univariate interpolation methods, so let us quickly summarize their advantages. More details regarding recent extensions and applications of rational Floater–Hormann interpolants can be found in [3].

Compared to classical rational interpolation, $F_n^d$ is guaranteed to not have any poles in $\mathbb{R}$, which is important in many applications. Moreover, interpolation with $F_n^d$ is linear in the data and does not require to solve a linear system.

The advantage over polynomial interpolation is that interpolation with $F_n^d$ is stable for a larger class of nodes, and in particular for equidistant nodes, where polynomial interpolation can be infeasible even for rather small $n \approx 20$.

Spline interpolation is probably the closest competitor, because approximation error and convergence rate of $F_n^d$ are similar to those of spline interpolation with (odd) degree $d$, and this carries over to the approximation of derivatives. Berrut et al. [2] show that

$$\|(F_n^d)^{(k)} - f^{(k)}\| \le Ch^{d+1-k}$$

for $k = 1, 2$ and $f$ being sufficiently smooth, where the constant $C$ may depend on the local mesh ratio (23) of the nodes, and they conjecture that a similar approximation result holds for $k \ge 3$. The advantage over spline interpolation is that $F_n^d$ is infinitely smooth, while the interpolating spline is only $d - 1$ times continuously differentiable.

However, the favourable properties of the rational interpolant $F_n^d$ may disappear if $d$ is chosen incorrectly. On the one hand, small values of $d$ lead to very stable interpolation, but rather low approximation order. On the other hand, large values of $d$ guarantee good convergence rates, but the interpolation process may become unstable for equidistant nodes, because the Lebesgue constant $\Lambda_n^d$ grows exponentially in $d$ for fixed $n$, which is not too surprising, as $F_n^d$ approaches the polynomial interpolant as $d \to n$. In practice, it is recommended [26, Chapter 3.4.1] to start with small values of $d$, say $d = 3$ and then try larger values to get better results. For the case when $f$ is analytic in a domain that contains the interpolation interval, Güttel and Klein [16] suggest an algorithm for choosing an optimal value of $d$.

## 3 Bivariate barycentric interpolation

The main idea behind the construction of the univariate rational barycentric interpolants in Section 2 can also be generalized to the bivariate setting. To this end, let $X = \{x_1, \dots, x_n\}$ be a set of $n$ distinct nodes in $\mathbb{R}^2$ with associated data $f_1, \dots, f_n$.

The classical *Shepard interpolant* [29]

$$S(x) = \sum_{i=1}^{n} \omega_i(x) f_i$$

with

$$\omega_i(x) = \frac{\dfrac{1}{\|x - x_i\|^\alpha}}{\displaystyle\sum_{j=1}^{n} \frac{1}{\|x - x_j\|^\alpha}}, \qquad i = 1, \dots, n$$

for some $\alpha > 0$ can be seen as a convex combination of local constant interpolants with weight functions $\omega_i(x)$. Like Berrut's first interpolant, $S$ does not reproduce linear functions in general and so it is not a barycentric interpolant.

To construct the simplest bivariate barycentric interpolant, we consider a triangulation $T = \{t_1, \dots, t_m\}$ of the nodes $X$ with triangles $t_j = [x_{j_1}, x_{j_2}, x_{j_3}]$. Analogously to (15) we then define

$$F(x) = \sum_{j=1}^{m} \mu_j(x) \pi_j(x), \tag{28}$$

where $\pi_j$ is the local linear interpolant to the data given at the vertices of the triangle $t_j$ and

$$\mu_j(x) = \frac{\lambda_j(x)}{\sum_{k=1}^{m} \lambda_k(x)}, \qquad j = 1, \dots, m,$$

are some weight functions that sum to one. Little [23] suggests to let

$$\lambda_j(x) = \frac{1}{\|x - x_{j_1}\|^2 \|x - x_{j_2}\|^2 \|x - x_{j_3}\|^2}, \qquad j = 1, \dots, m, \tag{29}$$

which guarantees $F$ to interpolate $f_i$ at $x_i$ and avoids the occurrence of poles, because the common denominator of the weight functions $\mu_j$ is positive. Since this *triangular Shepard interpolant* $F$ reproduces linear functions by construction, it clearly is a barycentric interpolant.

Little [23] observes that the triangular Shepard interpolant surpasses Shepard's interpolant in aesthetic behaviour, because it does not suffer from flat spots at the nodes and is generally "smoother". But he also notices that it requires the choice

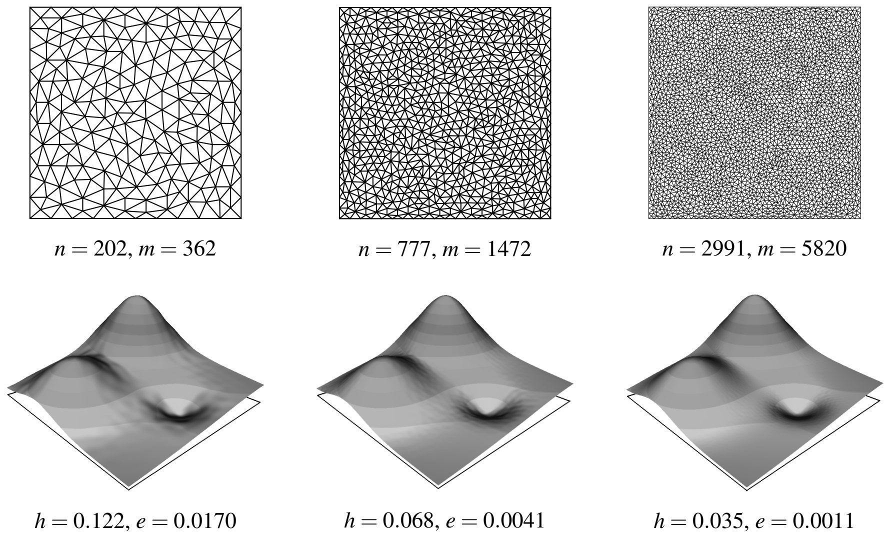


**Fig. 5** Examples of triangular Shepard interpolants to data sampled from Franke's test function at $n$ uniformly distributed nodes and with respect to the Delaunay triangulation of the nodes with $m$ triangles and maximum edge length $h$. The approximation error $e$ decreases roughly by a factor of 4 as $h$ decreases by a factor of 2.

of an appropriate triangulation $T$. One possible choice is to take the *Delaunay triangulation* [9] of $X$ and Figure 5 shows some examples for this choice and data sampled from Franke's classical test function [15]. In these examples, the approximation error seems to be $O(h^2)$, where $h$ is the maximum edge length of the triangles in $T$. Dell'Accio et al. [10] prove that the triangular Shepard interpolant has indeed quadratic approximation order for a very general class of triangulations, which includes the Delaunay triangulation.

While this construction can easily be extended to the multivariate setting and generalized to barycentric interpolants with arbitrary reproduction degree by taking convex combinations of higher order local polynomial interpolants with suitable weighting functions, it lacks two essential properties from the univariate interpolants. On the one hand, the degree of the bivariate rational interpolant is roughly twice the degree of the univariate analogue, because of the squared distances between $x$ and the nodes in the denominator of $\lambda_j$ in (29). The univariate setting allows us to take signed distances instead, which makes it harder to avoid poles but keeps the degree of the rational interpolant low. On the other hand, an equivalent of the elegant barycentric form in (12) is not known for the triangular Shepard interpolant, and its evaluation is therefore slightly less efficient.

## 4 Barycentric interpolation over polygons

A very special case of bivariate interpolation occurs if the data $f_1,\dots,f_n$ is given as the vertices $x_1,\dots,x_n$ of a planar polygon $\Omega$. In this setting, let us consider the $n$ triangles $t_i=[x_{i-1},x_i,x_{i+1}]$ for $i=1,\dots,n$, where the vertices are indexed cyclically (i.e., $x_{n+1}=x_1$ and $x_0=x_n$); see Figure 6.

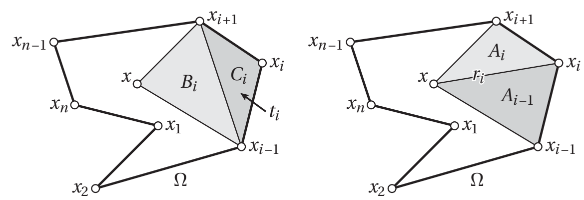


**Fig. 6** Notation used for the definition of the barycentric interpolant over a planar polygon $\Omega$ with vertices $x_1,\dots,x_n$.

As for the triangular Shepard interpolant, we then define $F$ as in (28) with $m=n$, except that we replace the functions $\lambda_j$ in (29) by

$$\lambda_i(x)=\varphi(r_i(x))\frac{C_i}{A_{i-1}(x)A_i(x)},\qquad i=1,\dots,n,$$

where $\varphi\colon\mathbb{R}^+\to\mathbb{R}$ is an arbitrary function, $r_i(x)=\|x-x_i\|$ is the distance between $x$ and $x_i$, $C_i$ is the signed area of $t_i$ and $A_{i-1}(x)$, $A_i(x)$ are the signed areas of the triangles $[x,x_{i-1},x_i]$, $[x,x_i,x_{i+1}]$, respectively; see Figure 6.

Denoting by $B_i(x)$ the signed area of the triangle $[x,x_{i-1},x_{i+1}]$ and remembering that $A_i(x)$, $-B_i(x)$, and $A_{i-1}(x)$ are homogeneous barycentric coordinates of $x$ with respect to $t_i$, we can write the linear interpolant to the data given at the vertices of $t_i$ as

$$\pi_i(x)=\frac{A_i(x)f_{i-1}-B_i(x)f_i+A_{i-1}(x)f_{i+1}}{A_{i-1}(x)-B_i(x)+A_i(x)}.$$

Since $C_i=A_{i-1}(x)-B_i(x)+A_i(x)$, we then have

$$\begin{aligned}\sum_{i=1}^n\lambda_i(x)\pi_i(x)&=\sum_{i=1}^n\varphi(r_i(x))\left(\frac{1}{A_{i-1}(x)}f_{i-1}-\frac{B_i(x)}{A_{i-1}(x)A_i(x)}f_i+\frac{1}{A_i(x)}f_{i+1}\right)\\&=\sum_{i=1}^n\left(\frac{\varphi(r_{i+1}(x))}{A_i(x)}-\frac{\varphi(r_i(x))B_i(x)}{A_{i-1}(x)A_i(x)}+\frac{\varphi(r_{i-1}(x))}{A_{i-1}(x)}\right)f_i\\&=\sum_{i=1}^n w_i(x)f_i,\end{aligned}$$

where

$$w_i(x)=\frac{\varphi(r_{i+1}(x))A_{i-1}(x)-\varphi(r_i(x))B_i(x)+\varphi(r_{i-1}(x))A_i(x)}{A_{i-1}(x)A_i(x)},\qquad i=0,\dots,n.$$

Likewise,

$$\sum_{i=1}^{n} \lambda_i(x) = \sum_{i=1}^{n} w_i(x),$$

and it turns out that we can rewrite $F$ in terms of the basis functions

$$b_i(x) = \frac{w_i(x)}{\sum_{j=1}^{n} w_j(x)}, \qquad i = 1, \ldots, n, \tag{30}$$

as

$$F(x) = \sum_{i=1}^{n} b_i(x) f_i.$$

Since $F$ reproduces linear functions by construction, it follows that the $b_i(x)$ in (30) satisfy conditions (2a) and (2b), and Floater et al. [14] show that they further satisfy (2c), if the polygon $\Omega$ is convex and the function $\varphi$ has the four properties

| | |
|---|---|
| Positivity: | $\varphi(r) \geq 0,$ |
| Monotonicity: | $\varphi'(r) \geq 0,$ |
| Convexity: | $\varphi''(r) \geq 0,$ |
| Sublinearity: | $\varphi(r) \geq r\varphi'(r).$ |

Under these assumptions, it also follows that $b_i(x)$ is positive for any $x$ in the interior of $\Omega$, so that $F(x)$ lies in the convex hull of the data $f_1, \ldots, f_n$, and that the $b_i(x)$ as well as $F(x)$ are linear along the edges of the polygon.

Two examples of functions that satisfy the four conditions above and thus give barycentric basis functions $b_i$ and corresponding barycentric interpolants $F$ are the functions $\varphi_1(r) = 1$ and $\varphi_2(r) = r$. Floater et al. [14] show that the $b_i$ corresponding to $\varphi_1$ are the *Wachspress coordinates* [31], which are important in the context of polygonal finite element methods. Instead, $\varphi_2$ leads to *mean value coordinates* [11], which turn out to be well-defined also for non-convex and even nested polygons [19] and are used in computer graphics for surface parameterization, image warping, shading, and many other applications. More details on both coordinates can be found in [12].

## References

1. Berrut, J.P.: Rational functions for guaranteed and experimentally well-conditioned global interpolation. Comput. Math. Appl. **15**(1), 1–16 (1988)
2. Berrut, J.P., Floater, M.S., Klein, G.: Convergence rates of derivatives of a family of barycentric rational interpolants. Appl. Numer. Math. **61**(9), 989–1000 (2011)
3. Berrut, J.P., Klein, G.: Recent advances in linear barycentric rational interpolation. J. Comput. Appl. Math. **259**(Part A), 95–107 (2014)
4. Berrut, J.P., Mittelmann, H.D.: Lebesgue constant minimizing linear rational interpolation of continuous functions over the interval. Comput. Math. Appl. **33**(6), 77–86 (1997)

5. Berrut, J.P., Trefethen, L.N.: Barycentric Lagrange interpolation. SIAM Rev. **46**(3), 501–517 (2004)
6. Bos, L., De Marchi, S., Hormann, K.: On the Lebesgue constant of Berrut's rational interpolant at equidistant nodes. J. Comput. Appl. Math. **236**(4), 504–510 (2011)
7. Bos, L., De Marchi, S., Hormann, K., Klein, G.: On the Lebesgue constant of barycentric rational interpolation at equidistant nodes. Numer. Math. **121**(3), 461–471 (2012)
8. Bos, L., De Marchi, S., Hormann, K., Sidon, J.: Bounding the Lebesgue constant for Berrut's rational interpolant at general nodes. J. Approx. Theory **169**, 7–22 (2013)
9. Delaunay, B.: Sur la sphère vide. A la mémoire de Georges Voronoï. Bull. Acad. Sci. URSS **6**, 793–800 (1934)
10. Dell'Accio, F., Di Tommaso, F., Hormann, K.: On the approximation order of triangular Shepard interpolation. Tech. rep., Department of Mathematics, Università della Calabria (2013)
11. Floater, M.S.: Mean value coordinates. Comput. Aided Geom. Des. **20**(1), 19–27 (2003)
12. Floater, M.S.: Wachspress and mean value coordinates. In: Approximation Theory XIV: San Antonio 2013, *Springer Proceedings in Mathematics*, vol. ?, pp. ?–? Springer, New York (2014)
13. Floater, M.S., Hormann, K.: Barycentric rational interpolation with no poles and high rates of approximation. Numer. Math. **107**(2), 315–331 (2007)
14. Floater, M.S., Hormann, K., Kós, G.: A general construction of barycentric coordinates over convex polygons. Adv. Comput. Math. **24**(1–4), 311–331 (2006)
15. Franke, R., Nielson, G.: Smooth interpolation of large sets of scattered data. Int. J. Numer. Methods Eng. **15**(11), 1691–1704 (1980)
16. Güttel, S., Klein, G.: Convergence of linear barycentric rational interpolation for analytic functions. SIAM J. Numer. Anal. **50**(5), 2560–2580 (2012)
17. Henrici, P.: Barycentric formulas for interpolating trigonometric polynomials and their conjugates. Numer. Math. **33**(2), 225–234 (1979)
18. Higham, N.J.: The numerical stability of barycentric Lagrange interpolation. IMA J. Numer. Anal. **24**(4), 547–556 (2004)
19. Hormann, K., Floater, M.S.: Mean value coordinates for arbitrary planar polygons. ACM Trans. Graph. **25**(4), 1424–1441 (2006)
20. Hormann, K., Klein, G., De Marchi, S.: Barycentric rational interpolation at quasi-equidistant nodes. Dolomites Res. Notes Approx. **5**, 1–6 (2012)
21. Ibrahimoglu, B.A., Cuyt, A.: Sharp bounds for Lebesgue constants of barycentric rational interpolation. Tech. rep., Department of Mathematics and Computer Science, Universiteit Antwerpen (2013)
22. Isaacson, E., Keller, H.B.: Analysis of Numerical Methods. Dover Publications, Mineola, NY (1994)
23. Little, F.F.: Convex combination surfaces. In: R.E. Barnhill, W. Boehm (eds.) Surfaces in Computer Aided Geometric Design, pp. 99–107. North-Holland, Amsterdam (1983)
24. Möbius, A.F.: Der barycentrische Calcul. Johann Ambrosius Barth Verlag, Leipzig (1827)
25. Powell, M.J.D.: Approximation Theory and Methods. Cambridge University Press, Cambridge (1981)
26. Press, W.H., Teukolsky, S.A., Vetterling, W.T., Flannery, B.P.: Numerical Recipes. The Art of Scientific Computing, third edn. Cambridge University Press, Cambridge (2007)
27. Rutishauser, H.: Vorlesungen über numerische Mathematik, Band 1: Gleichungssysteme, Interpolation und Approximation, *Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften, Mathematische Reihe*, vol. 50. Birkhäuser Verlag, Basel (1976)
28. Schneider, C., Werner, W.: Some new aspects of rational interpolation. Math. Comput. **47**(175), 285–299 (1986)
29. Shepard, D.: A two-dimensional interpolation function for irregularly-spaced data. In: Proceedings of the 23rd ACM National Conference, pp. 517–524. ACM Press, New York (1968)
30. Szabados, J., Vértesi, P.: Interpolation of Functions. World Scientific, Singapore (1990)
31. Wachspress, E.L.: A Rational Finite Element Basis, *Mathematics in Science and Engineering*, vol. 114. Academic Press, New York (1975)
32. Zhang, R.J.: An improved upper bound on the Lebesgue constant of Berrut's rational interpolation operator. J. Comput. Appl. Math. **255**, 652–660 (2014)