On the Enhancement of the Approximation Order of Triangular Shepard Method

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Abstract. Shepard's method is a well-known technique for interpolating large sets of scattered data. The classical Shepard operator reconstructs an unknown function as a normalized blend of the function values at the scattered points, using the inverse distances to the scattered points as weight functions. Based on the general idea of defining interpolants by convex combinations, Little suggested to extend the bivariate Shepard operator in two ways. On the one hand, he considers a triangulation of the scattered points and substitutes function values with linear polynomials which locally interpolate the given data at the vertices of each triangle. On the other hand, he modifies the classical point-based weight functions and defines instead a normalized blend of the locally interpolating polynomials with triangle-based weight functions which depend on the product of inverse distances to the three vertices of the corresponding triangle. The resulting triangular Shepard operator interpolates all data required for its definition and reproduces polynomials up to degree 1, whereas the classical Shepard operator reproduces only constants, and has quadratic approximation order. In this paper we discuss an improvement of the triangular Shepard operator.

SHEPARD AND TRIANGULAR SHEPARD OPERATORS

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of n distinct points in $\mathbb{R}^2$, called *nodes* or *sample points*, with associated data f_i sampled from some unknown function $f\colon \mathbb{R}^2 \to \mathbb{R}$, that is, $f_i = f(x_i)$, $i = 1, \ldots, n$. The *classical Shepard interpolation formula* [1]

$$S_\mu[f](x) = \sum_{i=1}^{n} A_{\mu,i}(x) f_i \tag{1}$$

uses point-based basis functions

$$A_{\mu,i}(x) = \frac{\dfrac{1}{|x - x_i|^\mu}}{\displaystyle\sum_{k=1}^{n} \frac{1}{|x - x_k|^\mu}}, \qquad i = 1, \ldots, n \tag{2}$$

in barycentric form, where $|\cdot|$ denotes the Euclidean norm and $\mu > 0$ is a parameter that controls the range of influence of the data values. A common choice is to take $\mu = 2$, so that the basis functions are rational and infinitely differentiable [2].

Since the basis functions $A_{\mu,i}$ are cardinal, non-negative, and form a partition of unity, the interpolation operator S_μ is stable [3] in the sense that

$$\min_i f_i \le S_\mu[f](x) \le \max_i f_i, \qquad x \in \mathbb{R}^2,$$

but for $\mu > 1$ it has flat spots at all nodes. Moreover, it reproduces only constant polynomials and its approximation order is at most $O(h)$, where h is the *mesh size* of the set of sample points [3].

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The Little's *triangular Shepard operator* is based on a triangulation of the nodes and an extension of Shepard's point-based basis functions (2) to triangle-based basis functions. The latter are then used in combination with linear polynomials that locally interpolate the given data at the vertices of each triangle. Polynomials based on the vertices of a triangle [4, 5, 6] are used in combination with Shepard basis functions in [7, 8, 9, 10].

To extend the point-based basis functions in (2) to triangle-based basis functions, let us consider a triangulation $T = \{t_1, t_2, \ldots, t_m\}$ of the nodes X. That is, each $t_j = [x_{j_1}, x_{j_2}, x_{j_3}]$ is a triangle with vertices in X and each node $x_i \in X$ is the vertex of at least one triangle, hence

$$\bigcup_{j=1}^{m} \{j_1, j_2, j_3\} = \{1, 2, \ldots, n\}. \tag{3}$$

For example, T can be the *Delaunay triangulation* [11] of X, but we also allow for general triangulations with overlapping or disjoint triangles [12].

The *triangle-based basis functions* with respect to the triangulation T are then defined by

$$B_{\mu,j}(x) = \frac{\displaystyle\prod_{\ell=1}^{3} \frac{1}{\left|x - x_{j_\ell}\right|^{\mu}}}{\displaystyle\sum_{k=1}^{m} \prod_{\ell=1}^{3} \frac{1}{\left|x - x_{k_\ell}\right|^{\mu}}}, \qquad j = 1, \ldots, m, \tag{4}$$

where $\mu > 0$ is again a control parameter. Like Shepard's basis functions, the triangle-based basis functions (4) are non-negative and form a partition of unity, but instead of being cardinal they and its gradient (that exists for $\mu > 1$) vanish at all nodes $x_i \in X$ that are not a vertex of the corresponding triangle t_j. As an immediate consequence of previous properties and the partition of unity property we have, for each $i = 1, \ldots, n$,

$$\begin{aligned} \textstyle\sum_{j \in J_i} B_{\mu,j}(x_i) &= 1, \\ \textstyle\sum_{j \in J_i} \nabla B_{\mu,j}(x_i) &= 0, \quad \mu > 1, \end{aligned} \tag{5}$$

where J_i is the set of indices of all triangles which have x_i as a vertex,

$$J_i = \Big\{k \in \{1, \ldots, m\} : i \in \{k_1, k_2, k_3\}\Big\}. \tag{6}$$

The second ingredient for Little's extension of the Shepard operator [13] are the linear polynomials that locally interpolate the given data at the vertices of each of the triangles. For $t_j \in T$, this polynomial $L_j \colon \mathbb{R}^2 \to \mathbb{R}$ can be written as

$$L_j(x) = \sum_{\ell=1}^{3} \lambda_{j,j_\ell}(x) f_{j_\ell}, \tag{7}$$

where $\lambda_{j,j_\ell}(x)$, $\ell = 1, 2, 3$ are the *barycentric coordinates* [14] of x with respect to the triangle $t_j = [x_{j_1}, x_{j_2}, x_{j_3}]$, that is,

$$\lambda_{j,j_1}(x) = \frac{A(x, x_{j_2}, x_{j_3})}{A(x_{j_1}, x_{j_2}, x_{j_3})}, \quad \lambda_{j,j_2}(x) = \frac{A(x_{j_1}, x, x_{j_3})}{A(x_{j_1}, x_{j_2}, x_{j_3})}, \quad \lambda_{j,j_3}(x) = \frac{A(x_{j_1}, x_{j_2}, x)}{A(x_{j_1}, x_{j_2}, x_{j_3})},$$

with $A(x, y, z)$ denoting the signed area of the triangle $[x, y, z]$. In general, $\lambda_{j,i}(x)$ with $j \in J_i$ and $i \in \{j_1, j_2, j_3\}$ is the unique linear polynomial with $\lambda_{j,i}(x_i) = 1$ and $\lambda_{j,i}(x) = 0$ for x on the line defined by the edge opposite x_i in the triangle t_j.

For any $\mu > 0$ the triangular Shepard operator is defined by

$$K_\mu[f](x) = \sum_{j=1}^{m} B_{\mu,j}(x) L_j(x). \tag{8}$$

The triangular Shepard operator satisfies the following remarkable properties [13]:

- $K_\mu\,[f]\,(x)$ interpolates function evaluations at each sample point;

- $K_\mu[f](x)$ reproduces polynomials of degree not greater than 1.

In [12] we studied the approximation order of the operator K_μ. Following Farwig [3] we let $\|\cdot\|$ be the maximum norm and $R_r(y) = \{x \in \mathbb{R}^2 : \|x - y\| \leq r\}$ be the axis-aligned closed square with centre y and edge length $2r$. With $V(t)$ denoting the set of vertices of a triangle $t \in T$, we then define

$$h' = \inf\{r > 0 : \forall x \in \Omega\ \exists t \in T : R_r(x) \cap V(t) \neq \emptyset\} \tag{9}$$

and

$$h'' = \inf\{r > 0 : \forall t \in T\ \exists x \in \Omega : t \subset R_r(x)\},$$

and finally

$$h = \max\{h', h''\}. \tag{10}$$

A small value of h' corresponds to a rather uniform triangle distribution, but does not exclude the presence of large triangles. The latter cannot occur if h'' and then also h are small, because each triangle is contained in a square with edge length $2h$. Note that in the maximum norm, the length of each triangle edge does not exceed $2h$. We further let

$$M = \sup_{x\in\Omega} \#\{t \in T : R_h(x) \cap V(t) \neq \emptyset\}, \tag{11}$$

where # is the cardinality operator, be the maximum number of triangles with at least one vertex in some square with edge length $2h$. Small values of M imply that there are no clusters of triangles. We further consider the class $C^{1,1}(\Omega)$ of differentiable functions $f\colon \Omega \to \mathbb{R}$ whose partial derivatives are Lipschitz-continuous of order 1, equipped with the seminorm

$$\|f\|_{1,1} = \sup\left\{ \frac{\left|\frac{\partial f}{\partial x^{1-\alpha}\partial y^{\alpha}}(u) - \frac{\partial f}{\partial x^{1-\alpha}\partial y^{\alpha}}(v)\right|}{|u - v|} : u, v \in \Omega, u \neq v, \alpha \in \{0, 1\} \right\}. \tag{12}$$

We get the following result:
Let Ω be a compact convex domain which contains X, $f \in C^{1,1}(\Omega)$, and $\mu > 4/3$. Then,

$$\left|f(x) - K_\mu[f](x)\right| \leq CM\|f\|_{1,1}h^2$$

for any $x \in \Omega$, with C a positive constant which depends on T and μ.

THE ENHANCED TRIANGULAR SHEPARD OPERATOR

If f is a differentiable function, the derivative of f along the directed line segment from x_{j_i} to x_{j_k} (side of the simplex) is denoted by

$$D_{ik}f := \left(x_{j_i} - x_{j_k}\right) \cdot \nabla f, \quad i, k = 1, 2, 3, i \neq k, \tag{13}$$

where $\cdot$ is the dot product and $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$. The composition of derivatives along the directed sides of the simplex (13) are denoted by

$$D_1^{\beta} = D_{21}^{\beta_1} D_{31}^{\beta_2}, \quad D_2^{\beta} = D_{12}^{\beta_1} D_{32}^{\beta_2}, \quad D_3^{\beta} = D_{13}^{\beta_1} D_{23}^{\beta_2}, \quad \beta = (\beta_1, \beta_2) \in \mathbb{N}^2. \tag{14}$$

The Quadratic Bernoulli polynomial [15, 16] of data $(x_{j_i}, f_{j_i}, \nabla f_{j_i}), i = 1, 2, 3$ on t_j, in barycentric coordinates, is defined as follows

$$\begin{aligned} P_j(x) &= f_{j_1}\lambda_{j,j_1}(x) + f_{j_2}\lambda_{j,j_2}(x) + f_{j_3}\lambda_{j,j_3}(x) + \tfrac{1}{2}\lambda_{j,j_1}\lambda_{j,j_2}\left(D_2^{(1,0)}f_{j_2} - D_2^{(1,0)}f_{j_1}\right) \\ &+ \tfrac{1}{2}\lambda_{j,j_1}\lambda_{j,j_3}\left(D_1^{(0,1)}f_{j_1} - D_1^{(0,1)}f_{j_3}\right) + \tfrac{1}{2}\lambda_{j,j_2}\lambda_{j,j_3}\left(D_3^{(0,1)}f_{j_3} - D_3^{(0,1)}f_{j_2}\right). \end{aligned} \tag{15}$$

The Quadratic Bernoulli polynomial on t_j satisfies the following properties

- it is a symmetric polynomial expansion;
- it interpolates functional evaluation at the vertices of t_j;
- it interpolates differences of derivatives along the directed sides of t_j;

- it is uniquely determined by the data in the space of polynomials of degree not greater than 2.

To study the approximation order of the Quadratic Bernoulli polynomial (15), we consider the class $C^{2,1}(\Omega)$ of differentiable functions $f:\Omega \to \mathbb{R}$ whose partial derivatives are Lipschitz-continuous of order 2, equipped with the seminorm

$$\|f\|_{2,1} = \sup\left\{ \frac{\left| \frac{\partial^2 f}{\partial x^{2-\alpha}\partial y^{\alpha}}(u) - \frac{\partial^2 f}{\partial x^{2-\alpha}\partial y^{\alpha}}(v) \right|}{|u-v|} : u, v \in \Omega, u \neq v, \alpha \in \{0,1,2\} \right\}. \tag{16}$$

Then, for each $x \in \Omega$ we can prove that

$$\left|f(x) - P_j(x)\right| \leq \|f\|_{2,1} \left(4\left|x - x_{j_1}\right|^3 + C_j\left(h_j^2\left|x - x_{j_1}\right| + h_j\left|x - x_{j_1}\right|^2\right)\right)$$

with $h_j = \max\{\left|x_{j_1} - x_{j_2}\right|, \left|x_{j_2} - x_{j_3}\right|, \left|x_{j_1} - x_{j_3}\right|\}$ and C_j a constant which depends only on the shape of t_j. Moreover, $P_j(x)$ approximates the first order derivatives of f at the vertices of t_j with $O(h_j^2)$.

For each $\mu > 2$ the enhanced Triangular Shepard operator is defined by

$$Q_\mu[f](x) = \sum_{j=1}^{m} B_{\mu,j}(x) P_j(x), \qquad x \in \Omega. \tag{17}$$

The enhanced triangular Shepard operator satisfies the following remarkable properties:

- $Q_\mu[f](x)$ interpolates function evaluations at each sample point;
- $Q_\mu[f](x)$ approximates first order derivatives at each sample point x_i with $O(h^2)$, where h is the maximum of all $h_j = \max\{\left|x_{j_1} - x_{j_2}\right|, \left|x_{j_2} - x_{j_3}\right|, \left|x_{j_1} - x_{j_3}\right|\}$ for all triangles t_j having x_i as a vertex;
- $Q_\mu[f](x)$ reproduces polynomials of degree not greater than 2.

With regard to the approximation order of the operator Q_μ we get the following result:
Let Ω be a compact convex domain which contains X. If $f \in C^{2,1}(\Omega)$ then for each $\mu > 5/3$ we have

$$\|f - Q_\mu[f]\| \leq CM\|f\|_{2,1} h^3,$$

where C is a positive constant which depends only on T and μ.

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REFERENCES

[1] D. Shepard, in *Proceedings of the 23rd ACM National Conference* (ACM Press, New York, 1968), pp. 517–524.
[2] R. E. Barnhill, R. P. Dube, and F. Little, Rocky Mountain Journal of Mathematics **13**, 365–382 (1983).
[3] R. Farwig, Mathematics of Computation **46**, 577–590 (1986).
[4] F. A. Costabile and F. Dell'Accio, Applied Numerical Mathematics **52**, 339–361 (2005).
[5] F. A. Costabile and F. Dell'Accio, Journal of Computational and Applied Mathematics **176**, 77 – 90 (2005).
[6] F. A. Costabile, F. Dell'Accio, and L. Guzzardi, Calcolo **45**, 177–192 (2008).
[7] R. Caira, F. Dell'Accio, and F. Di Tommaso, Journal of Computational and Applied Mathematics **236**, 1691–1707 (2012).
[8] F. A. Costabile, F. Dell'Accio, and F. Di Tommaso, Computer and Mathematics with Applications **64**, 3641–3655 (2012).
[9] F. A. Costabile, F. Dell'Accio, and F. Di Tommaso, Numerical Algorithms **64**, 157–180 (2013).
[10] F. Dell'Accio and F. Di Tommaso, Journal of Computational and Applied Mathematics **300**, 192–206 (2016).
[11] B. Delaunay, Bulletin de l'Académie des Sciences de l'URSS **6**, 793–800 (1934).
[12] F. Dell'Accio, F. Di Tommaso, and K. Hormann, IMA Journal of Numerical Analysis **36**, 359–379 (2016).
[13] F. Little, in *Surfaces in Computer Aided Geometric Design*, edited by R. E. Barnhill and W. Boehm (North-Holland, Amsterdam, 1983), pp. 99–107.
[14] A. F. Möbius, *Der barycentrische Calcul* (Johann Ambrosius Barth, Leipzig, 1827).
[15] F. Costabile and F. DellAccio, Journal of Computational and Applied Mathematics **210**, 116 – 135 (2007).
[16] F. Costabile and F. Dell'Accio, Numerical Algorithms **28**, 63–86 (2001).