Geometric conditions for tangent continuity of interpolatory planar subdivision curves

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Abstract

Curve subdivision is a technique for generating smooth curves from initial control polygons by repeated refinement. The most common subdivision schemes are based on linear refinement rules, which are applied separately to each coordinate of the control points, and the analysis of these schemes is well understood. Since the resulting limit curves are not sufficiently sensitive to the geometry of the control polygons, there is a need for geometric subdivision schemes. Such schemes take the geometry of the control polygons into account by using non-linear refinement rules and are known to generate limit curves with less artefacts. Yet, only few tools exist for their analysis, because the non-linear setting is more complicated. In this paper, we derive sufficient conditions for a convergent interpolatory planar subdivision scheme to produce tangent continuous limit curves. These conditions as well as the proofs are purely geometric and do not rely on any parameterization.

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1 Introduction

Linear subdivision schemes for curves are well-studied in the literature [1, 8], but the performance of these schemes is limited because they treat the coordinates of the control points independently of each other. A geometric subdivision scheme determines the points on the finer level by using refinement rules which depend non-linearly on all the components of the points on the coarser level. In this paper we consider interpolatory geometric schemes for curves in the plane and develop general tools for their analysis. While several such schemes are proposed in the literature, the proofs of their properties rely mainly on ad-hoc methods.

Marinov et al. [13] extend the linear four-point scheme [6] by adapting the tension parameter geometrically in several ways. They only show convergence, but their analysis relies on results from the linear theory [12] and is limited to the specific form of their schemes. Sabin and Dodgson [14] present a circle-preserving four-point scheme and conclude its convergence by noticing that this geometric scheme is asymptotically equivalent [7] to the linear four-point scheme. Another geometric variant of the linear four-point scheme is proposed by Dyn et al. [5]. Their scheme generates new points by evaluating locally interpolating cubic polynomials with respect to a geometry-dependent parameterization. Also for this scheme, only convergence is proved, but in this case the proof is geometric. The same technique is used by Hernández-Mederos et al. [11] to establish convergence of a variation of the schemes in [13]. In Section 3 we extend the ideas in [5] and derive necessary and sufficient geometric conditions for the convergence of general geometric schemes.

Numerical simulations indicate that all the schemes above generate C^1 limit curves, but there is no proof of this property. A formal proof of tangent continuity is available for specific geometric schemes which either are restricted to the refinement of convex polygons [9] or refine data consisting of control points and corresponding tangents [2, 3]. An existing general tool for proving tangent continuity of a non-linear scheme is by establishing its proximity to a tangent continuous linear scheme [15], but many geometric schemes do not satisfy this proximity condition, for example, the two schemes that we discuss in Section 6.

In Section 5 we give a sufficient geometric condition for a converging sequence of polygons to have a tangent continuous limit curve. While the usual approach to proving tangent continuity relies on a parameterization of the limit curve, our proof is purely geometric and based on the limit behaviour of the curve's secants. Moreover, the same geometric condition also guarantees the finite length of the limit curve between any two points on the curve (Section 4). We conclude the paper in Section 6 by applying our theory to two new geometric subdivision schemes. While the first is intentionally designed to be treated with our theory, the second is not, but we admit that we did not manage to apply our main result to other schemes, like the ones in [5, 11, 13, 14].

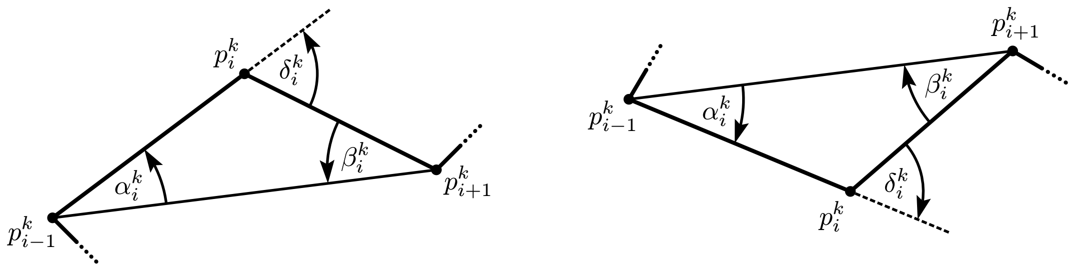

Figure 1: Definition of the signed angles α_i^k, β_i^k, δ_i^k at level k, for points p_{i-1}^k, p_i^k, p_{i+1}^k in clockwise (left) or counterclockwise (right) order.

2 Preliminaries

2.1 Directed lines and signed angles

To simplify our analysis we use the concepts of directed lines and signed angles, because they allow to handle the relations of angles between several lines in a common and concise way. Any two distinct points $p, q \in \mathbb{R}^2$ determine a *directed line* $\overrightarrow{pq}$ from p to q with *direction* $(q-p)/\|q-p\|$. The *signed angle* $\sphericalangle(s,t)$ between two directed lines s and t is the angle between their directions $\vec{s}$ and $\vec{t}$, measured from $\vec{s}$ to $\vec{t}$, where counterclockwise angles are positive and clockwise angles are negative. By convention, we consider angles modulo 2π in the range $(-\pi, \pi]$, which guarantees the uniqueness of the signed angle between two directed lines. Note that for parallel directed lines s and t, we have $\sphericalangle(s,t) = 0$ if $\vec{s} = \vec{t}$ and $\sphericalangle(s,t) = \pi$ if $\vec{s} = -\vec{t}$. Obviously, the signed angle between s and t is invariant under translations of either s or t, and

$$\sphericalangle(s,t) = -\sphericalangle(t,s), \qquad \sphericalangle(\overrightarrow{pq}, \overrightarrow{qp}) = \pi, \qquad \sphericalangle(\overrightarrow{p_1q_1}, \overrightarrow{p_2q_2}) = \sphericalangle(\overrightarrow{q_1p_1}, \overrightarrow{q_2p_2}).$$

A frequently used identity that relates the angles between three directed lines s, t, u is

$$\sphericalangle(s,t) = \sphericalangle(s,u) + \sphericalangle(u,t). \tag{1}$$

Moreover, we need the notion of convergence of directed lines and say that a sequence $\{s_n\}_{n\geq 0}$ of directed lines *converges* to the directed line $\bar{s}$, if the angles $\sphericalangle(s_n, \bar{s})$ tend to zero and if there exists for each n some point p_n on s_n such that $\lim_{n\to\infty} p_n$ is a point on $\bar{s}$.

2.2 Geometric subdivision

A curve subdivision scheme $\mathcal{S}$ generates a sequence $\{P^k\}_{k\geq 0}$ of polygons from an initial control polygon P^0 by repeated refinement. We denote the vertices of P^k by p_i^k, $i \in \mathbb{Z}$ and consider the case where all points p_i^k are planar. For any interpolatory scheme $\mathcal{S}$ we have

$$p_{2i}^{k+1} = p_i^k,$$

and some rule for generating the inserted points p_{2i+1}^{k+1} that depends on the points at level k.

Our analysis of such schemes relies on the signed angles

$$\alpha_i^k = \sphericalangle(\overrightarrow{p_{i-1}^k p_{i+1}^k}, \overrightarrow{p_{i-1}^k p_i^k}), \qquad \beta_i^k = \sphericalangle(\overrightarrow{p_i^k p_{i+1}^k}, \overrightarrow{p_{i-1}^k p_{i+1}^k}), \qquad \delta_i^k = \sphericalangle(\overrightarrow{p_i^k p_{i+1}^k}, \overrightarrow{p_{i-1}^k p_i^k}),$$

as shown in Figure 1. Note that δ_i^k is the exterior angle between two consecutive edges at level k, and by (1) it satisfies

$$\delta_i^k = \alpha_i^k + \beta_i^k. \tag{2}$$

A geometric subdivision scheme may also specify how to determine the inserted point p_{2i+1}^{k+1} in terms of rules for the angles α_{2i+1}^{k+1} and β_{2i+1}^{k+1}. With these angles given, we have (see Figure 2)

$$\begin{aligned} p_{2i+1}^{k+1} &= \frac{\cot\alpha_{2i+1}^{k+1} p_{i+1}^k + \cot\beta_{2i+1}^{k+1} p_i^k + (p_{i+1}^k - p_i^k)^\perp}{\cot\alpha_{2i+1}^{k+1} + \cot\beta_{2i+1}^{k+1}} \\ &= \frac{\cos\alpha_{2i+1}^{k+1}\sin\beta_{2i+1}^{k+1} p_{i+1}^k + \sin\alpha_{2i+1}^{k+1}\cos\beta_{2i+1}^{k+1} p_i^k + \sin\alpha_{2i+1}^{k+1}\sin\beta_{2i+1}^{k+1}(p_{i+1}^k - p_i^k)^\perp}{\sin\delta_{2i+1}^{k+1}}, \end{aligned} \tag{3}$$

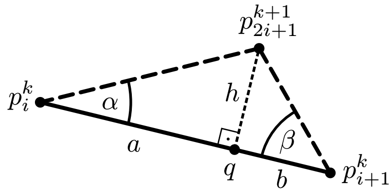

Figure 2: Illustration of Equation (3). Here, $\alpha = |\alpha_{2i+1}^{k+1}|$, $\beta = |\beta_{2i+1}^{k+1}|$, and $q = (a p_{i+1}^k + b p_i^k)/(a+b)$. As long as $h \neq 0$, we have $a = h\cot\alpha$ and $b = h\cot\beta$, and so $h = (a+b)/(\cot\alpha + \cot\beta)$.

where $x^\perp$ denotes the vector x rotated by 90 degrees (counterclockwise). In order for the rule (3) to be well-defined, the scheme must guarantee that α_{2i+1}^{k+1} and β_{2i+1}^{k+1} are both nonzero, have the same sign, and $|\alpha_{2i+1}^{k+1} + \beta_{2i+1}^{k+1}| = |\delta_{2i+1}^{k+1}| < \pi$.

An interesting family of geometric subdivision schemes are the ones that place the inserted point on the perpendicular bisector of the corresponding edge. This construction implies $\alpha_{2i+1}^{k+1} = \beta_{2i+1}^{k+1}$ and (3) simplifies to

$$p_{2i+1}^{k+1} = \frac{p_i^k + p_{i+1}^k}{2} + \tan\left(\frac{\delta_{2i+1}^{k+1}}{2}\right)\frac{(p_{i+1}^k - p_i^k)^\perp}{2}, \tag{4}$$

which is well-defined as long as $|\alpha_{2i+1}^{k+1}| = |\beta_{2i+1}^{k+1}| = |\delta_{2i+1}^{k+1}/2| < \pi/2$. The new subdivision schemes that we investigate in Section 6 are examples of such *bisector schemes*.

3 Convergence

In order to define the convergence of a subdivision scheme, we go back to the standard parametric definition [1]. For each level $k \geq 0$, we represent the polygon P^k by the continuous piecewise linear function $f^k : \mathbb{R} \to \mathbb{R}^2$ that interpolates the points p_i^k at the dyadic parameter values $2^{-k} i$, $i \in \mathbb{Z}$. We then say that a subdivision scheme $\mathcal{S}$ is *convergent* if the sequence $\{f^k\}_{k\geq 0}$ converges uniformly in the L_∞-norm. The limit $f^\infty = \lim_{k\to\infty} f^k$ is a parametric representation of the *limit curve* $\mathcal{P}$. By the uniform convergence theorem, $\mathcal{P}$ is continuous.

The aim of this section is to study necessary conditions and sufficient conditions for a scheme to be convergent. We derive two kinds of geometric conditions, based either on distances or on angles.

3.1 Distance conditions

Let

$$d_i^k = \|(p_i^k + p_{i+1}^k)/2 - p_{2i+1}^{k+1}\|$$

be the distance between the midpoint of the edge $[p_i^k, p_{i+1}^k]$ and the corresponding inserted point p_{2i+1}^{k+1} as shown in Figure 3. Moreover, let

$$d^k = \sup_{i\in\mathbb{Z}} d_i^k$$

be the supremum of these distances at level k and

$$\boldsymbol{d} = \{d^k\}_{k\geq 0},$$

be the sequence of these values. Observing that

$$\|f^{k+1} - f^k\|_\infty = \sup_{t\in\mathbb{R}} \|f^{k+1}(t) - f^k(t)\| = d^k, \tag{5}$$

we conclude the following necessary condition for convergence.

Theorem 1. *If $\mathcal{S}$ is convergent, then $\boldsymbol{d}$ is a null sequence.*

Instead of checking that $\boldsymbol{d}$ converges to zero, it may at times be easier to analyse the sequence $\boldsymbol{e} = \{e^k\}_{k\geq 0}$ of least upper bounds $e^k = \sup_{i\in\mathbb{Z}} e_i^k$ of edge lengths $e_i^k = \|p_{i+1}^k - p_i^k\|$ at level k (see Figure 3).

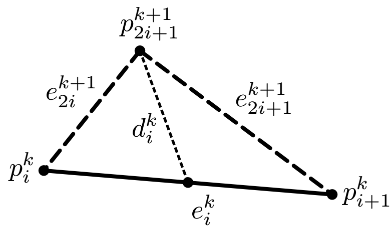

Figure 3: Definition of distances d_i^k and edge lengths e_i^k.

Proposition 2. *The sequence* $\boldsymbol{e}$ *is a null sequence if and only if* $\boldsymbol{d}$ *is a null sequence.*

Proof. By the triangle inequality,

$$d_i^k \leq \frac{e_i^k}{2} + e_{2i}^{k+1}$$

for all $i \in \mathbb{Z}$ and therefore

$$d^k \leq \frac{e^k}{2} + e^{k+1} \tag{6}$$

for $k \geq 0$. Hence, if $\boldsymbol{e}$ is a null sequence, then so is $\boldsymbol{d}$.

On the other hand, we have again by the triangle inequality (see Figure 3) that

$$\max(e_{2i}^{k+1}, e_{2i+1}^{k+1}) \leq \frac{e_i^k}{2} + d_i^k,$$

and so

$$e^{k+1} \leq \frac{e^k}{2} + d^k. \tag{7}$$

Recursive application of this inequality gives

$$e^k \leq \frac{e^0}{2^k} + \sum_{j=1}^{k} \frac{d^{k-j}}{2^{j-1}}.$$

Now, if $\boldsymbol{d}$ is a null sequence, then there exists some upper bound $\bar{d}$ on $\boldsymbol{d}$, so that

$$e^k \leq \frac{e^0}{2^k} + \bar{d} \sum_{j=1}^{k} \frac{1}{2^{j-1}} < e^0 + 2\bar{d} =: M$$

for any $k \geq 0$. Moreover, there exist for any $\varepsilon > 0$ some k_0 such that $d^k < \varepsilon/4$ for $k \geq k_0$ and some l_0 such that $M/2^{l_0} < \varepsilon$. For any $l > l_0$ we then get from (7) that

$$e^{k_0+l} \leq \frac{e^{k_0}}{2^l} + \sum_{j=1}^{l} \frac{d^{k_0+l-j}}{2^{j-1}} < \frac{M}{2^l} + \frac{\varepsilon}{4} \sum_{j=1}^{l} \frac{1}{2^{j-1}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

While the condition that the sequences $\boldsymbol{d}$ and $\boldsymbol{e}$ are null sequences is necessary for the convergence of the subdivision scheme, the following example demonstrates that this condition is not sufficient.

Example 1. Consider the interpolatory subdivision scheme that generates new points by the rule

$$p_{2i+1}^{k+1} = \frac{p_i^k + p_{i+1}^k}{2} + q \cdot \begin{cases} 1/(k+1), & \text{if } i = \lfloor 2^k/3 \rfloor, \\ 0, & \text{otherwise,} \end{cases}$$

for some $q \in \mathbb{R}^2$. Figure 4 illustrates the effect of this scheme for the initial data $p_0^0 = (0,0)$, $p_1^0 = (1,0)$ with $q = (0,1)$. At any level k, all edges are refined by inserting the midpoint, except for the unique edge that crosses the vertical line $\{(1/3, y) : y \in \mathbb{R}\}$, where a vertical offset of length $1/(k+1)$ is added to the midpoint. It is clear that $d^k = 1/(k+1)$ in this example and it can be seen from Figure 4 that

$$e^k \leq \frac{1}{2^k} + \frac{1}{k} + \frac{1}{k-1}$$

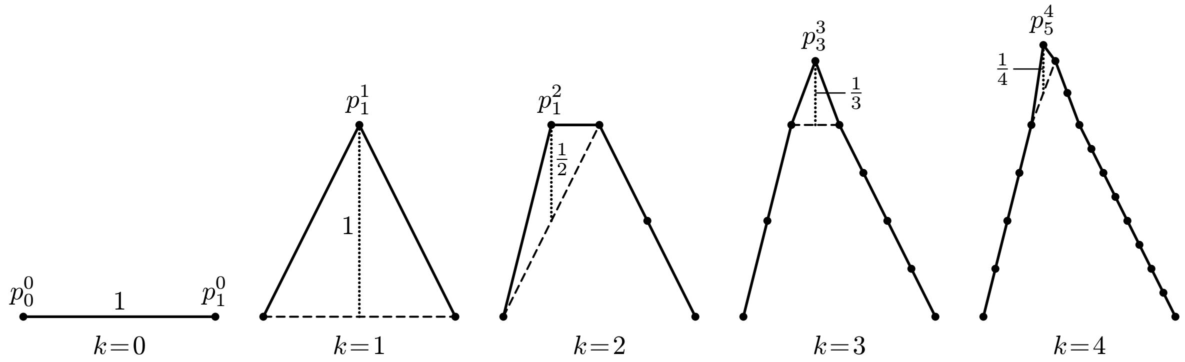

Figure 4: Illustration of the divergent subdivision scheme in Example 1.

for $k \geq 2$, so that both sequences $\boldsymbol{d}$ and $\boldsymbol{e}$ converge to zero. On the other hand, due to the divergence of the harmonic series, the y-coordinates of those points, that are inserted with the additional vertical offset, diverge to ∞ with k. Thus, the sequence of piecewise linear functions f^k does not converge uniformly in the L_∞-norm.

Therefore, we need a stronger condition to guarantee convergence.

Theorem 3. *If $\boldsymbol{d}$ is summable then $\mathcal{S}$ is convergent.*

Proof. By (5) we can bound the maximum distance between f^k and f^{k+l} for any $l \geq 1$,

$$\|f^{k+l} - f^k\|_\infty \leq \sum_{j=0}^{l-1} \|f^{k+j+1} - f^{k+j}\|_\infty = \sum_{j=0}^{l-1} d^{k+j}.$$

Now, if $\boldsymbol{d}$ is summable then it follows that $\{f^k\}_{k\geq 0}$ is a Cauchy sequence in the L_∞-norm and so the subdivision scheme $\mathcal{S}$ is convergent. □

Again, whenever the summability of $\boldsymbol{d}$ is hard to verify, we can equivalently check for the summability of $\boldsymbol{e}$.

Proposition 4. *The sequence $\boldsymbol{e}$ is summable if and only if $\boldsymbol{d}$ is summable.*

Proof. For any $n \geq 0$, summing the inequalities (6) for $0 \leq k \leq n$ gives

$$\sum_{k=0}^{n} d^k \leq \frac{1}{2}\sum_{k=0}^{n} e^k + \sum_{k=1}^{n+1} e^k \leq \frac{3}{2}\sum_{k=0}^{n+1} e^k,$$

and so the summability of $\boldsymbol{d}$ follows from the summability of $\boldsymbol{e}$.

On the other hand, summing the inequalities (7) for $0 \leq k \leq n$ gives

$$\sum_{k=0}^{n} e^k \leq e^0 + \sum_{k=0}^{n} e^{k+1} \leq e^0 + \frac{1}{2}\sum_{k=0}^{n} e^k + \sum_{k=0}^{n} d^k$$

and further

$$\sum_{k=0}^{n} e^k \leq 2e^0 + 2\sum_{k=0}^{n} d^k.$$

Thus, if $\boldsymbol{d}$ is summable, then so is $\boldsymbol{e}$. □

A standard approach for verifying the summability of $\boldsymbol{e}$ is to show that it is bounded from above by a convergent geometric sequence, that is, there exists some $0 < \mu < 1$ such that $e^{k+1} \leq \mu e^k$ for all $k \geq 0$. For example, the parameterization-based four-point schemes in [5] satisfy this condition. More generally, the weaker condition $e^{k+l} \leq \mu e^k$ for some $l \geq 1$ is also sufficient for the summability of $\boldsymbol{e}$.

An even stricter condition is to require the subdivision scheme to be *displacement-safe* [13], that is, there exists some $0 < \eta < 1$ such that $d_i^k \leq \eta e_i^k/2$ for all $i \in \mathbb{Z}$. Due to (7) this clearly implies $\boldsymbol{e}$ to be a geometric sequence with $\mu = (1+\eta)/2 < 1$, and it also guarantees that the orthogonal projection of the new point p_{2i+1}^{k+1} always lies in the interior of the corresponding edge $[p_i^k, p_{i+1}^k]$ at level k. For example, the centripetal four-point scheme in [5] satisfies this condition with $\eta = 1/2$.

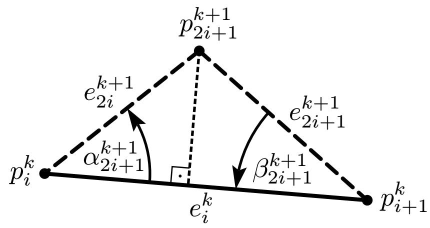

Figure 5: Relation between the edge lengths e_i^k, e_{2i}^{k+1}, e_{2i+1}^{k+1} and the angles α_{2i+1}^{k+1}, β_{2i+1}^{k+1}.

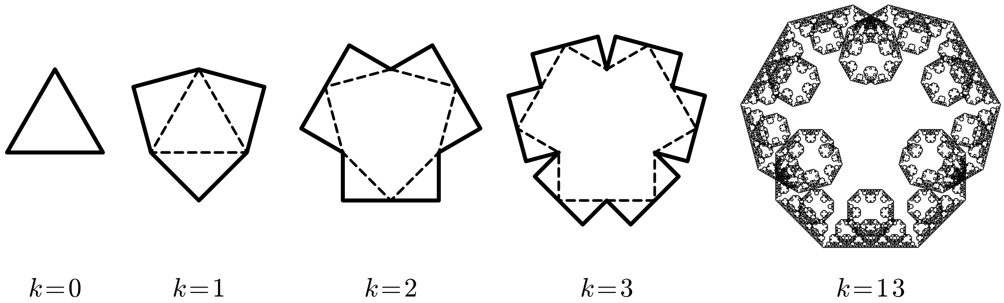

Figure 6: Illustration of the convergent bisector scheme in Example 2.

3.2 Angle conditions

Analogously to the definition of d^k and e^k in the previous section, let us denote

$$\alpha^k = \sup_{i\in\mathbb{Z}} |\alpha_i^k|, \qquad \beta^k = \sup_{i\in\mathbb{Z}} |\beta_i^k|, \qquad \delta^k = \sup_{i\in\mathbb{Z}} |\delta_i^k|,$$

and let

$$\boldsymbol{\alpha} = \{\alpha^k\}_{k\geq 0}, \qquad \boldsymbol{\beta} = \{\beta^k\}_{k\geq 0}, \qquad \boldsymbol{\delta} = \{\delta^k\}_{k\geq 0}.$$

At least for bisector schemes, the behaviour of these sequences is closely related to the convergence of the subdivision scheme.

Theorem 5. *If $\mathcal{S}$ is a bisector scheme and $\boldsymbol{\delta}$ is a null sequence, then $\mathcal{S}$ is convergent.*

Proof. For any bisector scheme, it is clear (see Figure 5) that

$$e_{2i}^{k+1} = e_{2i+1}^{k+1} = \frac{e_i^k}{2\cos(\delta_{2i+1}^{k+1}/2)}. \tag{8}$$

Now, if $\boldsymbol{\delta}$ is a null sequence, then there exists some k_0 such that $\delta^k \leq \pi/2$ for $k \geq k_0$ and so

$$e^{k+1} \leq \frac{e^k}{2\cos(\delta^{k+1}/2)} \leq \frac{e^k}{2\cos(\pi/4)} = \mu e^k$$

with $\mu = 1/\sqrt{2} < 1$, which guarantees the summability of $\boldsymbol{e}$. □

As in the case of distance conditions, there is again an equivalent sufficient condition that can be verified alternatively. This condition follows immediately from the fact that α_i^k, β_i^k, and δ_i^k always have the same sign (see Figure 1), and so

$$\max(\alpha^k, \beta^k) \leq \delta^k \leq \alpha^k + \beta^k \tag{9}$$

for all k.

Proposition 6. *The sequence $\boldsymbol{\delta}$ is a null sequence, if and only if both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are null sequences.*

However, while these conditions are sufficient for the convergence of a bisector scheme, they are not necessary, as demonstrated by the following example.

Example 2. Let us consider the bisector scheme that adds right triangles over each edge (see Figure 6). That is, it generates new points according to (4) with $\delta_{2i+1}^{k+1} = \pi/2$ for all i and at all levels k. Therefore, $\delta^k \geq \pi/2$ and $\boldsymbol{\delta}$ is not a null sequence. Still, it follows from (8) that $e^k \leq \mu e^{k+1}$ with $\mu = 1/\sqrt{2}$, hence $\boldsymbol{e}$ is summable and the scheme converges by Theorem 3 and Proposition 4.

4 Finite length

Assume now that $\mathcal{S}$ is a convergent subdivision scheme. Then a natural question to ask is whether or not the length of the limit curve is finite. Since our formulation allows for an infinite number of initial edges, we introduce the notion of *local finite length*, namely that the sequence $\boldsymbol{\ell} = \{\ell^k\}_{k\geq 0}$ with

$$\ell^k = \sup_{i\in\mathbb{Z}} \left(\sum_{j=2^k i}^{2^k(i+1)-1} e_j^k \right)$$

converges to some L. This implies that the length of the limit curve between any two consecutive initial points is finite, because the length of the polygonal line between these two points is monotonically increasing with k (see Figure 3) and bounded from above by L.

Example 2 shows that convergent schemes do not necessarily generate limit curves with local finite length, because in this example $\ell^{k+1} = \sqrt{2}\ell^k$. So, while the summability of $\boldsymbol{e}$ (or $\boldsymbol{d}$) does not guarantee limit curves with local finite length, the summability of $\boldsymbol{\delta}$ turns out to be sufficient.

Theorem 7. *If $\mathcal{S}$ is convergent and $\boldsymbol{\delta}$ is summable, then the limit curve has local finite length.*

Proof. We first observe (see Figure 5) that

$$e_{2i}^{k+1} \cos\alpha_{2i+1}^{k+1} + e_{2i+1}^{k+1} \cos\beta_{2i+1}^{k+1} = e_i^k. \tag{10}$$

Since $\mathcal{S}$ is convergent, $\boldsymbol{e}$ is a null sequence by Theorem 1 and Proposition 2. This together with the summability of $\boldsymbol{\delta}$ implies the existence of some k_0 such that e^{k_0} is finite and $\delta^k < \pi/2$ for all $k \geq k_0$. It then follows from (9) and (10) that

$$e_{2i}^{k+1} + e_{2i+1}^{k+1} \leq \frac{e_i^k}{\cos\delta^{k+1}},$$

and so

$$\ell^{k+1} \leq \frac{1}{\cos\delta^{k+1}} \ell^k \leq \left(\prod_{j=k_0+1}^{k+1} \frac{1}{\cos\delta^j} \right) \ell^{k_0}.$$

Now, observing that $\ell^{k_0} \leq 2^{k_0} e^{k_0}$, we conclude that the limit curve has local finite length if the infinite product

$$Q = \prod_{j=k_0+1}^{\infty} \frac{1}{\cos\delta^j}$$

is finite. But this follows from the summability of $\boldsymbol{\delta}$ and the inequality $1/\cos(x) - 1 \leq x$ for sufficiently small non-negative x. □

It follows directly from (9) that the condition on $\boldsymbol{\delta}$ in Theorem 7 can be replaced by an equivalent condition on the sequences $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

Proposition 8. *The sequence $\boldsymbol{\delta}$ is summable if and only if both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are summable.*

5 Tangent continuity

An important question concerning convergent subdivision schemes is the smoothness of the limit curves. The simplest notion of geometric smoothness (G^1) is the existence of a continuously varying directed tangent along the curve, and it turns out that it is again the summability of $\boldsymbol{\delta}$ that guarantees this property.

By the standard definition, the tangent to a curve $\mathcal{C}$ at some $p \in \mathcal{C}$ is the limit of the secant lines through p and $q \in \mathcal{C}$ as q approaches p. As this definition also allows for a cusp at p, we resort to the slightly different notion of a *directed tangent* that excludes this kind of singularity at the cost of assuming that $\mathcal{C}$ has a consistent local orientation, which is given in the setting of subdivision curves. A directed line t is the directed tangent at p, if the directed lines $\overrightarrow{pq}$ converge to t as q approaches p from the right and so do the directed lines $\overrightarrow{qp}$ as q approaches p from the left.

To formalize the definition of a G^1 curve, we first introduce the notion of the *positive ε-cone* and the *negative ε-cone* at a point q relative to a directed line s,

$$C^+(q,s,\varepsilon) = \{x : |\sphericalangle(s, \overrightarrow{qx})| \leq \varepsilon\} \qquad \text{and} \qquad C^-(q,s,\varepsilon) = \{x : |\sphericalangle(s, \overrightarrow{xq})| \leq \varepsilon\}.$$

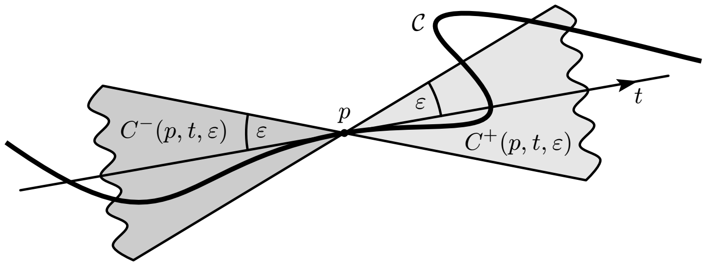

Figure 7: Definition of the positive (light grey) and the negative (dark grey) ε-cone around t at p.

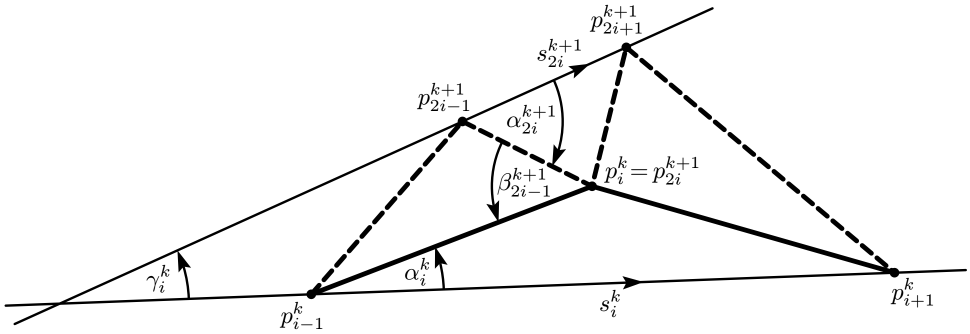

Figure 8: Definition of secants s_i^k and angles γ_i^k.

Then a directed line t through p is the directed tangent to $\mathcal{C}$ at p, if for any $\varepsilon > 0$ and any sequence $\boldsymbol{q} = \{q_n\}_{n\geq 0}$ of points $q_n \in \mathcal{C}$ converging to p from the right or from the left, there exists some n_0 such that q_n for $n > n_0$ is contained in $C^{+}(p,t,\varepsilon)$, if $\boldsymbol{q}$ approaches p from the right, and in $C^{-}(p,t,\varepsilon)$, if $\boldsymbol{q}$ approaches p from the left (see Figure 7). Finally, a curve $\mathcal{C}$ is G^1, if it has a directed tangent at each point $p \in \mathcal{C}$, and if the directed tangent at $q \in \mathcal{C}$ converges to the one at p as q approaches p along $\mathcal{C}$.

The proof of our sufficient condition for G^1 limit curves is based on the notion of secants at level k and the analysis of their behaviour as k increases. Let

$$s_i^k = \overrightarrow{p_{i-1}^k p_{i+1}^k}$$

be the *secant related to* p_i^k *at level* k. Note that if $\boldsymbol{\delta}$ is summable, then $\delta^k < \pi$ for sufficiently large k, and so $p_{i-1}^k \neq p_{i+1}^k$. Hence, all secants are well-defined and our analysis starts at this level, denoted for simplicity by $k = 0$. We let $\gamma_i^k = \sphericalangle(s_i^k, s_{2i}^{k+1})$ be the angle between the two secants related to p_i^k at levels k and $k+1$, and define $\gamma^k = \sup_{i\in\mathbb{Z}} |\gamma_i^k|$ and $\boldsymbol{\gamma} = \{\gamma^k\}_{k\geq 0}$. Not surprisingly, the summability of $\boldsymbol{\gamma}$ is again related to that of $\boldsymbol{\delta}$.

Proposition 9. *If* $\boldsymbol{\delta}$ *is summable, then so is* $\boldsymbol{\gamma}$.

Proof. By (1),

$$\gamma_i^k = \alpha_i^k - \beta_{2i-1}^{k+1} - \alpha_{2i}^{k+1},$$

as can also be seen in Figure 8. Hence the claim follows from Proposition 8. □

We first show that the sequence $\{s_{2^j i}^{k+j}\}_{j\geq 0}$ of all secants related to p_i^k converges to a *limit secant* $\bar{s}_i^k$.

Theorem 10. *If* $\mathcal{S}$ *is convergent and* $\boldsymbol{\delta}$ *is summable, then there exists a limit secant* $\bar{s}_i^k$ *through* p_i^k *for all* $i \in \mathbb{Z}$ *and* $k \geq 0$.

Proof. It follows from Proposition 9 that $\Gamma_i^k = \sum_{j=0}^{\infty} \gamma_{2^j i}^{k+j}$ is finite, and from the definition of γ_i^k and (1) that $\Gamma_i^k = \lim_{j\to\infty} \sphericalangle(s_i^k, s_{2^j i}^{k+j})$. Now we let $\bar{s}_i^k$ be the directed line which passes through p_i^k such that $\sphericalangle(s_i^k, \bar{s}_i^k) = \Gamma_i^k$ and show that it is indeed the limit of the sequence $\{s_{2^j i}^{k+j}\}_{j\geq 0}$. On the one hand,

$$\lim_{j\to\infty} \sphericalangle(s_{2^j i}^{k+j}, \bar{s}_i^k) = \lim_{j\to\infty} \big(\sphericalangle(s_{2^j i}^{k+j}, s_i^k) + \sphericalangle(s_i^k, \bar{s}_i^k)\big) = -\lim_{j\to\infty} \sphericalangle(s_i^k, s_{2^j i}^{k+j}) + \Gamma_i^k = 0. \tag{11}$$

On the other hand, since $p_{2^j i+1}^{k+j}$ is a point on $s_{2^j i}^{k+j}$ and $\|p_{2^j i+1}^{k+j} - p_i^k\| = e_{2^j i}^{k+j} \leq e^{k+j}$, we have $\lim_{j\to\infty} p_{2^j i+1}^{k+j} = p_i^k$ by Theorem 1 and Proposition 2. □

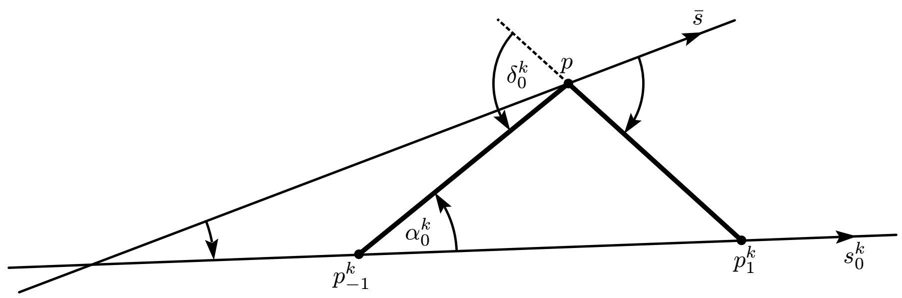

Figure 9: Illustration of Equation (12).

Next we show that the limit secant $\bar{s}_i^k$ is actually the directed tangent to the limit curve $\mathcal{P}$ at p_i^k. Without loss of generality, it is sufficient to consider only the point $p = p_0^0$ with limit secant $\bar{s} = \bar{s}_0^0$ and the limit curve to its "right". By symmetry, all results also apply to the "left" of p. For the sake of simplicity, the following results are all stated and proved for this special case, but hold for any point p_i^k. Our first observation is that from a certain subdivision level on, the "right neighbour" of p is in $C^+(\varepsilon) = C^+(p, \bar{s}, \varepsilon)$.

Lemma 11. *If $\mathcal{S}$ is convergent and $\boldsymbol{\delta}$ is summable, then there exists for any $\varepsilon > 0$ some k_0 such that $p_1^k \in C^+(\varepsilon)$ for $k \geq k_0$.*

Proof. By (11) and Proposition 8 there exists for any $\varepsilon > 0$ some k_0 such that

$$|\sphericalangle(\bar{s}, s_0^k)| < \frac{\varepsilon}{3}, \qquad \alpha^k < \frac{\varepsilon}{3}, \qquad \delta^k < \frac{\varepsilon}{3}$$

for $k \geq k_0$. Hence the statement follows from the identity

$$\sphericalangle(\bar{s}, \overrightarrow{pp_1^k}) = \sphericalangle(\bar{s}, s_0^k) + \alpha_0^k - \delta_0^k, \tag{12}$$

obtained by (1) and illustrated in Figure 9, because it implies

$$\left|\sphericalangle(\bar{s}, \overrightarrow{pp_1^k})\right| \leq |\sphericalangle(\bar{s}, s_0^k)| + |\alpha_0^k| + |\delta_0^k| < \varepsilon.$$

□

We now extend this observation and show that all points generated by the scheme between p and some right neighbour of p lie in $C^+(\varepsilon)$.

Lemma 12. *If $\mathcal{S}$ is convergent and $\boldsymbol{\delta}$ is summable, then there exists for any $\varepsilon > 0$ some k_0 such that $p_i^k \in C^+(\varepsilon)$ for $1 \leq i \leq 2^{k-k_0}$ and $k \geq k_0$.*

Proof. First we prove a simple geometric observation (see Figure 10), which is the key to our inductive proof of the lemma. Consider an edge $e = [p_1, p_2]$, a point $p \notin e$, and the cone C in between the directed lines $\overrightarrow{pp_1}$ and $\overrightarrow{pp_2}$. Let q_1 be a point on the perpendicular bisector of e and q_2 be the reflection of q_1 about e. Without loss of generality, p and q_2 both lie in the right half-plane with respect to the directed line $\overrightarrow{p_1p_2}$. Moreover, let δ be the (unsigned) angles of the triangle $\triangle(p_1, p_2, q_1)$ at p_1 and p_2. In case q_1 does not lie inside C, then the angle between $\overrightarrow{pq_1}$ and $\overrightarrow{pp_1}$ is $\theta_1 < \eta_1 \leq \delta$. Similarly, if $q_2 \notin C$, then the angle between $\overrightarrow{pq_2}$ and $\overrightarrow{pp_2}$ is $\theta_2 < \eta_2 < 2\delta$. Now, if we enlarge the opening angle of C by 2δ in both directions and call this larger cone $C_{2\delta}$, then it follows that the rhombus $\lozenge(p_1, p_2, q_1, q_2)$ is contained in $C_{2\delta}$.

By the summability of $\boldsymbol{\delta}$ and Lemma 11 there exists for any $\varepsilon > 0$ some k_0 such that

$$p_1^k \in C^+\left(\frac{\varepsilon}{2}\right) \qquad \text{and} \qquad \varepsilon^k = \frac{\varepsilon}{2} + 2\sum_{j=k_0+1}^{k} \delta^j < \varepsilon \tag{13}$$

for $k \geq k_0$. Under these conditions, we now prove by induction over k that

$$p_i^k \in C^+(\varepsilon^k), \qquad 1 \leq i \leq 2^{k-k_0} \tag{14}$$

for any $k \geq k_0$. The base case $k = k_0$ follows immediately from (13), so let us assume that (14) is true for some $k > k_0$. At the next refinement level $k+1$, the points $p_{2i}^{k+1} = p_i^k$ for $1 \leq i \leq 2^{k-k_0}$ lie inside $C^+(\varepsilon^k) \subset C^+(\varepsilon^{k+1})$

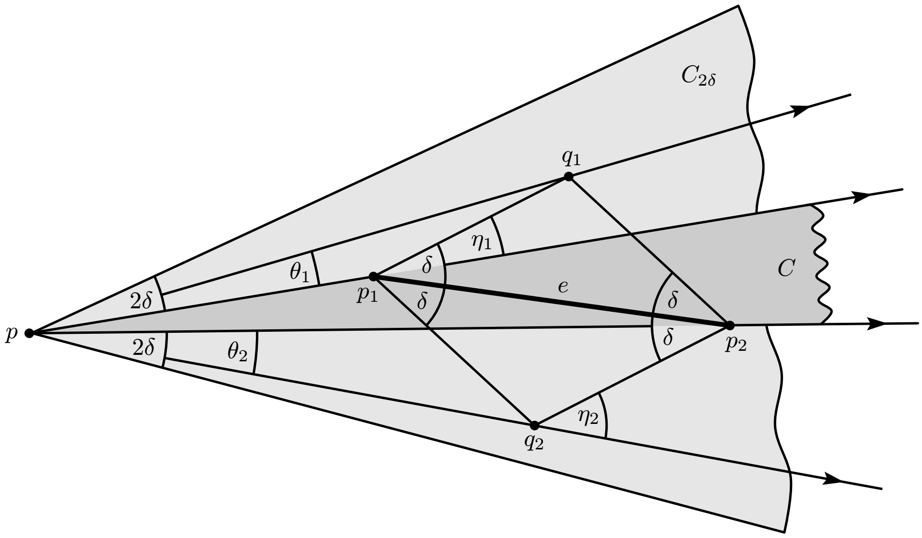

Figure 10: The rhombus $\Diamond(q_1, p_1, q_2, p_2)$ with diagonal e inside the cone C lies inside the larger cone $C_{2\delta}$ (for visualization purposes, the angles 2δ appear smaller than the angles δ).

by the induction hypothesis, and $p_1^{k+1} \in C^+(\varepsilon^{k+1})$ by (13). Any remaining point is an inserted point p_{2i+1}^{k+1}, corresponding to the edge $[p_i^k, p_{i+1}^k]$. Due to (9), it is contained in the rhombus with diagonal $[p_i^k, p_{i+1}^k]$ and angles $2\delta^{k+1}$ at p_i^k and p_{i+1}^k. By the initial geometric observation, it follows that $p_{2i+1}^{k+1} \in C^+(\varepsilon^{k+1})$, which completes the inductive step. □

The statement of Lemma 12 can be extended to all the points on the part of the limit curve strictly between p and $p_1^{k_0}$, which we denote by $\mathcal{P}(p, p_1^{k_0})$, because the limit curve $\mathcal{P}$ is continuous and the points generated by an interpolatory scheme are dense on $\mathcal{P}$.

Corollary 13. *If $\mathcal{S}$ is convergent and $\boldsymbol{\delta}$ is summable, then there exists for any $\varepsilon > 0$ some k_0 such that $\mathcal{P}(p, p_1^{k_0}) \in C^+(\varepsilon)$.*

This is the key to finally proving the existence of a directed tangent at p.

Theorem 14. *If $\mathcal{S}$ is convergent and $\boldsymbol{\delta}$ is summable, then the limit secant $\bar{s}$ is the directed tangent to the limit curve at p.*

Proof. By Corollary 13, there exists for any $\varepsilon > 0$ some k_0 such that $\mathcal{P}(p, p_1^{k_0}) \subset C^+(\varepsilon)$. Therefore, if $\boldsymbol{q} = \{q_j\}_{j\geq 0}$ is a sequence of points $q_j \in \mathcal{P}$ which converges to p from the right, then there exists some j_0 such that $q_j \in \mathcal{P}(p, p_1^{k_0})$ for $j > j_0$, and so $q_j \in C^+(\varepsilon) = C^+(p, \bar{s}, \varepsilon)$. By symmetry, any sequence of points on $\mathcal{P}$ that converges to p from the left, is eventually in $C^-(p, \bar{s}, \varepsilon)$. Thus, $\bar{s}$ is the directed tangent to $\mathcal{P}$ at p by definition. □

As remarked above, the reasoning that leads to Theorem 14 applies to any point p_i^k, hence the directed tangent to $\mathcal{P}$ at p_i^k is $\bar{s}_i^k$. The next step is to verify that these tangents change continuously. We first show that the angle between the directed tangent $\bar{s}$ at p and any edge in a certain neighbourhood of p is arbitrarily small.

Lemma 15. *If $\mathcal{S}$ is convergent and $\boldsymbol{\delta}$ is summable, then there exists for any $\varepsilon > 0$ some k_0 such that $\left|\sphericalangle(\bar{s}, \overrightarrow{p_i^k p_{i+1}^k})\right| \leq \varepsilon$ for $k \geq k_0$ and $0 \leq i < 2^{k-k_0}$.*

Proof. By the summability of $\boldsymbol{\delta}$ and Lemma 11, there exists for any $\varepsilon > 0$ some k_0 such that

$$\left|\sphericalangle(\bar{s}, \overrightarrow{p_0^k p_1^k})\right| \leq \frac{\varepsilon}{2} \qquad \text{and} \qquad \varepsilon^k = \frac{\varepsilon}{2} + \sum_{j=k_0+1}^{k} \delta^j < \varepsilon \tag{15}$$

for $k \geq k_0$. Under these conditions, we now prove by induction over k that

$$\left|\sphericalangle(\bar{s}, \overrightarrow{p_i^k p_{i+1}^k})\right| \leq \varepsilon^k, \qquad 0 \leq i < 2^{k-k_0} \tag{16}$$

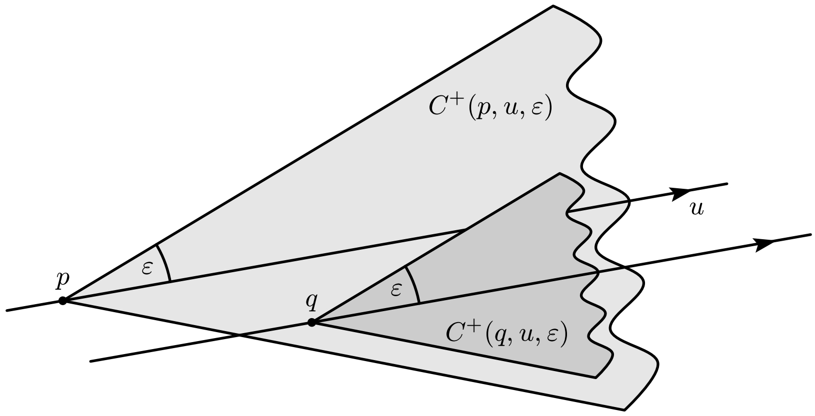

Figure 11: For any point $q \in C^+(p,u,\varepsilon)$, the ε-cone $C^+(q,u,\varepsilon)$ is contained in $C^+(p,u,\varepsilon)$.

for any $k \geq k_0$. The base case $k = k_0$ follows immediately from (15), so let us assume that (16) is true for some $k \geq k_0$. Each of the edges $[p_i^k, p_{i+1}^k]$ is replaced at the next refinement level $k+1$ by two new edges $[p_{2i}^{k+1}, p_{2i+1}^{k+1}]$ and $[p_{2i+1}^{k+1}, p_{2i+2}^{k+1}]$. By (1), we then have

$$\sphericalangle(\bar{s}, \overrightarrow{p_{2i}^{k+1} p_{2i+1}^{k+1}}) = \sphericalangle(\bar{s}, \overrightarrow{p_i^k p_{i+1}^k}) + \alpha_{2i+1}^{k+1} \qquad \text{and} \qquad \sphericalangle(\bar{s}, \overrightarrow{p_{2i+2}^{k+1} p_{2i+2}^{k+1}}) = \sphericalangle(\bar{s}, \overrightarrow{p_i^k p_{i+1}^k}) - \beta_{2i+1}^{k+1},$$

and the validity of (16) for $k+1$ follows from the triangle inequality and (9). □

A direct consequence of this lemma is that also the angle between $\bar{s}$ and the line through any two points generated by the scheme in a certain neighbourhood of p is arbitrarily small.

Corollary 16. *If $\mathcal{S}$ is convergent and $\boldsymbol{\delta}$ is summable, then there exists for any $\varepsilon > 0$ some k_0 such that $\left|\sphericalangle(\bar{s}, \overrightarrow{p_i^k p_j^k})\right| \leq \varepsilon$ for $k \geq k_0$ and $0 \leq i < j \leq 2^{k-k_0}$.*

Proof. We first observe that

$$p_{l+1}^k \in C^+(p_l^k, \bar{s}, \varepsilon)$$

for $0 \leq l < 2^{k-k_0}$, which follows directly from Lemma 15. By a simple geometric consideration (see Figure 11), this implies

$$C^+(p_{l+1}^k, \bar{s}, \varepsilon) \subset C^+(p_l^k, \bar{s}, \varepsilon).$$

Hence,

$$p_j^k \in C^+(p_{j-1}^k, \bar{s}, \varepsilon) \subset \cdots \subset C^+(p_i^k, \bar{s}, \varepsilon).$$

□

We now extend this observation to the angle between $\bar{s}$ and any directed tangent $\bar{s}_i^k$ at nearby points on the limit curve.

Lemma 17. *If $\mathcal{S}$ is convergent and $\boldsymbol{\delta}$ is summable, then there exists for any $\varepsilon > 0$ some k_0 such that $|\sphericalangle(\bar{s}, \bar{s}_i^k)| \leq \varepsilon$ for $k \geq k_0$ and $0 < i < 2^{k-k_0}$.*

Proof. By Corollary 16 and Theorem 10, there exists for any $\varepsilon > 0$ some k_0 such that

$$\left|\sphericalangle(\bar{s}, \overrightarrow{p_{i-1}^k p_{i+1}^k})\right| \leq \varepsilon/2 \qquad \text{and} \qquad |\sphericalangle(s_i^k, \bar{s}_i^k)| \leq \varepsilon/2$$

for $k \geq k_0$ and $0 < i < 2^{k-k_0}$. Since $s_i^k = \overrightarrow{p_{i-1}^k p_{i+1}^k}$, the statement follows by the triangle inequality. □

Again, the result of Lemma 17 holds more generally for any point p_i^k and its directed tangent $\bar{s}_i^k$. Thus, the directed tangent to the limit curve $\mathcal{P}$ varies continuously on the set $\boldsymbol{P} = \{p_i^k : k \geq 0, i \in \mathbb{Z}\}$, which is dense on $\mathcal{P}$. However, we still need to show that this behaviour extends to all points on the limit curve.

Theorem 18. *If $\mathcal{S}$ is convergent and $\boldsymbol{\delta}$ is summable, then the limit curve is G^1.*

Proof. Let us continuously extend the tangents $\bar{s}_i^k$ to all the points of the limit curve $\mathcal{P}$. That is, we attach to any point $q \in \mathcal{P} \setminus \boldsymbol{P}$ the directed line

$$t = \lim_{n\to\infty} \bar{s}_n,$$

where $\{p_n\}_{n\geq 0}$ is any sequence in $\boldsymbol{P}$ that converges to q and $\bar{s}_n$ is the directed tangent at p_n. As remarked above, Lemma 17 guarantees the existence and uniqueness of this limit. To prove the statement of the theorem, it now remains to show that t is indeed the directed tangent to the limit curve at q. Obviously t passes through q.

By the convergence of $\mathcal{S}$ and Corollary 13, there exist for any $\varepsilon > 0$ some level k_0 and an index i such that $\mathcal{P}(p_i^{k_0}, p_{i+1}^{k_0})$ contains q and is contained in $C^+(p_i^{k_0}, \bar{s}_i^{k_0}, \varepsilon)$. Without loss of generality, we further assume that k_0 is sufficiently large so that the conditions of Corollary 16 and Lemma 17 are satisfied.

Now consider any two points $p, r \in \boldsymbol{P}$ such that the order of the points along $\mathcal{P}$ is $p_i^{k_0}, p, q, r, p_{i+1}^{k_0}$. Further let $\bar{s}$ be the directed tangent at p. It then follows from Lemma 17 and Corollary 16 that

$$|\sphericalangle(\bar{s}, \overrightarrow{pr})| \leq |\sphericalangle(\bar{s}, \bar{s}_i^{k_0})| + |\sphericalangle(\bar{s}_i^{k_0}, \overrightarrow{pr})| \leq 2\varepsilon. \tag{17}$$

Hence, r is contained in $C^+(p, \bar{s}, 2\varepsilon)$.

Now let us go back to an arbitrary sequence $\{p_n\}_{n\geq 0}$ in $\boldsymbol{P}$ that converges to q and the corresponding sequence of directed tangents $\{\bar{s}_n\}_{n\geq 0}$ as above. For sufficiently large n we have

$$p_n \in \mathcal{P}(p_i^{k_0}, r), \qquad |\sphericalangle(\overrightarrow{p_n r}, \overrightarrow{qr})| < \varepsilon, \qquad |\sphericalangle(t, \bar{s}_n)| < \varepsilon.$$

This, together with (17) implies

$$|\sphericalangle(t, \overrightarrow{qr})| \leq |\sphericalangle(t, \bar{s}_n)| + |\sphericalangle(\bar{s}_n, \overrightarrow{p_n r})| + |\sphericalangle(\overrightarrow{p_n r}, \overrightarrow{qr})| < 4\varepsilon,$$

that is, $r \in C^+(q, t, 4\varepsilon)$. Since $r \in \boldsymbol{P}$ is an arbitrary point between q and $p_{i+1}^{k_0}$ and since $\mathcal{S}$ is convergent, we conclude that

$$\mathcal{P}(q, p_{i+1}^{k_0}) \subset C^+(q, t, 4\varepsilon),$$

and similarly $\mathcal{P}(p_i^{k_0}, q) \subset C^-(q, t, 4\varepsilon)$. Therefore, t is the directed tangent to $\mathcal{P}$ at q. □

The following is an immediate consequence of Theorems 5 and 18.

Corollary 19. *If $\mathcal{S}$ is a bisector scheme and $\boldsymbol{\delta}$ is summable then $\mathcal{S}$ is convergent and the limit curve is G^1.*

6 Two geometric four-point schemes with tangent continuity

Let us now show the applicability of our machinery by analysing two new subdivision schemes. Both of them are bisector schemes and the position of a new point at level $k+1$ depends on four successive points at level k. Hence, they can be considered geometric variations of the linear four-point scheme [4, 6]. For both schemes, the key idea is to derive a relation between the exterior angles δ_i^k at some level k and the exterior angles at some higher level $k+l$ for $l \geq 1$. This relation is used to show that $\delta^{k+l} \leq \mu\delta^k$ for some $\mu < 1$, which ensures that the sequence $\boldsymbol{\delta}$ is summable. Therefore, by Corollary 19, the schemes converge and the limit curves are G^1.

Two important relations between the angles at levels k and $k+1$ that we utilize for the analysis of both schemes are

$$\delta_{2i}^{k+1} = \delta_i^k - \beta_{2i-1}^{k+1} - \alpha_{2i+1}^{k+1} \qquad \text{and} \qquad \delta_{2i+1}^{k+1} = \alpha_{2i+1}^{k+1} + \beta_{2i+1}^{k+1}, \tag{18}$$

as illustrated in Figure 12. Note that these identities hold *not only* for bisector schemes, but rather for *any* interpolatory subdivision scheme.

6.1 An angle-based geometric four-point scheme

The first scheme is based on quadrisecting and averaging the exterior angles. At any level k, we let

$$\alpha_{2i+1}^{k+1} = \beta_{2i+1}^{k+1} = \frac{\delta_i^k + \delta_{i+1}^k}{8}. \tag{19}$$

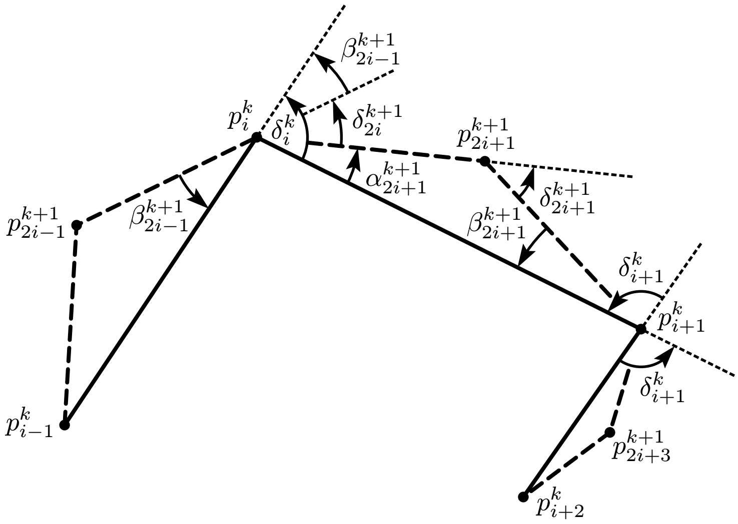

Figure 12: Explanation of the identities in (18).

The coordinates of the new points p_{2i+1}^{k+1} are then computed by the formula in (4). According to (18), we have

$$\delta_{2i}^{k+1} = \delta_i^k - \frac{\delta_{i-1}^k + \delta_i^k}{8} - \frac{\delta_i^k + \delta_{i+1}^k}{8} = \frac{-\delta_{i-1}^k + 6\delta_i^k - \delta_{i+1}^k}{8} \qquad \text{and} \qquad \delta_{2i+1}^{k+1} = \frac{\delta_i^k + \delta_{i+1}^k}{4} \tag{20}$$

for the exterior angles at level $k+1$. Applying this recurrence relation twice yields

$$\delta_{4i}^{k+2} = \frac{-\delta_{i-1}^k + 4\delta_i^k - \delta_{i+1}^k}{8}, \qquad \delta_{4i+1}^{k+2} = \frac{-\delta_{i-1}^k + 8\delta_i^k + \delta_{i+1}^k}{32},$$

$$\delta_{4i+2}^{k+2} = \frac{\delta_{i-1}^k + 7\delta_i^k + 7\delta_{i+1}^k + \delta_{i+2}^k}{64}, \qquad \delta_{4i+3}^{k+2} = \frac{\delta_{i-1}^k + 8\delta_i^k - \delta_{i+1}^k}{32}.$$

Therefore,

$$\delta^{k+2} \le \max\left\{\frac{3}{4}, \frac{5}{16}, \frac{1}{4}, \frac{5}{16}\right\}\delta^k = \frac{3}{4}\delta^k,$$

which implies that $\boldsymbol{\delta}$ is summable.

Remark 1. This scheme produces a circle as the limit curve whenever P^0 is a regular polygon. In this case, all initial exterior angles are the same, $\delta_i^0 = \delta^0$, and it follows from (20) that this property is kept at all levels $k \ge 1$ with $\delta^k = 2^{-k}\delta^0$. As the scheme is interpolatory, it generates a sequence of regular polygons with common radius and centre, which clearly converges to the circle with the same radius and centre.

Remark 2. The recurrence relation in (20) has the same coefficients as the one for the differences of the first divided differences of the linear four-point scheme with $\omega = 1/16$; see [6]. Moreover, the latter recurrence relation for general ω corresponds to the bisector scheme with the rule (19) replaced by

$$\alpha_{2i+1}^{k+1} = \beta_{2i+1}^{k+1} = 2\omega(\delta_i^k + \delta_{i+1}^k).$$

The analysis in [10] then guarantees the summability of $\boldsymbol{\delta}$ for $0 < \omega < \omega^* \approx 0.19273$.

6.2 A circle-based geometric four-point scheme

The second scheme is based on the observation that the insertion rule of the linear four-point scheme with $\omega = 1/16$ can be derived from fitting and evaluating two quadratic polynomials and averaging the results. To determine the new point p_{2i+1}^{k+1} analogously in the geometric setting, we first consider the two circles that interpolate the points $p_{i-1}^k, p_i^k, p_{i+1}^k$ and $p_i^k, p_{i+1}^k, p_{i+2}^k$, respectively, and intersect them with the perpendicular bisector of the edge $[p_i^k, p_{i+1}^k]$. This yields two points, q_l and q_r (see Figure 13), which need to be averaged in some way. Instead of taking their midpoint, we average the angles $\theta_l = \sphericalangle(\overrightarrow{p_i^k p_{i+1}^k}, \overrightarrow{p_i^k q_l})$ and $\theta_r = \sphericalangle(\overrightarrow{p_i^k p_{i+1}^k}, \overrightarrow{p_i^k q_r})$,

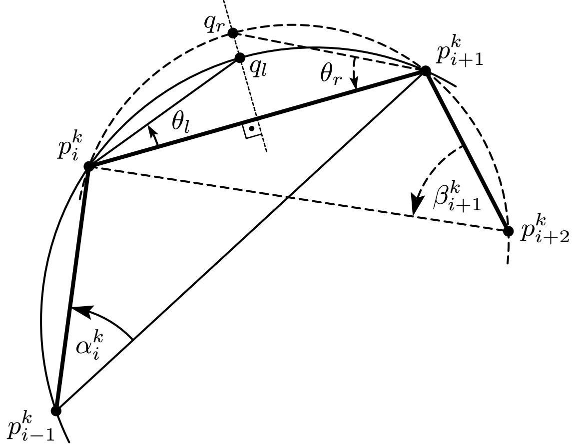

Figure 13: Construction of the points q_l and q_r for the refinement rule of the circle-based scheme.

and let p_{2i+1}^{k+1} be the point on the perpendicular bisector with $\sphericalangle(\overrightarrow{p_i^k p_{i+1}^k}, \overrightarrow{p_i^k p_{2i+1}^{k+1}}) = (\theta_l + \theta_r)/2$. Some basic geometric considerations reveal that $\theta_l = \alpha_i^k/2$ and $\theta_r = \beta_{i+1}^k/2$, hence this circle-based scheme is characterized by the refinement rule

$$\alpha_{2i+1}^{k+1} = \beta_{2i+1}^{k+1} = \frac{\alpha_i^k + \beta_{i+1}^k}{4}. \tag{21}$$

This construction clearly guarantees that the scheme reproduces circles, that is, if all points of P^0 lie on a common circle C, then $\mathcal{P} = C$.

The reason for averaging angles rather than taking the midpoint of q_l and q_r is that this choice, by (18), gives the recurrence relations

$$\delta_{2i}^{k+1} = \frac{-\alpha_{i-1}^k + 3\delta_i^k - \beta_{i+1}^k}{4} \qquad \text{and} \qquad \delta_{2i+1}^{k+1} = \frac{\alpha_i^k + \beta_{i+1}^k}{2}. \tag{22}$$

Using (9) and noticing that either i or $i+1$ is odd, and hence either $\alpha_i^k = \delta_i^k/2$ or $\beta_{i+1}^k = \delta_{i+1}^k/2$ for $k > 0$, we have the bounds

$$|\delta_{2i}^{k+1}| \le \frac{5}{4}\delta^k \qquad \text{and} \qquad |\delta_{2i+1}^{k+1}| \le \frac{3}{4}\delta^k. \tag{23}$$

Since this is not sufficient to conclude the summability of $\boldsymbol{\delta}$, let us consider a double step. According to (22),

$$\delta_{4i}^{k+2} = -\frac{1}{4}\alpha_{2i-1}^{k+1} + \frac{3}{4}\delta_{2i}^{k+1} - \frac{1}{4}\beta_{2i+1}^{k+1}, \qquad \delta_{4i+1}^{k+2} = \frac{1}{2}\alpha_{2i}^{k+1} + \frac{1}{2}\beta_{2i+1}^{k+1},$$

$$\delta_{4i+2}^{k+2} = -\frac{1}{4}\alpha_{2i}^{k+1} + \frac{3}{4}\delta_{2i+1}^{k+1} - \frac{1}{4}\beta_{2i+2}^{k+1}, \qquad \delta_{4i+3}^{k+2} = \frac{1}{2}\alpha_{2i+1}^{k+1} + \frac{1}{2}\beta_{2i+2}^{k+1}.$$

For δ_{4i}^{k+2}, we apply the recurrence relations (21) and (22) again and use (2) to get

$$\delta_{4i}^{k+2} = -\frac{1}{4}\alpha_{i-1}^k + \frac{1}{2}\delta_i^k - \frac{1}{4}\beta_{i+1}^k \qquad \Longrightarrow \qquad |\delta_{4i}^{k+2}| \le \delta^k. \tag{24}$$

For the other three terms, it is better to recall that $\alpha_{2i+1}^{k+1} = \beta_{2i+1}^{k+1} = \delta_{2i+1}^{k+1}/2$ and to utilize the bounds in (9) and (23) to get

$$|\delta_{4i+1}^{k+2}| \le \frac{13}{16}\delta^k, \qquad |\delta_{4i+2}^{k+2}| \le \frac{19}{16}\delta^k, \qquad |\delta_{4i+3}^{k+2}| \le \frac{13}{16}\delta^k. \tag{25}$$

Again, this is not sufficient and more iterations are needed, but now we can exploit the bounds in (24) and (25), together with the recurrence relations (21) and (22), to establish bounds of the kind $|\delta_{8i+j}^{k+3}| \le \mu_{3,j}\delta^k$ for $j = 0,\dots,7$. Iterating this strategy allows to determine the constants $\mu_l = \max_{j=0,\dots,2^l-1}\mu_{l,j}$, which are listed in Table 1, and since $\mu_{10} < 1$, we finally conclude that $\boldsymbol{\delta}$ is summable.

l	1	2	3	4	5	6	7	8	9	10
μ_l	$\frac{5}{4}$	$\frac{19}{16}$	$\frac{37}{32}$	$\frac{291}{256}$	$\frac{283}{256}$	$\frac{4435}{4096}$	$\frac{8651}{8192}$	$\frac{67663}{65536}$	$\frac{33031}{32768}$	$\frac{1032775}{1048576}$

Table 1: Constants μ_l of the bounds $\delta^{k+l} \le \mu_l \delta^k$ for $l = 1, \dots, 10$.

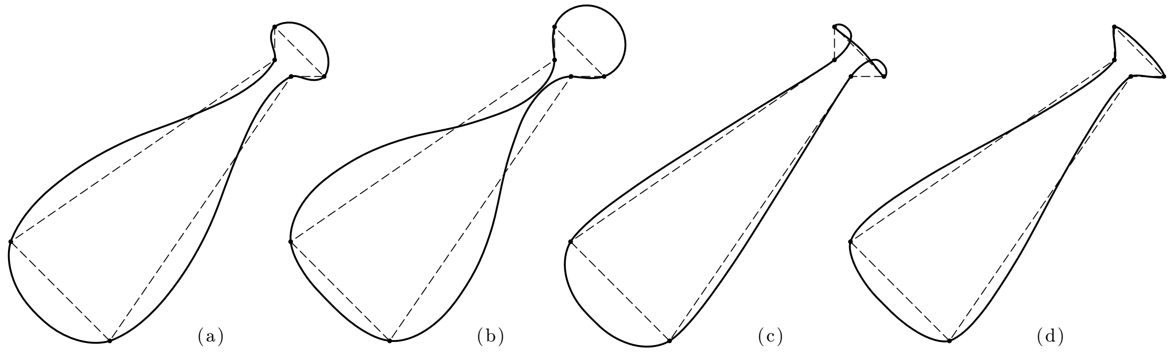

Figure 14: Comparison of the angle-based scheme (a), the circle-based scheme (b), the linear four-point scheme (c), and the centripetal four-point scheme (d) for a control polygon with varying edge lengths.

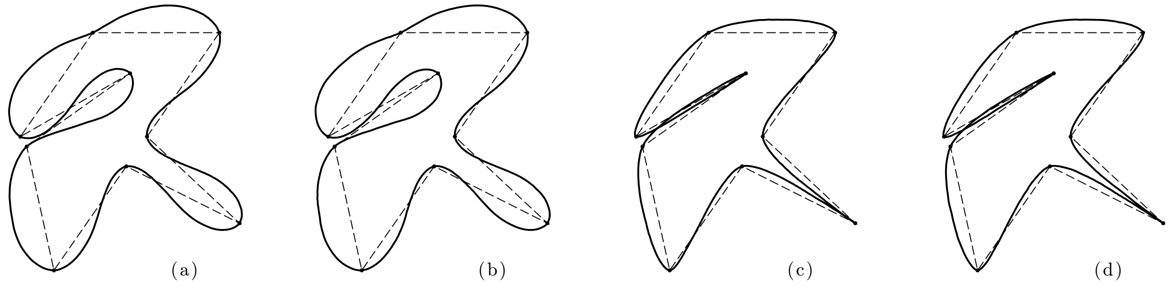

Figure 15: Comparison of the angle-based scheme (a), the circle-based scheme (b), the linear four-point scheme (c), and the centripetal four-point scheme (d) for a control polygon with sharp corners.

6.3 Comparison

We have implemented the two geometric four-point schemes described above, as well as the classical linear four-point scheme [4] and the non-linear four-point scheme based on centripetal parameterizations [5]. Figure 14 shows an example where the edge lengths of the initial control polygon vary significantly. In this situation, the linear four-point scheme leads to unwanted loops, while the centripetal scheme avoids such artefacts and gives a limit curve that is relatively close to all initial edges. The angle-based scheme from Section 6.1 behaves similarly, but results in a more roundish shape with smaller curvature peaks at most of the initial points. The limit curve of the circle-based scheme from Section 6.2 is yet more gently curved, but its distance to the initial edges is significantly larger. In fact, it follows from (21) that α_{2i+1}^{k+1} is close to $\pi/2$ if both α_i^k and β_{i+1}^k are close to π, and so p_{2i+1}^{k+1} can be arbitrarily far away from the edge $[p_i^k, p_{i+1}^k]$. For the angle-based scheme, this does not happen because $|\alpha_{2i+1}^{k+1}|$ is bounded by $\pi/4$, according to (19).

Figure 15 shows another example where all edges of the initial control polygon have the same length, but one of the initial exterior angles δ_i^0 is close to π and another is close to $-\pi$. The limit curves of the linear as well as the centripetal four-point scheme have a loop or even a cusp at these points. In case of the linear four-point scheme, this artefact stems from the fact that the coordinates of the control points are treated independently of each other. Hence, it is possible that both components of the limit curve's derivative vanish at some point p_i^k, allowing for a cusp without tangent continuity, even though the scheme is C^1. Although the refinement rule of the centripetal scheme depends on both coordinates of the points, it only takes the edge lengths but not the angles of the control polygon into account. Since all initial edge lengths are equal in this example, the limit curve is very close to the one of the linear scheme. Instead, the limit curves of the angle-based and the circle-based schemes are both G^1 and so they cannot have any cusps.

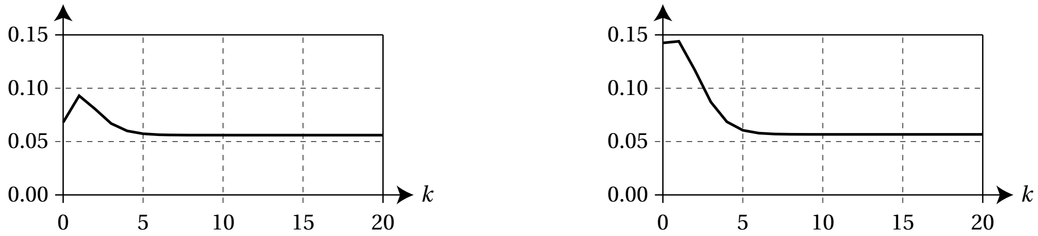

Figure 16: Plots of $\|P^{k+1} - SP^k\|_\infty / e^k$, where S denotes the subdivision operator of the linear four-point scheme, P^k is the polygon after applying k subdivision steps with the angle-based scheme from Section 6.1 (left) and the circle-based scheme from Section 6.2 (right) to the initial control polygon in Figure 14, and e^k is the maximum edge length in P^k. In both cases, this ratio appears to converge to a constant.

This example also shows that our theory cannot be used to prove that the limit curves of the linear and the centripetal four-point scheme are G^1 without some restriction on the geometry of the control polygon which prevents the occurrence of cusps. On the other hand, we are not aware of any other approach for showing that the geometric schemes from Sections 6.1 and 6.2 generate tangent continuous limit curves. In fact, Figure 16 shows that both schemes are not in sufficient proximity to the linear four-point scheme (see [15, Definition 7] for the original notion of proximity, which can be relaxed to $\|P^{k+1} - SP^k\|_\infty = \mathcal{O}\big((e^k)^\eta\big)$ with $1 < \eta \leq 2$) to conclude that their limit curves are C^1.

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