On the Lebesgue constant of Berrut's rational interpolant at equidistant nodes

Len Bos · Stefano De Marchi · Kai Hormann

Abstract

It is well known that polynomial interpolation at equidistant nodes can give bad approximation results and that rational interpolation is a promising alternative in this setting. In this paper we confirm this observation by proving that the Lebesgue constant of Berrut's rational interpolant grows only logarithmically in the number of interpolation nodes. Moreover, the numerical results suggest that the Lebesgue constant behaves similarly for interpolation at Chebyshev as well as logarithmically distributed nodes.

Citation Info

Journal
Journal of Computational and Applied Mathematics

Volume
236(4), September 2011

Pages
504–510

1 Introduction

Suppose we want to approximate a function $f\colon [a,b] \to \mathbb{R}$ by some function g that interpolates f at the $n+1$ distinct *interpolation nodes*

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Given a set of *basis functions* b_i which satisfy the *Lagrange property*

$$b_i(x_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

the interpolant g can be written as

$$g(x) = \sum_{j=0}^{n} b_j(x) f(x_j).$$

The *Lebesgue constant* of this interpolation operator is

$$\Lambda_n = \max_{a \le x \le b} \Lambda_n(x),$$

where $\Lambda_n(x)$ is the associated *Lebesgue function*

$$\Lambda_n(x) = \sum_{j=0}^{n} |b_j(x)|. \tag{1}$$

The Lebesgue constant has been studied intensively in the case of polynomial interpolation, that is, when b_i are the Lagrange basis polynomials (see [3, 4, 9] and references therein). In the special case of equidistant nodes, the Lebesgue constant for polynomial interpolation grows exponentially [8, 10], which is one of the reasons why other interpolation methods should be used in this setting. One popular alternative is rational interpolation and two recent results by Carnicer [5] and Wang, Moin, and Iaccarino [11] confirm that rational interpolation at equidistant nodes can have a much smaller Lebesgue constant than polynomial interpolation. However, both papers report only numerical observations and do not give any theoretical bounds.

In this paper we investigate the rational interpolant that was introduced by Berrut [1] with basis functions

$$b_i(x) = \frac{(-1)^i}{x - x_i} \Bigg/ \sum_{j=0}^{n} \frac{(-1)^j}{x - x_j}, \qquad i = 0, \dots, n \tag{2}$$

and show that the associated Lebesgue constant grows logarithmically in the number of interpolation nodes. More precisely, we prove in Section 2 that the Lebesgue constant is bounded by $2 + \ln(n)$ from above and asymptotically by $\frac{2}{\pi}\ln(n+1)$ from below, which improves the lower bound given by Berrut and Mittelmann [2].

The more interesting bound is of course the upper bound as it gives information on the stability of the interpolation process and the conditioning of the interpolation problem.

The numerical results in Section 3 further indicate that the Lebesgue constant of Berrut's rational interpolant at equidistant nodes is even smaller than the Lebesgue constant for polynomial interpolation at Chebyshev nodes. Moreover, we observe that the Lebesgue constant of Berrut's interpolant behaves similarly if Chebyshev or logarithmically distributed nodes are considered instead of equidistant nodes.

2 The main result

Let us start by recalling some well-known bounds for the partial sums of the Leibniz series and the harmonic series, namely

$$\frac{\pi}{4} - \frac{1}{2n+3} \le \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \le \frac{\pi}{4} + \frac{1}{2n+3} \tag{3}$$

and

$$\ln(n+1) \le \sum_{k=1}^{n} \frac{1}{k} \le \ln(2n+1) \tag{4}$$

for any $n \in \mathbb{N}$. Moreover, it follows from (4) that

$$\sum_{k=0}^{n} \frac{1}{2k+1} = \sum_{k=1}^{2n+1} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{2k} \ge \ln(2n+2) - \frac{1}{2}\ln(2n+1) \ge \frac{1}{2}\ln(2n+3), \tag{5}$$

where the last inequality is due to the concavity of the logarithm function.

We are now ready to prove our main result, that the Lebesgue constant grows logarithmically in the number of nodes, by establishing logarithmic upper and lower bounds. For simplicity we assume without loss of generality that the interpolation interval is $[0,1]$, so the interpolation nodes are equally spaced with distance $h = 1/n$, that is,

$$x_j = jh = \frac{j}{n}, \qquad j = 0, \ldots, n. \tag{6}$$

Our first result concerns the lower bound of the Lebesgue constant.

Theorem 1. *The Lebesgue constant for interpolation with the basis functions $b_i(x)$ in* (2) *at the nodes x_j in* (6) *satisfies*

$$\Lambda_n \ge c_n \ln(n+1)$$

for $c_n = 2n/(4+n\pi)$ with $\lim_{n\to\infty} c_n = 2/\pi$.

Proof. By the general definition of the Lebesgue function in (1) we have

$$\Lambda_n(x) = \frac{\displaystyle\sum_{j=0}^{n} \frac{1}{|x - j/n|}}{\left|\displaystyle\sum_{j=0}^{n} \frac{(-1)^j}{x - j/n}\right|} = \frac{\displaystyle\sum_{j=0}^{n} \frac{1}{|2nx - 2j|}}{\left|\displaystyle\sum_{j=0}^{n} \frac{(-1)^j}{2nx - 2j}\right|} =: \frac{N(x)}{D(x)}$$

for the basis functions in (2) and the nodes in (6). Our goal now is to bound the numerator $N(x)$ from below and the denominator $D(x)$ from above.

Let us first assume that n is even, say $n = 2k$, and let $x = (n+1)/(2n)$. Using (5) we get

$$\begin{aligned}
N\left(\frac{n+1}{2n}\right) &= \sum_{j=0}^{n} \frac{1}{|n+1-2j|} = \sum_{j=0}^{2k} \frac{1}{|2(k-j)+1|} = \sum_{j=0}^{k} \frac{1}{|2(k-j)+1|} + \sum_{j=k+1}^{2k} \frac{1}{|2(k-j)+1|} \\
&= \sum_{j=0}^{k} \frac{1}{2j+1} + \sum_{j=0}^{k-1} \frac{1}{2j+1} \\
&\ge \frac{1}{2}\ln(2k+3) + \frac{1}{2}\ln(2k+1) \\
&\ge \ln(2k+1) = \ln(n+1)
\end{aligned}$$

for the numerator, and by the triangle inequality and (3) we get

$$
\begin{aligned}
D\left(\frac{n+1}{2n}\right) &= \left|\sum_{j=0}^{n} \frac{(-1)^j}{n+1-2j}\right| = \left|\sum_{j=0}^{2k} \frac{(-1)^j}{2(k-j)+1}\right| = \left|\sum_{j=0}^{k} \frac{(-1)^j}{2(k-j)+1} + \sum_{j=k+1}^{2k} \frac{(-1)^j}{2(k-j)+1}\right| \\
&\le \left|(-1)^k \sum_{j=0}^{k} \frac{(-1)^j}{2j+1}\right| + \left|(-1)^k \sum_{j=0}^{k-1} \frac{(-1)^j}{2j+1}\right| = \sum_{j=0}^{k} \frac{(-1)^j}{2j+1} + \sum_{j=0}^{k-1} \frac{(-1)^j}{2j+1} \\
&\le \left(\frac{\pi}{4} + \frac{1}{2k+3}\right) + \left(\frac{\pi}{4} + \frac{1}{2k+1}\right) \\
&\le \frac{\pi}{2} + \frac{2}{2k+1} = \frac{\pi}{2} + \frac{2}{n+1},
\end{aligned}
$$

for the denominator. Therefore,

$$
\Lambda_n\left(\frac{n+1}{2n}\right) = \frac{N\left(\frac{n+1}{2n}\right)}{D\left(\frac{n+1}{2n}\right)} \ge \frac{2\ln(n+1)}{\pi + \frac{4}{n+1}}. \tag{7}
$$

Similarly, if n is odd, say $n = 2k+1$, then at $x = 1/2$ we have

$$
\begin{aligned}
N\left(\frac{1}{2}\right) &= \sum_{j=0}^{n} \frac{1}{|n-2j|} = \sum_{j=0}^{2k+1} \frac{1}{|2(k-j)+1|} = \sum_{j=0}^{k} \frac{1}{|2(k-j)+1|} + \sum_{j=k+1}^{2k+1} \frac{1}{|2(k-j)+1|} = 2\sum_{j=0}^{k} \frac{1}{2j+1} \\
&\ge \ln(2k+3) = \ln(n+2)
\end{aligned}
$$

and

$$
\begin{aligned}
D\left(\frac{1}{2}\right) &= \left|\sum_{j=0}^{n} \frac{(-1)^j}{n-2j}\right| = \left|\sum_{j=0}^{2k+1} \frac{(-1)^j}{2(k-j)+1}\right| = \left|\sum_{j=0}^{k} \frac{(-1)^j}{2(k-j)+1} + \sum_{j=k+1}^{2k+1} \frac{(-1)^j}{2(k-j)+1}\right| = 2\sum_{j=0}^{k} \frac{(-1)^j}{2j+1} \\
&\le 2\left(\frac{\pi}{4} + \frac{1}{2k+3}\right) = \frac{\pi}{2} + \frac{2}{n+2},
\end{aligned}
$$

and therefore

$$
\Lambda_n\left(\frac{1}{2}\right) = \frac{N\left(\frac{1}{2}\right)}{D\left(\frac{1}{2}\right)} \ge \frac{2\ln(n+2)}{\pi + \frac{4}{n+2}} \ge \frac{2\ln(n+1)}{\pi + \frac{4}{n+1}}, \tag{8}
$$

where the last inequality follows because $2\ln(x)/(\pi + 4/x)$ is a nondecreasing function for $x \ge 1$. From (7) and (8) we finally conclude that

$$
\Lambda_n = \max_{0 \le x \le 1} \Lambda_n(x) \ge \frac{2\ln(n+1)}{\pi + \frac{4}{n+1}} \ge \frac{2\ln(n+1)}{\pi + \frac{4}{n}} = \frac{2n}{4+n\pi}\ln(n+1)
$$

for any $n \in \mathbb{N}$. □

Note that the bound in Theorem 1 is a considerable improvement of the corresponding result given by Berrut and Mittelmann [2, Theorem 3.1], namely $\Lambda_n \ge 1/(2n^2)$. Our next result concerns the upper bound of the Lebesgue constant.

Theorem 2. *The Lebesgue constant for interpolation with the basis functions $b_i(x)$ in* (2) *at the nodes x_j in* (6) *satisfies*

$$
\Lambda_n \le 2 + \ln(n).
$$

Proof. If $x = x_k$ for any k, then it follows from the interpolation property of the basis functions that $\Lambda_n(x) = 1$. So let $x_k < x < x_{k+1}$ for some k and consider the function

$$
\Lambda_{n,k}(x) = \frac{(x-x_k)(x_{k+1}-x)\displaystyle\sum_{j=0}^{n} \frac{1}{|x-x_j|}}{\left|(x-x_k)(x_{k+1}-x)\displaystyle\sum_{j=0}^{n} \frac{(-1)^j}{x-x_j}\right|} =: \frac{N_k(x)}{D_k(x)}. \tag{9}
$$

Our goal now is to bound the numerator $N_k(x)$ from above and the denominator $D_k(x)$ from below.

We first focus on the numerator,

$$\begin{aligned}
N_k(x) &= (x-x_k)(x_{k+1}-x)\sum_{j=0}^{n}\frac{1}{|x-x_j|}\\
&= (x-x_k)(x_{k+1}-x)\left(\sum_{j=0}^{k-1}\frac{1}{x-x_j}+\frac{1}{x-x_k}+\frac{1}{x_{k+1}-x}+\sum_{j=k+2}^{n}\frac{1}{x_j-x}\right)\\
&= (x_{k+1}-x)+(x-x_k)+(x-x_k)(x_{k+1}-x)\left(\sum_{j=0}^{k-1}\frac{1}{x-x_j}+\sum_{j=k+2}^{n}\frac{1}{x_j-x}\right)\\
&= (x_{k+1}-x_k)+(x-x_k)(x_{k+1}-x)\left(\sum_{j=0}^{k-1}\frac{1}{x-x_j}+\sum_{j=k+2}^{n}\frac{1}{x_j-x}\right).
\end{aligned}$$

As the nodes x_j are equally spaced at distance $h=1/n$ we have

$$\frac{1}{x_i-x_j}=\frac{1}{h(i-j)}=\frac{n}{i-j}$$

for any $i\neq j$ and

$$(x-x_k)(x_{k+1}-x)\le\left(\frac{h}{2}\right)^2=\frac{1}{4n^2}$$

for $x_k<x<x_{k+1}$. Therefore, using also (4), we get

$$\begin{aligned}
N_k(x) &\le \frac{1}{n}+\frac{1}{4n^2}\left(\sum_{j=0}^{k-1}\frac{1}{x_k-x_j}+\sum_{j=k+2}^{n}\frac{1}{x_j-x_{k+1}}\right)=\frac{1}{n}+\frac{1}{4n^2}\left(\sum_{j=0}^{k-1}\frac{n}{k-j}+\sum_{j=k+2}^{n}\frac{n}{j-k-1}\right)\\
&= \frac{1}{n}+\frac{1}{4n}\left(\sum_{j=1}^{k}\frac{1}{j}+\sum_{j=1}^{n-k-1}\frac{1}{j}\right)\\
&\le \frac{1}{n}+\frac{1}{4n}\big(\ln(2k+1)+\ln(2n-2k-1)\big)=\frac{1}{n}+\frac{1}{4n}\ln\big((2k+1)(2n-(2k+1))\big)\\
&\le \frac{1}{n}+\frac{1}{4n}\ln\big((2n/2)^2\big)=\frac{1}{n}+\frac{1}{2n}\ln(n).
\end{aligned}$$

We now turn to the denominator in (9), ignoring the absolute value and assuming both k and n to be even for the moment, so that

$$\begin{aligned}
D_k(x) &= (x-x_k)(x_{k+1}-x)\sum_{j=0}^{n}\frac{(-1)^j}{x-x_j}\\
&= (x-x_k)(x_{k+1}-x)\left(\sum_{j=0}^{k-1}\frac{(-1)^j}{x-x_j}+\frac{1}{x-x_k}+\frac{1}{x_{k+1}-x}-\sum_{j=k+2}^{n}\frac{(-1)^j}{x_j-x}\right)\\
&= \frac{1}{n}+(x-x_k)(x_{k+1}-x)\left(\sum_{j=0}^{k-1}\frac{(-1)^j}{x-x_j}-\sum_{j=k+2}^{n}\frac{(-1)^j}{x_j-x}\right).
\end{aligned}$$

Pairing the positive and negative terms in the rightmost factor adequately then gives

$$\begin{aligned}
S_k(x) &= \sum_{j=0}^{k-1}\frac{(-1)^j}{x-x_j}-\sum_{j=k+2}^{n}\frac{(-1)^j}{x_j-x}\\
&= \frac{1}{x-x_0}+\left(\frac{1}{x-x_2}-\frac{1}{x-x_1}\right)+\cdots+\left(\frac{1}{x-x_{k-2}}-\frac{1}{x-x_{k-3}}\right)-\frac{1}{x-x_{k-1}}\\
&\quad -\frac{1}{x_{k+2}-x}+\left(\frac{1}{x_{k+3}-x}-\frac{1}{x_{k+4}-x}\right)+\cdots+\left(\frac{1}{x_{n-1}-x}-\frac{1}{x_n-x}\right).
\end{aligned}\tag{10}$$

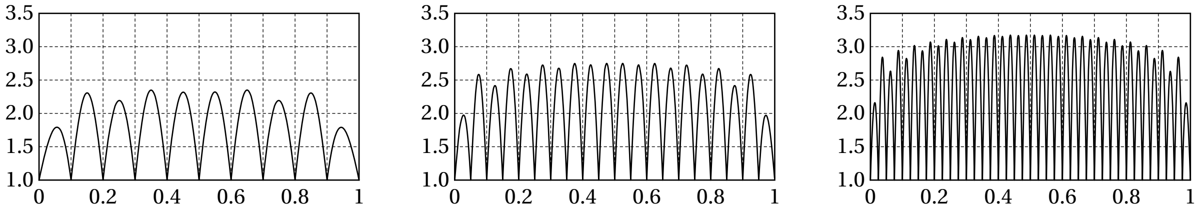

Figure 1: Lebesgue function of Berrut's interpolant at $n+1$ equidistant nodes for $n=10,20,40$.

Since both the leading term and all paired terms are positive, we have

$$S_k(x) > -\frac{1}{x-x_{k-1}} - \frac{1}{x_{k+2}-x} \geq -\frac{1}{x_k-x_{k-1}} - \frac{1}{x_{k+2}-x_{k+1}} = -2n$$

and further

$$D_k(x) = \frac{1}{n} + (x-x_k)(x_{k+1}-x)S_k(x) \geq \frac{1}{n} + \frac{1}{4n^2}(-2n) = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}.$$

This bound also holds if n is odd as this only adds a single positive term $1/(x_n-x)$ to $S_k(x)$ in (10). If k and n are both odd, then a similar reasoning shows that

$$\begin{aligned} D_k(x) = -\frac{1}{n} + (x-x_k)(x_{k+1}-x)\Bigg[\left(\frac{1}{x-x_0} - \frac{1}{x-x_1}\right) + \cdots + \left(\frac{1}{x-x_{k-3}} - \frac{1}{x-x_{k-2}}\right) + \frac{1}{x-x_{k-1}} \\ + \frac{1}{x-x_{k+2}} + \left(\frac{1}{x_{k+4}-x} - \frac{1}{x_{k+3}-x}\right) + \cdots + \left(\frac{1}{x_n-x} - \frac{1}{x_{n-1}-x}\right)\Bigg] \\ < -\frac{1}{n} + \frac{1}{4n^2}(2n) = -\frac{1}{2n}, \end{aligned}$$

because all paired terms are negative, and likewise for k odd and n even. Therefore, we have $|D_k(x)| \geq 1/(2n)$ regardless of the parity of k and n, and combining the bounds for the numerator and denominator in (9) yields

$$\Lambda_n = \max_{k=0,\ldots,n-1}\left(\max_{x_k<x<x_{k+1}} \frac{N_k(x)}{D_k(x)}\right) \leq \frac{\frac{1}{n} + \frac{1}{2n}\ln(n)}{\frac{1}{2n}} = 2 + \ln(n).$$

□

3 Numerical experiments

Besides the theoretical results in the previous section, we also performed a number of numerical experiments to further analyse the behaviour of the Lebesgue function and the Lebesgue constant.

Figure 1 shows the Lebesgue function for interpolation at $n+1$ equidistant nodes for some small values of n. The plots suggest that the maximum is always obtained near the centre of the interpolation interval, which explains why we analyse the Lebesgue function at $x=1/2$ for odd n and at $x=1/2+h/2$ for even n in the proof of Theorem 1.

We further computed the Lebesgue constant numerically for $1 \leq n \leq 200$ by evaluating

$$\Lambda_n \approx \max_{0\leq k\leq N} \Lambda_n\left(\frac{k}{N}\right)$$

for $N = 10\,000\,n$. Figure 2 shows these values as well as the lower and upper bounds from Theorem 1 and Theorem 2. Our results suggest that the sequences $(\Lambda_{2k-1})_{k\in\mathbb{N}}$ and $(\Lambda_{2k})_{k\in\mathbb{N}}$ are strictly increasing and that $\Lambda_{2k} < \Lambda_{2k-1}$ for $k \geq 2$. Thus, interpolation at an odd number of equidistant nodes (i.e. n even) is slightly more stable than interpolation at an even number of nodes, which could be related to the fact that Berrut's rational interpolant reproduces only constant functions for even n, but linear functions for odd n (which follows from Theorem 3 in [6]). Moreover, we observe that for $n \geq 10$ the Lebesgue constant of this rational interpolant is even smaller than the lower bound for the Lebesgue constant for polynomial interpolation at Chebyshev nodes that was found by Rivlin [7], namely $\frac{2}{\pi}\ln(n+1)+\alpha$ with $\alpha \approx 0.9625$.

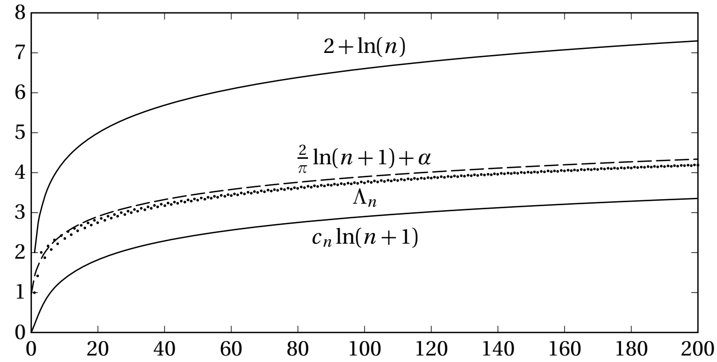

Figure 2: Lebesgue constant of Berrut's interpolant at $n+1$ equidistant nodes for $1 \le n \le 200$, compared to our lower and upper bounds and Rivlin's lower bound for polynomial interpolation at Chebyshev nodes.

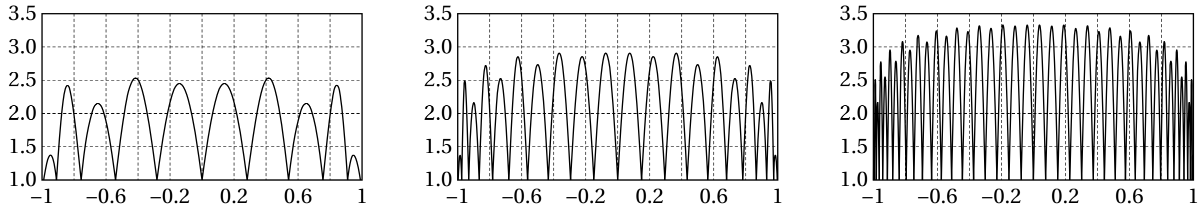

Figure 3: Lebesgue function of Berrut's interpolant at $n+1$ Chebyshev nodes for $n = 10, 20, 40$.

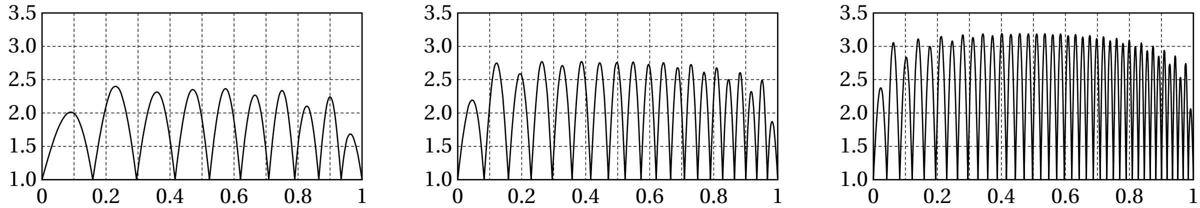

Figure 4: Lebesgue function of Berrut's interpolant at $n+1$ logarithmic nodes for $n = 10, 20, 40$.

Interestingly, the Lebesgue constant behaves very similarly if we consider interpolation at the Chebyshev nodes

$$x_j = \cos\left(\frac{2j+1}{2n+2}\pi\right), \qquad j = 0, \dots, n$$

and the logarithmically distributed nodes

$$x_j = \ln\left(1 + \frac{i}{n}(e-1)\right), \qquad j = 0, \dots, n.$$

The corresponding Lebesgue functions for some small values of n are plotted in Figure 3 and Figure 4, respectively. Figure 5 shows the numerically computed Lebesgue constants, together with the two functions $\frac{2}{\pi}\ln(n+1) + 0.6$ and $\frac{2}{\pi}\ln(n+1) + 1.2$ as a reference to help comparing the two plots. As for equidistant nodes, it seems that $(\Lambda_{2k-1})_{k \in \mathbb{N}}$ and $(\Lambda_{2k})_{k \in \mathbb{N}}$ are strictly increasing and that $\Lambda_{2k} < \Lambda_{2k-1}$ for $k \ge 2$ in both cases.

Overall, we observe that the Lebesgue constants for Chebyshev points are greater than those for logarithmically distributed points, which in turn are slightly greater than the Lebesgue constants for equidistant points. However, in all three cases, the asymptotic growth seems to be $\frac{2}{\pi}\ln(n+1)$.

Acknowledgements

This work was done with support of the *60% funds, year 2010* of the University of Padua.

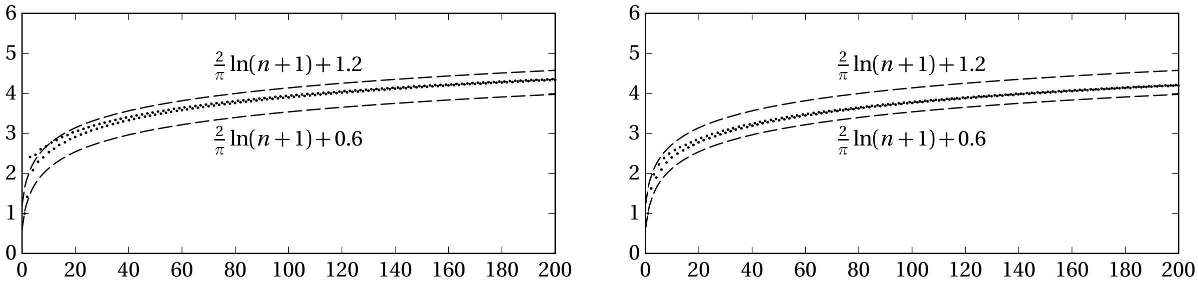

Figure 5: Lebesgue constant of Berrut's interpolant at $n+1$ Chebyshev (left) and logarithmic nodes (right) for $1 \leq n \leq 200$, compared to two logarithmic functions.

References

[1] J.-P. Berrut. Rational functions for guaranteed and experimentally well-conditioned global interpolation. *Computers & Mathematics with Applications*, 15(1):1–16, 1988.

[2] J.-P. Berrut and H. D. Mittelmann. Lebesgue constant minimizing linear rational interpolation of continuous functions over the interval. *Computers & Mathematics with Applications*, 33(6):77–86, Mar. 1997.

[3] L. Brutman. On the Lebesgue function for polynomial interpolation. *SIAM Journal on Numerical Analysis*, 15(4):694–704, Aug. 1978.

[4] L. Brutman. Lebesgue functions for polynomial interpolation—a survey. *Annals of Numerical Mathematics*, 4:111–127, 1997.

[5] J. M. Carnicer. Weighted interpolation for equidistant nodes. *Numerical Algorithms*, 55(2–3):223–232, Nov. 2010.

[6] M. S. Floater and K. Hormann. Barycentric rational interpolation with no poles and high rates of approximation. *Numerische Mathematik*, 107(2):315–331, Aug. 2007.

[7] T. J. Rivlin. The Lebesgue constants for polynomial interpolation. In H. G. Garnir, K. R. Unni, and J. H. Williamson, editors, *Functional Analysis and its Applications*, volume 399 of *Lecture Notes in Mathematics*, pages 422–437. Springer, Berlin, 1974.

[8] A. Schönhage. Fehlerfortpflanzung bei Interpolation. *Numerische Mathematik*, 3(1):62–71, Dec. 1961.

[9] S. J. Smith. Lebesgue constants in polynomial interpolation. *Annales Mathematicae et Informaticae*, 33:109–123, 2006.

[10] L. N. Trefethen and J. A. C. Weideman. Two results on polynomial interpolation in equally spaced points. *Journal of Approximation Theory*, 65(3):247–260, June 1991.

[11] Q. Wang, P. Moin, and G. Iaccarino. A rational interpolation scheme with superpolynomial rate of convergence. *SIAM Journal on Numerical Analysis*, 47(6):4073–4097, 2010.