Multinode Rational Operators for Univariate Interpolation

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Abstract. Birkhoff (or lacunary) interpolation is an extension of polynomial interpolation that appears when observation gives irregular information about function and its derivatives. A Birkhoff interpolation problem is not always solvable even in the appropriate polynomial or rational space. In this talk we split up the initial problem in subproblems having a unique polynomial solution and use multinode rational basis functions in order to obtain a global interpolant.

INTRODUCTION

Let $X = \{x_1, x_2, \dots, x_n\}$ be a set of pairwise distinct real numbers for which we assume that $x_1 < x_2 < \dots < x_n$. In the problem of interpolation of given data $f_{i,j} = f^{(j)}(x_i)$, $i = 1, \dots, n$, $j \in \mathcal{J}_i \subset \mathbb{N}$, by a polynomial p of appropriate degree,

$$p^{(j)}(x_i) = f_{i,j}$$

we mainly distinguish between *Hermite interpolation* and *Birkhoff interpolation*. We have an Hermite interpolation problem if, for each i, the indices j in the set $\mathcal{J}_i$ form an unbroken sequence, i.e. $\mathcal{J}_i = \{0, 1, \dots, j_i\}$, Birkhoff interpolation otherwise. It is, however, convenient to consider Hermite interpolation to be a special case of lacunary interpolation and to deal with Hermite-Birkhoff interpolation. In contrast to Hermite interpolation, a Birkhoff interpolation problem does not always have a unique solution or, even worse, does not have a solution [1]. In this paper we propose to split up the unsolvable problems in two or more uniquely solvable subproblems, whose solutions can be blended together. Here we consider the case of *multinode basis functions* [2] as blending functions. An approach to Birkhoff interpolation using Shepard basis functions can be found in [3, 4, 5, 6, 7, 8, 9]. To this goal we consider a covering $\mathcal{F} = \{F_1, F_2, \dots, F_m\}$ of X by subsets $F_k \subset X$ such that, for each $k = 1, \dots, m$, the corresponding Hermite-Birkhoff interpolation subproblems $p^{(j)}(x_i) = f_{i,j}, x_i \in F_k, j \in \mathcal{J}_i$ have a unique solution and we associate to each F_k, $k = 1, \dots, m$, a multinode basis function. The latter are then used in combination with the local Hermite-Birkhoff polynomials that interpolate the data associated to F_k. Finally, we provide numerical experiments which show the approximation order.

MULTINODE BASIS FUNCTIONS

Let us consider a covering $\mathcal{F} = \{F_1, F_2, \dots, F_m\}$ of X by its not empty subsets $F_k \subset X$, that is

$$\bigcup_{k=1}^{m} F_k = X, \quad F_k \neq \emptyset, \quad \text{for each } k = 1, \dots, m. \tag{1}$$

The multinode basis functions with respect to the covering $\mathcal{F}$ are defined by

$$B_{\mu,k}(x) = \frac{\prod\limits_{x_i \in F_k} |x - x_i|^{-\mu}}{\sum\limits_{l=1}^{m} \prod\limits_{x_i \in F_l} |x - x_i|^{-\mu}}, \quad k = 1, \dots, m, \tag{2}$$

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where $\mu > 0$ is a parameter that determines the differentiability class of the basis and controls the range of influence of the data values. The multinode basis functions (2) are non-negative and form a partition of unity, that is

$$\sum_{k=1}^{m} B_{\mu,k}(x) = 1; \tag{3}$$

but instead of being cardinal they vanish at all nodes x_j that are not in F_k, that is

$$B_{\mu,k}\left(x_j\right) = 0, \quad \mu > 0, \tag{4}$$

for any $k = 1, \ldots, m$ and $j \notin F_k$, and

$$\sum_{k \in K_i} B_{\mu,k}(x_i) = 1, \quad \mu > 0 \tag{5}$$

where

$$K_i = \{l \in \{1, \ldots, m\} : x_i \in F_l\} \neq \emptyset, \tag{6}$$

is the set of indices of all subsets of $\mathcal{F}$ that contain x_i. For $\mu > 0$ even integer the multinode basis functions (2) are rational and have no real poles, otherwise their class of differentiability is $\mu - 1$ for μ odd integer and $[\mu]$, the largest integer not greater than μ, in all remaining cases. Moreover, all derivatives of order $\ell > 0$ vanish at all nodes x_j that are not in F_k,

$$B_{\mu,k}^{(\ell)}\left(x_j\right) = 0, \tag{7}$$

for any $k = 1, \ldots, m$ and $j \notin F_k$ and

$$\sum_{k \in K_i} B_{\mu,k}^{(\ell)}(x_i) = 0, \quad \mu > 1. \tag{8}$$

MULTINODE GLOBAL INTERPOLATION OPERATOR

Let us consider the Hermite-Birkhoff interpolation problem

$$p^{(j)}[f](x_i) = f^{(j)}(x_i), \quad i = 1, \ldots, n, \ j \in \mathcal{J}_i, \tag{9}$$

and let us assume that, for each $k = 1, \ldots, m$, the Hermite-Birkhoff interpolation subproblems

$$P_k^{(j)}[f](x_i) = f^{(j)}(x_i), \quad x_i \in F_k, \ j \in \mathcal{J}_i, \tag{10}$$

have a unique solution $P_k[f]$ in their appropriate polynomial spaces $\mathcal{P}_x^{q_k}$, $q_k = \sum_{x_i \in F_k} \#(\mathcal{J}_i) - 1$. As soon as we have provided a solution for all local Hermite-Birkhoff interpolation problems, we define the multinode global interpolation operator by

$$M_\mu[f, \mathcal{F}](x) = \sum_{k=1}^{m} B_{\mu,k}(x) P_k[f](x) \tag{11}$$

where $P_k[f](x)$ is the polynomial solution of the Hermite-Birkhoff interpolation problem on F_k. The operator $M_\mu[f, \mathcal{F}](x)$ has remarkable properties. Firstly, it reproduces polynomials up to the degree $q_{\min} = \min_k q_k$ and by setting $\mathcal{F} = \{X\}$, $M_\mu[f, \mathcal{F}](x)$ coincides with that polynomial solution if the global problem has a unique polynomial solution. Secondly, the operator $M_\mu[f, \mathcal{F}]$ interpolates the functional data

$$M_\mu[f, \mathcal{F}](x_i) = f(x_i), \quad \textit{for each } i : 0 \in \mathcal{J}_i \tag{12}$$

and, if $\mathcal{F}$ is a partition of X (i.e. $F_\alpha \cap F_\beta = \emptyset$ for each $\alpha \neq \beta$) the operator $M_\mu[f, \mathcal{F}]$ interpolates all data used in its definition, i.e.

$$M_\mu^{(j)}[f, \mathcal{F}](x_i) = f^{(j)}(x_i), \quad \textit{for each } k = 1, \ldots, m, \ x_i \in F_k, \ j \in \mathcal{J}_i.$$

However, we notice that the operator $M_\mu[f, \mathcal{F}]$ could not interpolate all derivative data at some x_κ if $\sharp(K_\kappa) > 1$ and the sequence of indices in $\mathcal{J}_\kappa$ is broken. For example, let us assume

$$\sharp(K_\kappa) = 2, \quad F_\alpha \cap F_\beta = \{x_\kappa\}, \quad \mathcal{J}_\kappa = \{0, 2, \ldots, \ell - 1, \ell\}, \ell \geq 2$$

and

$$B^{(\ell-1)}_{\mu,\alpha}(x_\kappa)P'_\alpha[f](x_\kappa) + B^{(\ell-1)}_{\mu,\beta}(x_\kappa)P'_\alpha[f](x_\kappa) \neq 0.$$

We notice that

$$P'_\alpha[f](x_\kappa) \neq P'_\beta[f](x_\kappa)$$

since property (8). From

$$P^{(\ell)}_\alpha[f](x_\kappa) = P^{(\ell)}_\beta[f](x_\kappa) = f^{(\ell)}(x_\kappa)$$

by properties (4) and (5) easily follows

$$\sum_{k=1}^{m} B_{\mu,k}(x_\kappa)P^{(\ell)}_k[f](x_\kappa) = B_{\mu,\alpha}(x_\kappa)f^{(\ell)}(x_\kappa) + B_{\mu,\beta}(x_\kappa)f^{(\ell)}(x_\kappa) = f^{(\ell)}(x_\kappa).$$

On the other hand

$$\sum_{k=1}^{m}\sum_{\iota=0}^{\ell-1}\binom{\ell}{\iota}B^{(\ell-\iota)}_{\mu,k}(x_\kappa)P^{(\iota)}_k[f](x_\kappa) = \sum_{\iota=0}^{\ell-1}\binom{\ell}{\iota}\left(B^{(\ell-\iota)}_{\mu,\alpha}(x_\kappa)P^{(\iota)}_\alpha[f](x_\kappa) + B^{(\ell-\iota)}_{\mu,\beta}(x_\kappa)P^{(\iota)}_\alpha[f](x_\kappa)\right)$$

by property (7). Let us fix our attention to the right hand side of previous equality. For each $\iota \in \mathcal{J}_\kappa$ we get

$$B^{(\ell-\iota)}_{\mu,\alpha}(x_\kappa)P^{(\iota)}_\alpha[f](x_\kappa) + B^{(\ell-\iota)}_{\mu,\beta}(x_\kappa)P^{(\iota)}_\alpha[f](x_\kappa) = \left(B^{(\ell-\iota)}_{\mu,\alpha}(x_\kappa) + B^{(\ell-\iota)}_{\mu,\beta}(x_\kappa)\right)f^{(\iota)}(x_\kappa) = 0$$

by property (8), but

$$B^{(\ell-1)}_{\mu,\alpha}(x_\kappa)P'_\alpha[f](x_\kappa) + B^{(\ell-1)}_{\mu,\beta}(x_\kappa)P'_\alpha[f](x_\kappa) \neq 0$$

and consequently

$$M^{(\ell)}_\mu[f,\mathcal{F}](x_\kappa) \neq f^{(\ell)}(x_\kappa).$$

In order to avoid this trouble, we proceed as follows. For each $\kappa = 1,\ldots,n$ let be $\nu_\kappa = \sharp(K_\kappa)$ and $F_{\alpha_1},\ldots,F_{\alpha_{\nu_\kappa}}$ the subset of X which contain x_κ. As above, let us denote by $P_{\alpha_1}[f],\ldots,P_{\alpha_{\nu_\kappa}}[f]$ the polynomial solutions of the Hermite-Birkhoff interpolation problems on $F_{\alpha_1},\ldots,F_{\alpha_{\nu_\kappa}}$ respectively. For all $j = 0,1,\ldots,\max(\mathcal{J}_\kappa)$ we set

$$\tilde{f}^{(j)}(x_\kappa) = \frac{1}{\nu_\kappa}\left(P^{(j)}_{\alpha_1}[f](x_\kappa) + \ldots + P^{(j)}_{\alpha_{\nu_\kappa}}[f](x_\kappa)\right) \tag{13}$$

and we note that

$$\tilde{f}^{(j)}(x_\kappa) = f^{(j)}(x_\kappa) \tag{14}$$

as soon as $j \in \mathcal{J}_\kappa$. For each $k = 1,\ldots,m$ we call the Hermite interpolation problem

$$\tilde{P}^{(j)}_k[f](x_i) = \tilde{f}^{(j)}(x_i), \quad x_i \in F_k,\ j = 0,1,\ldots\max(\mathcal{J}_i), \tag{15}$$

hermitian completion of the Hermite-Birkhoff interpolation problem (10). It is well known that each interpolation problem (15) has a unique solution $\tilde{P}_k[f](x)$ in the polynomial space $\mathcal{P}^{d_k}_x$, $d_k = \sharp(F_k) + \sum_{x_i\in F_k}\max(\mathcal{J}_i) - 1$, for which there are explicit formulas in Lagrange or Newton form [10]. Nevertheless, if $q_k < d_k$ and $p \in \mathcal{P}^{q_k}_x$, $\tilde{P}_k[p]$ may be different from p, since we have completed the lacunary data using solutions of several interpolation problems. We set

$$\tilde{M}_\mu[f,\mathcal{F}](x) = \sum_{k=1}^{m} B_{\mu,k}(x)\tilde{P}_k[f](x). \tag{16}$$

The operator $\tilde{M}_\mu[\cdot,\mathcal{F}]$ preserves the reproducing polynomial property of $M_\mu[\cdot,\mathcal{F}]$, that is reproduces polynomials up to the degree $q_{\min} = \min_k q_k$ and interpolates all data used in its definition, that is

$$\tilde{M}^{(j)}_\mu[f,\mathcal{F}](x_i) = f^{(j)}(x_i), \quad \textit{for each } k = 1,\ldots,m,\ x_i \in F_k,\ j \in \mathcal{J}_i.$$

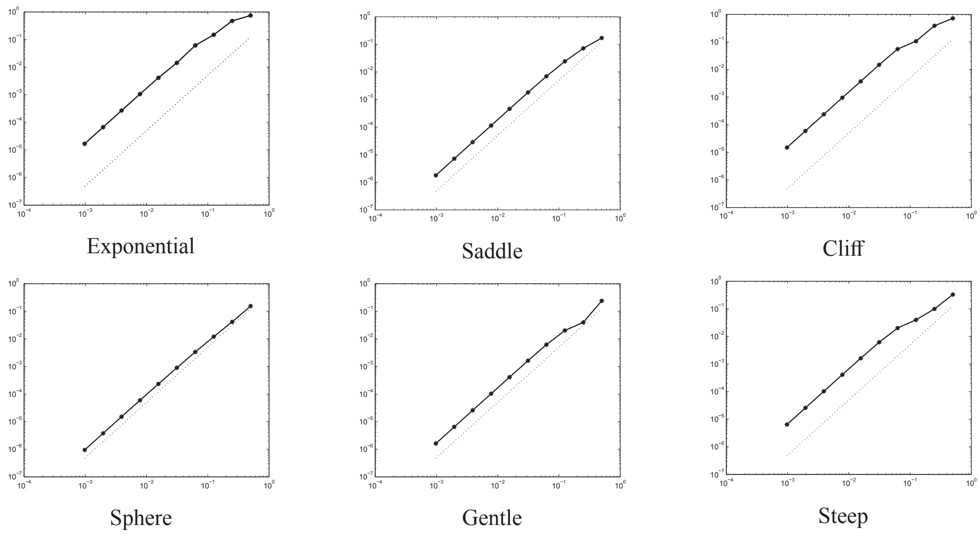

FIGURE 1. Log-log-plot of the approximation error e_{max} over the interval width for the 6 test functions. As reference, the dotted line indicates a perfect quadratic trend.

NUMERICAL RESULTS

In the following we numerically test the approximation order of the multinode rational interpolation operator. We carried out a series of experiments with different sets of equispaced nodes on $[0, 1]$ and test functions $f_i, i = 1, \dots, 6$ as in [11]. More precisely, we consider different coverings $\mathcal{F}$ of the nodeset X with increasing number of subsets F_k. For each of the 6 test functions f_i we constructed the multinode rational interpolant $M_\mu [f_i, \mathcal{F}](x)$ and we determined the maximum approximation error $e_{\max}$ by evaluating $\left|f_i(x) - M_4 [f_i, \mathcal{F}](x)\right|$ at 100, 000 random points $x \in [0, 1]$ and recording the maximum value. In the Figure 1 we display the log-log-plot of the approximation error e_{max} over the interval width for the 6 test functions. As reference, the dotted line indicates a perfect quadratic trend.

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