On the norms of the Dubuc–Deslauriers subdivision schemes

Chongyang Deng · Kai Hormann · Zhifeng Zhang

Abstract

Conti et al. [1, Remark 3.4] conjecture that the norm of the interpolatory $2n$-point Dubuc–Deslauriers subdivision scheme is bounded from above by 4 for any $n \in \mathbb{N}$. We disprove their conjecture by showing that the norm grows logarithmically in n and therefore diverges as n increases.

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1 Introduction

Let $S_{\boldsymbol{a}}$ be a univariate binary subdivision scheme with mask $\boldsymbol{a} = (a_i)_{i\in\mathbb{Z}}$. The norm of $S_{\boldsymbol{a}}$ is given by

$$\|S_{\boldsymbol{a}}\| = \max\left\{\sum_{i\in\mathbb{Z}} |a_{2i}|, \sum_{i\in\mathbb{Z}} |a_{2i+1}|\right\},$$

and if $S_{\boldsymbol{a}}$ is convergent, then $\|S_{\boldsymbol{a}}\| \ge 1$, due to the well-known necessary convergence condition $\sum_{i\in\mathbb{Z}} a_{2i} = \sum_{i\in\mathbb{Z}} a_{2i+1} = 1$ [4]. If the scheme is also interpolatory with $a_{2i} = \delta_{i,0}$, $i \in \mathbb{Z}$, then

$$\|S_{\boldsymbol{a}}\| = \sum_{i\in\mathbb{Z}} |a_{2i+1}|. \tag{1}$$

For example, this is the case for the family of interpolatory $2n$-point Dubuc–Deslauriers schemes $S_{\boldsymbol{a}^{[n]}}$, $n \in \mathbb{N}$, which are based on the evaluation of locally interpolating polynomials of degree $2n-1$ [3]. Conti et al. [1, Remark 3.4] conjecture that

$$\|S_{\boldsymbol{a}^{[n]}}\| < 4$$

for all $n \in \mathbb{N}$, after having observed the slow growth of $\|S_{\boldsymbol{a}^{[n]}}\|$ for $n \le 2000$. In Theorem 1 we disprove this conjecture by showing that the norm of the $2n$-point scheme grows logarithmically in n, which implies $\lim_{n\to\infty} \|S_{\boldsymbol{a}^{[n]}}\| = \infty$ and also explains the slow growth.

2 Bounding the norms of the Dubuc–Deslauriers schemes

Using the explicit formula for the non-zero coefficients of the $2n$-point scheme with positive odd indices given by de Villiers et al. [2],

$$a_{2i-1}^{[n]} = \frac{(-1)^{i+1}}{2^{4n-1}} \binom{2n}{n} \binom{2n}{n+i} \frac{n+i}{2i-1}, \qquad i = 1,2,\dots,n,$$

we first observe that

$$\begin{aligned} \left|a_{2i-1}^{[n+1]}\right| - \left|a_{2i-1}^{[n]}\right| &= \frac{1}{2^{4n+1}} \binom{2n}{n} \binom{2n}{n+i} \left[\frac{(2n+1)^2}{(n+1-i)(2i-1)} - \frac{4(n+i)}{2i-1}\right] \\ &= \frac{1}{2^{4n+1}} \binom{2n}{n} \binom{2n}{n+i} \frac{(n+i)-(n+1-i)}{n+1-i} \\ &= \frac{1}{2^{4n+1}} \binom{2n}{n} \left[\binom{2n}{n+i-1} - \binom{2n}{n+i}\right]. \end{aligned}$$

Denoting the sum of the absolute values of these coefficients by

$$\sigma_n = \sum_{i=1}^{n} \left|a_{2i-1}^{[n]}\right|,$$

we then have

$$\delta_n = \sigma_{n+1} - \sigma_n = \frac{1}{2^{4n+1}} \binom{2n}{n} \sum_{i=1}^{n+1} \left[\binom{2n}{n+i-1} - \binom{2n}{n+i} \right] = \frac{1}{2^{4n+1}} \binom{2n}{n}^2.$$

By (1) and the symmetry of the coefficients of the $2n$-point schemes in the sense that $a_i^{[n]} = a_{-i}^{[n]}$, $i \in \mathbb{Z}$, we finally get

$$\|S_{\boldsymbol{a}^{[n]}}\| = 2\sigma_n = 1 + 2 \sum_{k=1}^{n-1} \delta_k, \qquad n \in \mathbb{N}, \tag{2}$$

which we now bound from below and from above.

Theorem 1. *The norm of the $2n$-point Dubuc–Deslauriers scheme satisfies*

$$1 + \frac{1}{4} \ln n \le \|S_{\boldsymbol{a}^{[n]}}\| \le \frac{3}{2} + \frac{1}{2} \ln n \tag{3}$$

for $n \in \mathbb{N}$.

Proof. First observe that

$$\frac{1}{2^{2n}} \binom{2n}{n} = \prod_{k=1}^{n} \frac{2k-1}{2k}.$$

Then, on the one hand,

$$2\delta_n = \prod_{k=1}^{n} \frac{(2k-1)^2}{(2k)^2} = \frac{1}{2} \Bigg(\underbrace{\prod_{k=2}^{n} \frac{(2k-1)^2}{(2k-2)(2k)}}_{\ge 1} \Bigg) \frac{1}{2n} \ge \frac{1}{4n},$$

and on the other hand,

$$2\delta_n = \frac{2n-1}{4n^2} \underbrace{\prod_{k=1}^{n-1} \frac{(2k-1)(2k+1)}{(2k)^2}}_{\le 1} \le \frac{2n-1}{4n^2} \le \frac{1}{2n}.$$

The bounds in (3) then follow from (2) and the well-known bounds

$$\ln(n+1) < \sum_{k=1}^{n} \frac{1}{k} < 1 + \ln n$$

for the n-th partial sum of the harmonic series. □

While it is certainly possible, but beyond the scope of this paper, to get tighter bounds[1], Theorem 1 shows that $\|S_{\boldsymbol{a}^{[n]}}\|$ grows logarithmically in n and is unbounded from above. Using *Maxima* [5], we further found that $\|S_{\boldsymbol{a}^{[n]}}\| \ge 4$ for $n \ge 10063$.

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[1] For example, as pointed out by one of the reviewers, $\lim_{n\to\infty}(2n\delta_n) = 1/\pi$, by Wallis' product, which shows that $\|S_{\boldsymbol{a}^{[n]}}\|$ grows as $\ln(n)/\pi$ asymptotically.