Blue Noise through Optimal Transport

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Geometry Processing and Finite/Boundary Element Methods
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INTRODUCTION

Improving Hodge-star accuracy

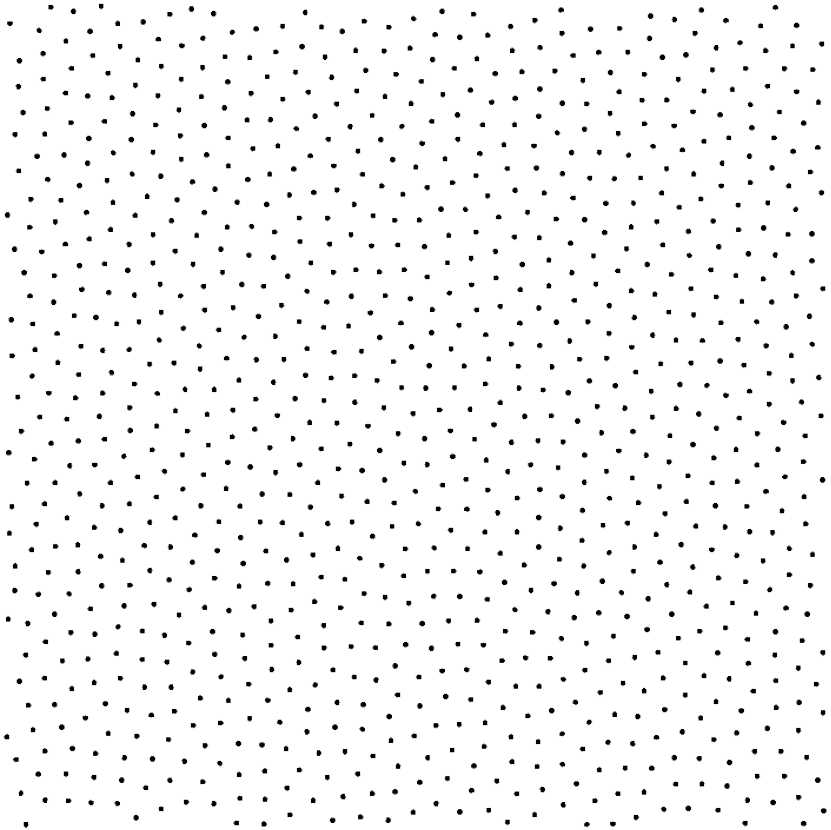
$$\star_{diag} \simeq \star$$

Prescribing Hodge-star values

$$\star_{diag} \equiv m\mathbb{I} \simeq \star$$

OUTLINE

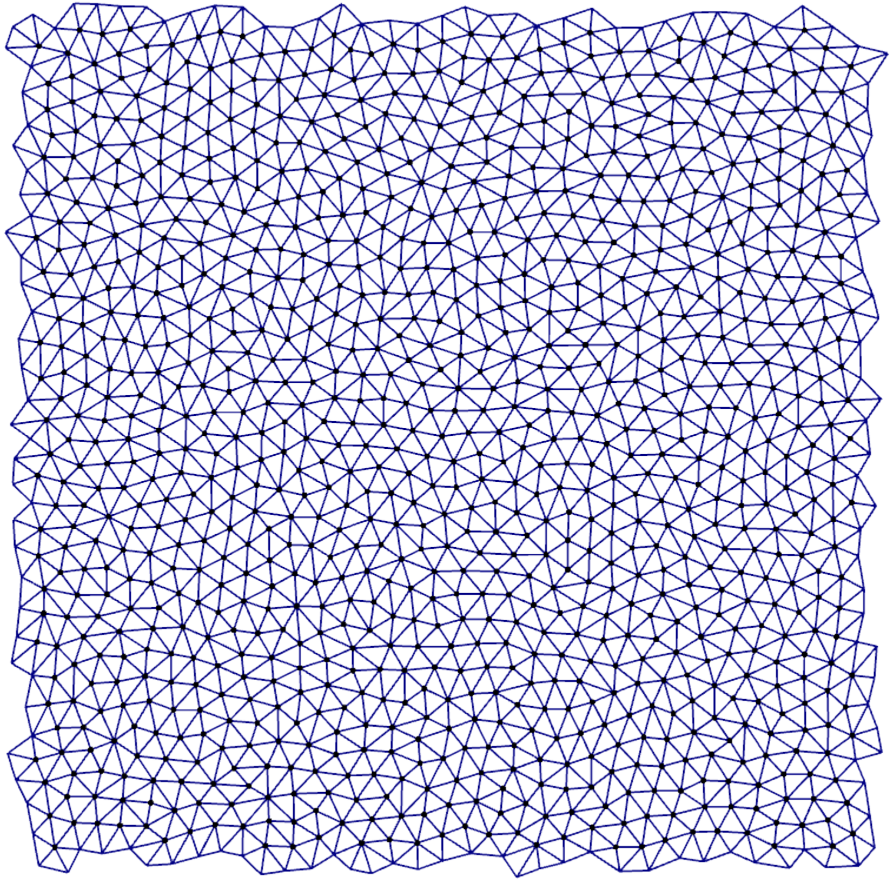
Blue Noise



OUTLINE

Blue Noise

Meshing

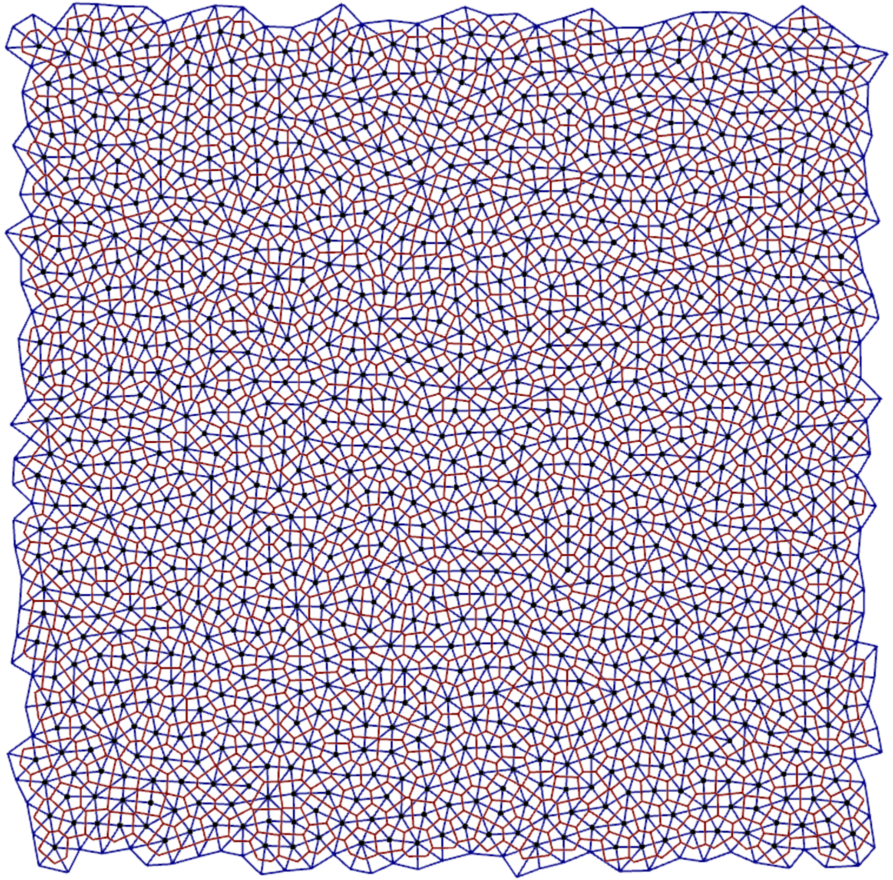


Outline

Blue Noise

Meshing

Properties



Outline

Blue Noise

Meshing

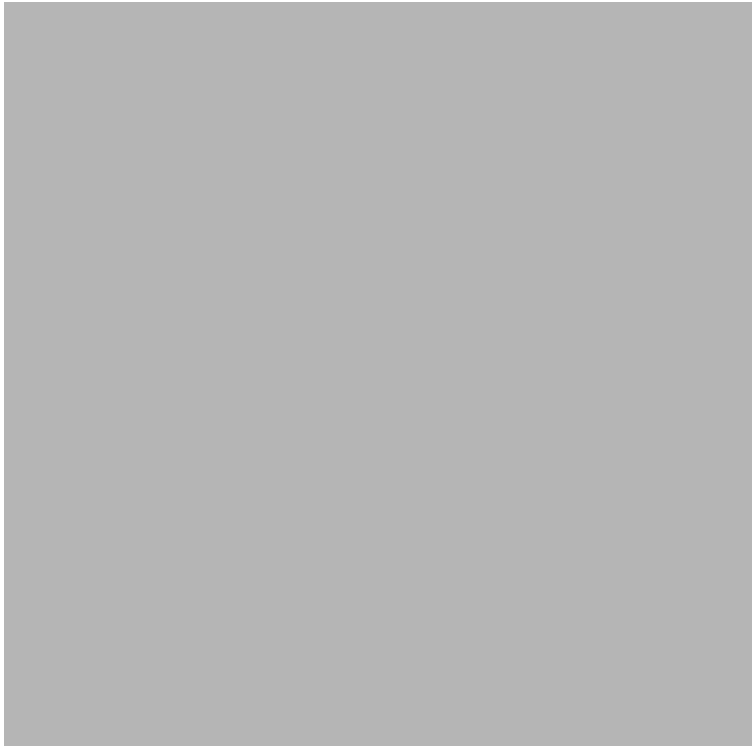
Properties

Numerics and Results

Blue Noise

What is Blue Noise?

Blue Noise is an even, isotropic, yet unstructured distribution of points.



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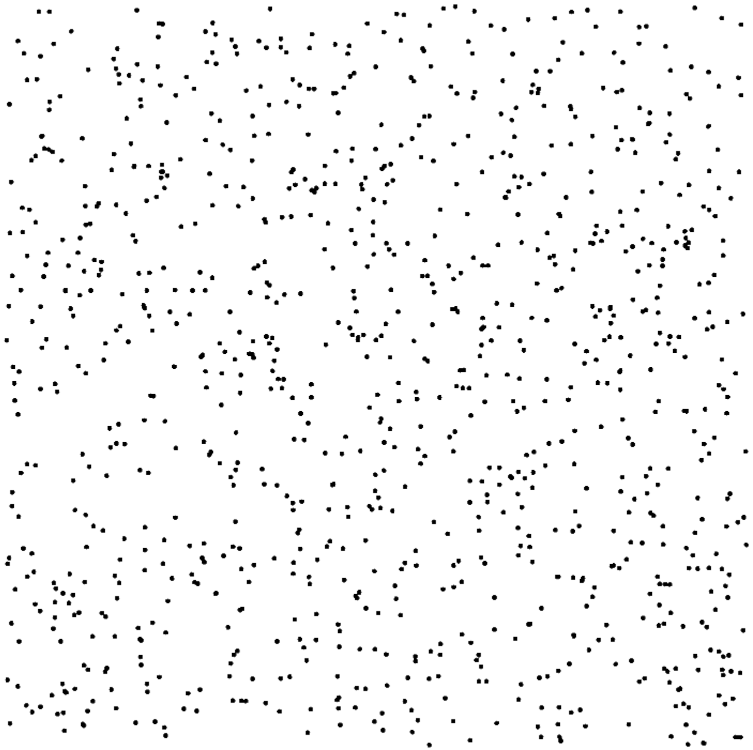

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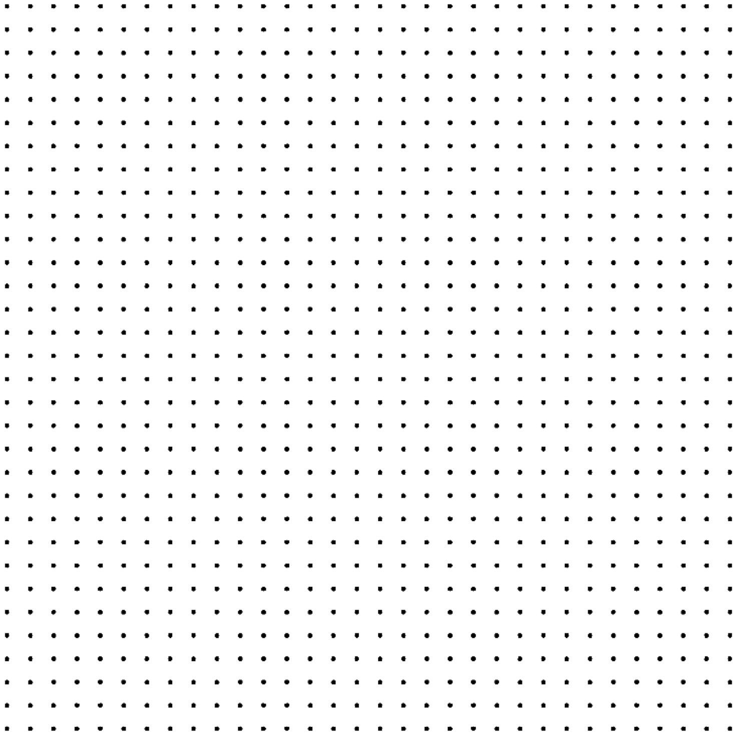

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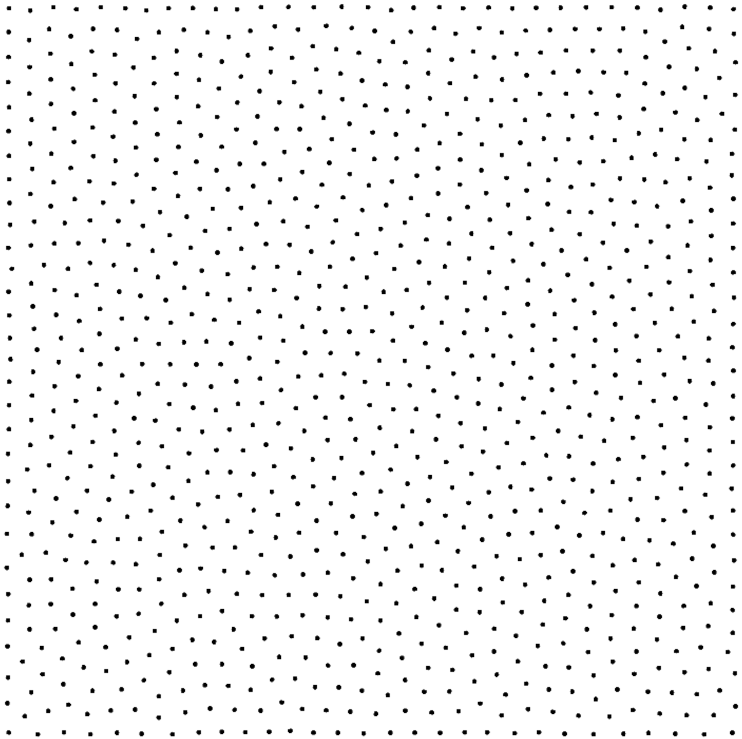

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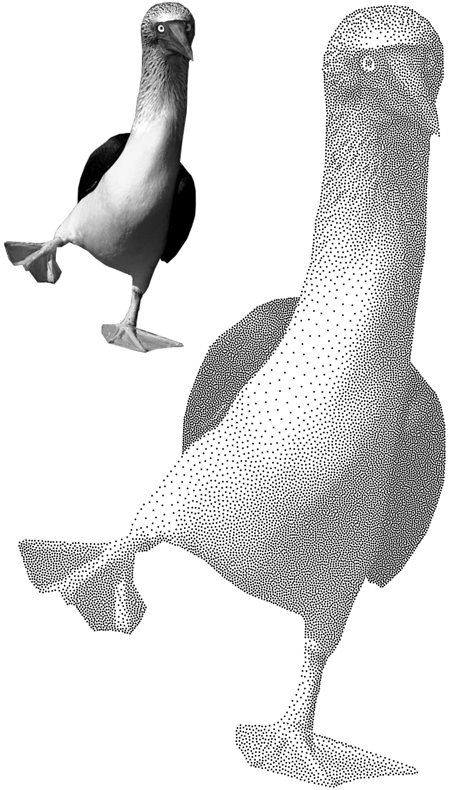
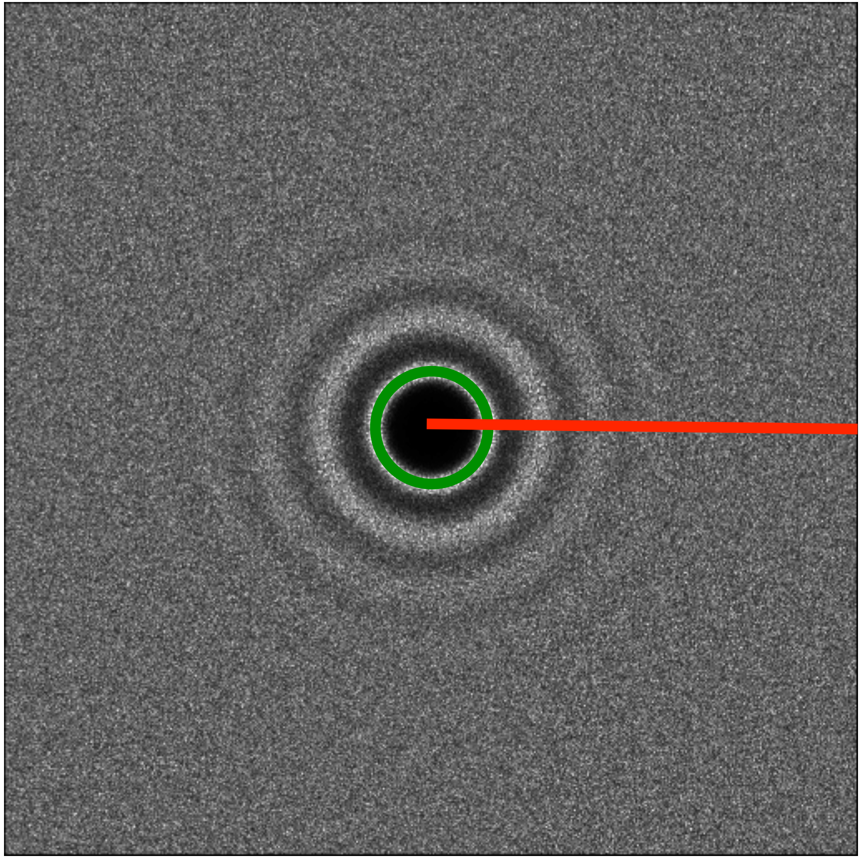
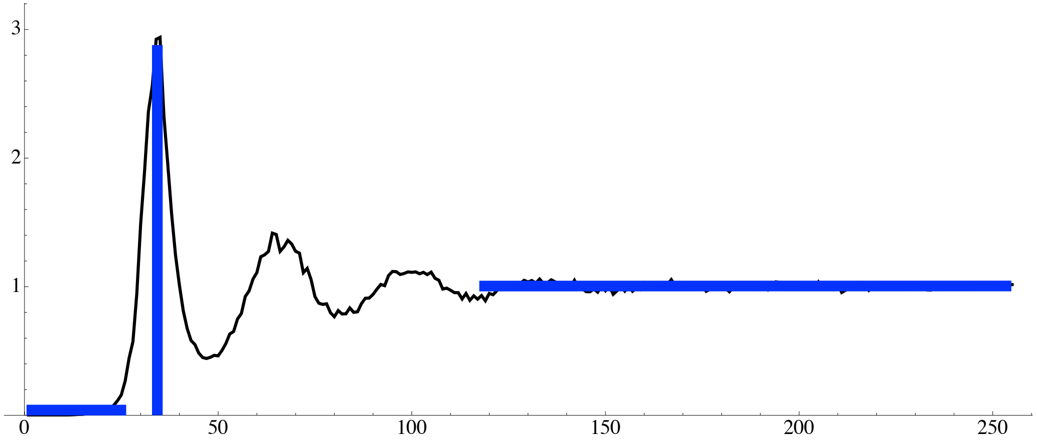


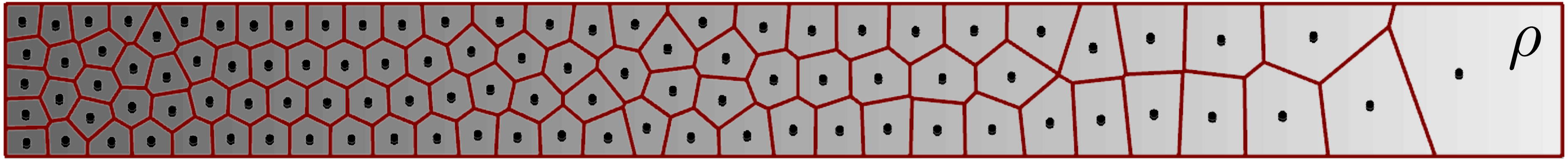
Spectral Characterization





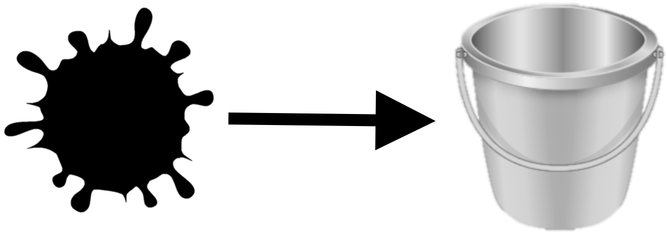
[Lagae and Dutre 2006]

Blue Noise as Meshing



Isotropic

$$\mathcal{E}_i = \int_{\mathcal{V}_i} \rho(\mathbf{x}) \|\mathbf{x} - \mathbf{x}_i\|^2 d\mathbf{x}$$



Even

$$m_i = \int_{\mathcal{V}_i} \rho(\mathbf{x}) \; d\mathbf{x} \equiv m$$

Unstructured

$$r_i = \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} |\mathcal{E}_i - \mathcal{E}_j|$$

APPROACH

Isotropic + Even

$$\min_{\mathbf{X},\mathcal{V}} \mathcal{E}(\mathbf{X},\mathcal{V}) = \sum_i \mathcal{E}_i \text{ s.t. } m = m_i$$

Solutions

Unstructured

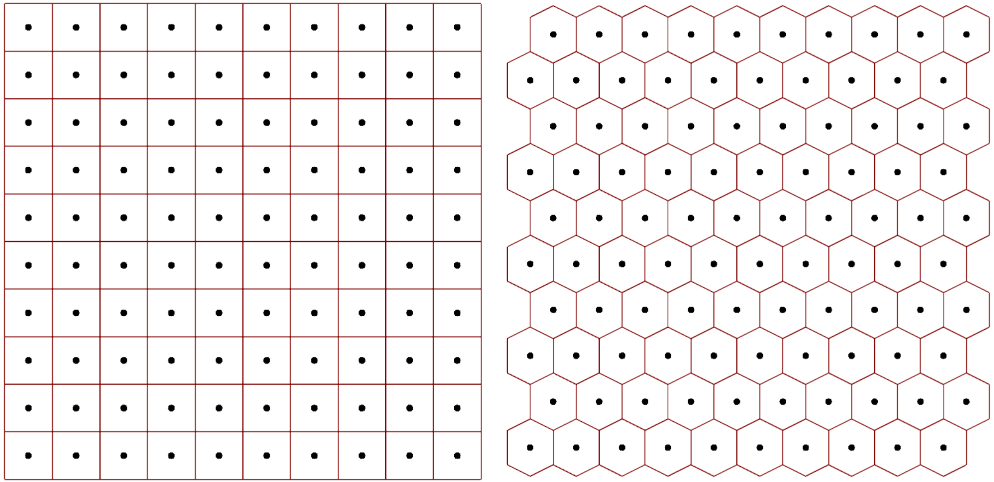
$$r_i > \tau$$

Link to HOT meshes

$$\min_{\mathbf{X},\mathcal{V}} \mathcal{E}(\mathbf{X},\mathcal{V}) \text{ s.t. } m = m_i$$

$$\mathcal{E}(\mathbf{X},\mathcal{V}) = \sum_i \int_{\mathcal{V}_i} \rho(\mathbf{x}) \|\mathbf{x} - \mathbf{x}_i\|^2 d\mathbf{x} \quad \equiv \text{CVT-Energy}$$

Voronoi Diagram

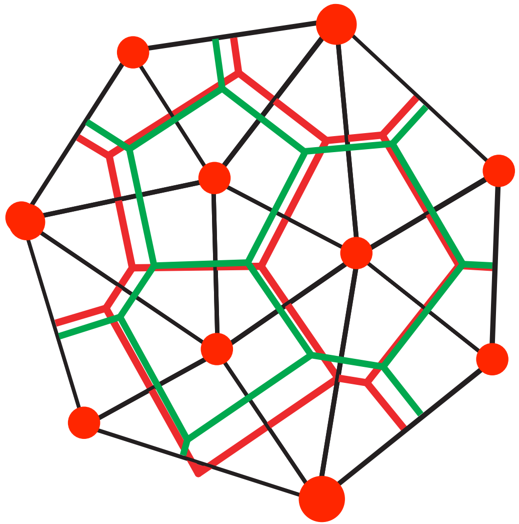


Link to HOT meshes

$$\min_{\mathbf{X},W} \mathcal{E}(\mathbf{X},W) \text{ s.t. } m = m_i$$

$$\mathcal{E}(\mathbf{X},W) = \sum_i \int_{\mathcal{V}_i^w} \rho(\mathbf{x}) \|\mathbf{x} - \mathbf{x}_i\|^2 d\mathbf{x} \equiv \star^0\text{-HOT}_{2,2}$$

Power Diagram



Recap

Even and *isotropic* point distribution

$\Updownarrow$

Minimize $\bigstar^0$-HOT$_{2,2}$ with constraints $m = m_i$

$$m\mathbb{I} \equiv \bigstar_{diag} \simeq \bigstar$$

Properties

Constrained Minimization

$$\min_{\mathbf{X},W} \mathcal{E}(\mathbf{X},W) \text{ s.t. } m = m_i$$

$$\underset{\mathbf{X},W,\Lambda}{\text{Extremize}}\ \mathcal{L}(\mathbf{X},W,\Lambda) = \mathcal{E}(\mathbf{X},W) + \sum_i \lambda_i(m_i - m)$$

$$0 = \nabla_{w_i}\mathcal{L}(\mathbf{X}^*,W^*,\Lambda^*) = \Delta^w \underbrace{(\Lambda^* + W^*)}_{\Lambda^* + W^* = \text{constant}}$$

$$\underset{\mathbf{X},W}{\text{Extremize}}\ \mathcal{F}(\mathbf{X},W) = \mathcal{E}(\mathbf{X},W) - \sum_i w_i(m_i - m)$$

Derivatives

$$\nabla_{\mathbf{x}_i} \mathcal{F}(\mathbf{X}, \underline{W}) = 2m_i(\mathbf{x}_i - \mathbf{b}_i^w)$$

$$\nabla_{\mathbf{x}_i} \mathcal{F} = 0 \implies \mathbf{x}_i = \mathbf{b}_i^w \implies \text{Centroidal Power Diagram}$$

$$\nabla_{w_i} \mathcal{F}(\underline{\mathbf{X}}, W) = m - m_i$$

$$\nabla_{w_i} \mathcal{F} = 0 \implies m = m_i \implies \text{Capacity-Constrained Assignment}$$

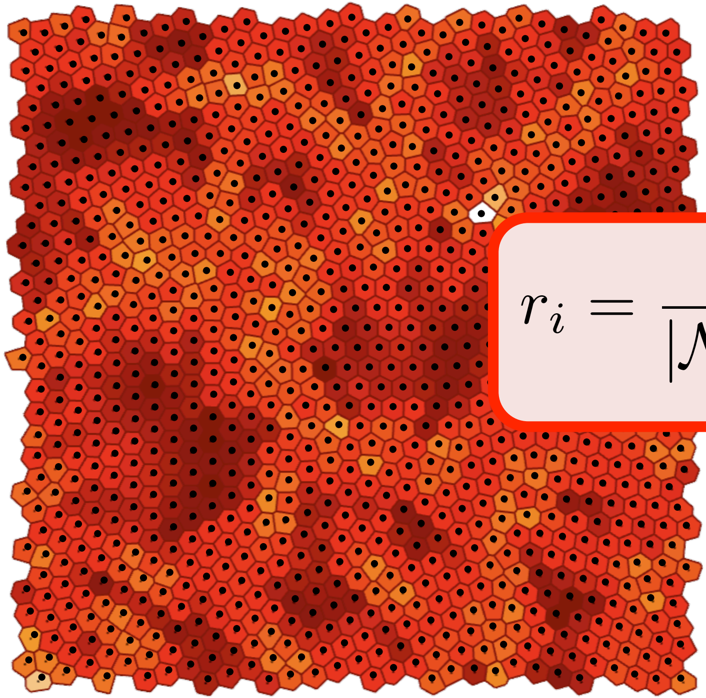
$$\nabla_w^2 \mathcal{F}(\underline{\mathbf{X}}, W) = -\Delta^w = -d \star^1 d$$

$$\Delta^w \text{ is PSD} \implies \mathcal{F}(\underline{\mathbf{X}}, W) \text{ is concave}$$

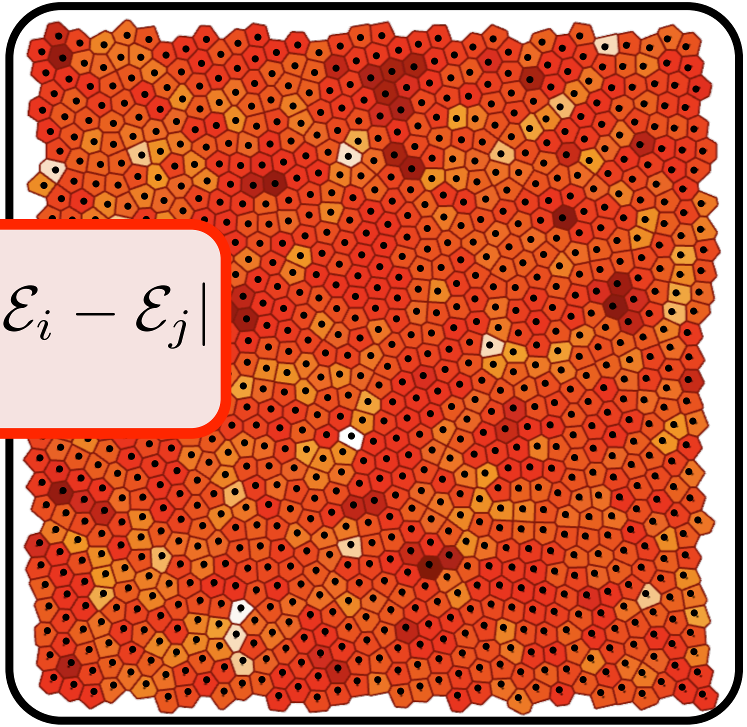
[Aurenhammer et al. 1998, Bosc 2010, Merigot 2011]

Unstructured Solution

$$\underset{\mathbf{X},W}{\text{Extremize}}\ \mathcal{F}(\mathbf{X},W) \implies \text{many solutions}$$



$$r_i = \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} |\mathcal{E}_i - \mathcal{E}_j|$$

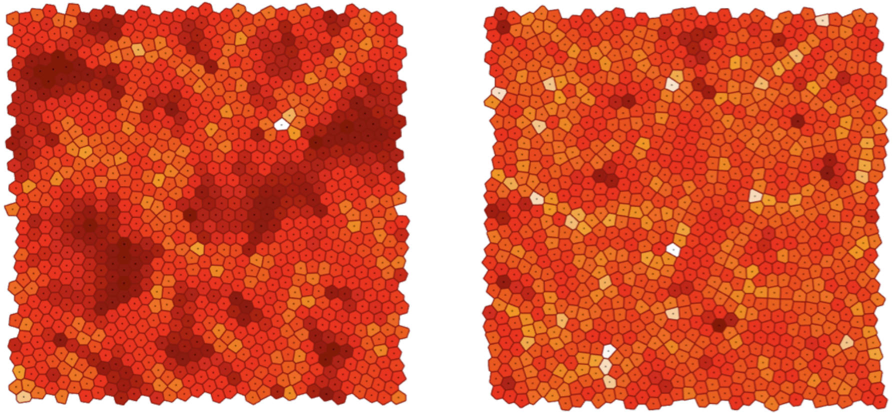


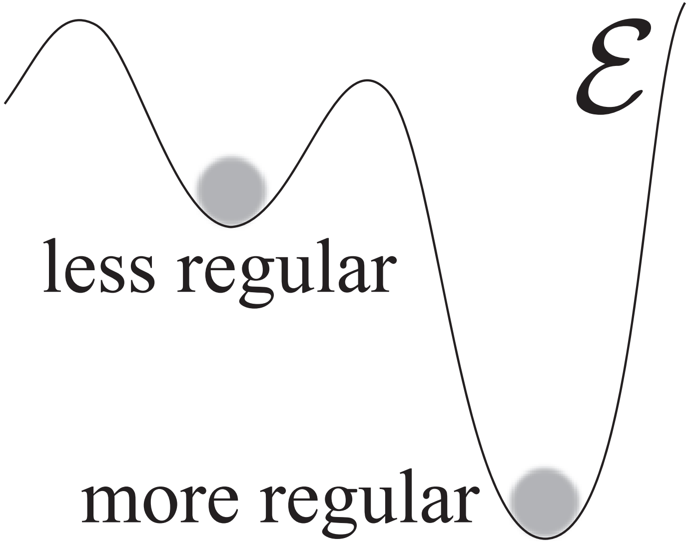
Unstructured = Shallow

$$\mathcal{F}(\mathbf{X}, W) = \mathcal{E}(\mathbf{X}, W) - \sum_i w_i(m_i - m)$$

$$\nabla_{w_i}\mathcal{F} = 0 \implies m = m_i$$

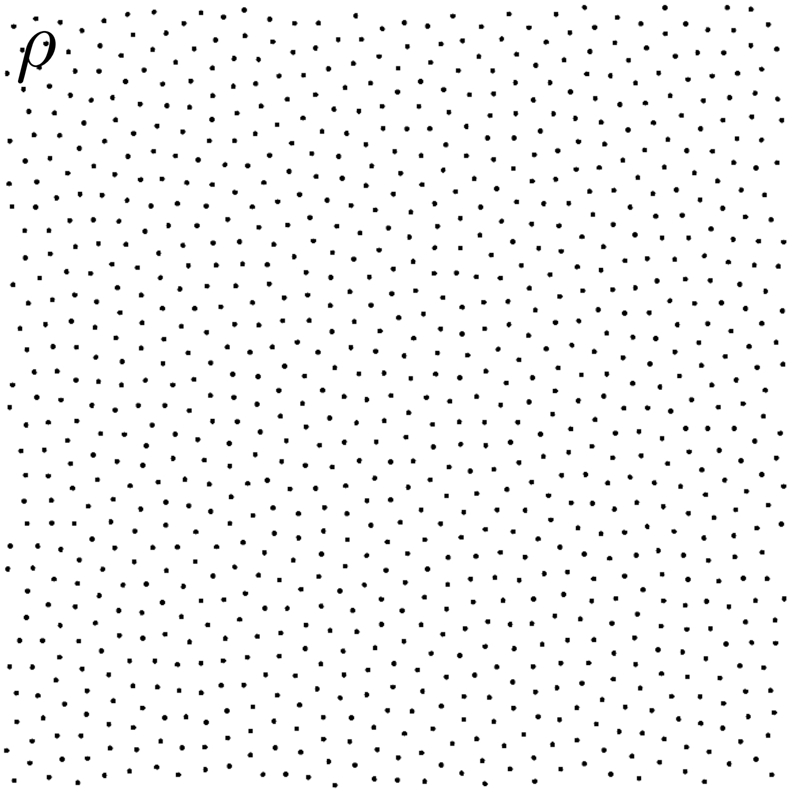
$$\mathcal{F}(\mathbf{X}^*, W^*) = \mathcal{E}(\mathbf{X}^*, W^*)$$





Numerics

ALGORITHM AT A GLANCE

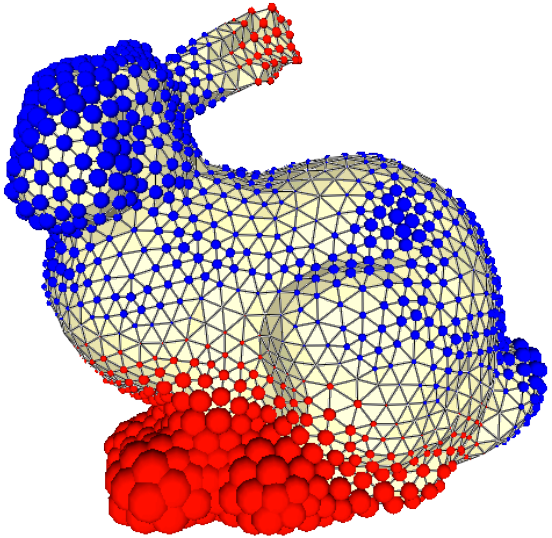


- Enforce Constraints (Weight Update)
- Minimize Energy (Position Update)

- If result has local structures
  Break structures

Weight Update

Newton's method:



Repeat:

1. $\Delta^w \delta_w = (m - m_1 \ldots m - m_n)^t$
2. Find time-step α along δ_w
3. $W \leftarrow W + \alpha \delta_w$

Until $\|\nabla_W \mathcal{F}\|^2 < \varepsilon$

Position Update

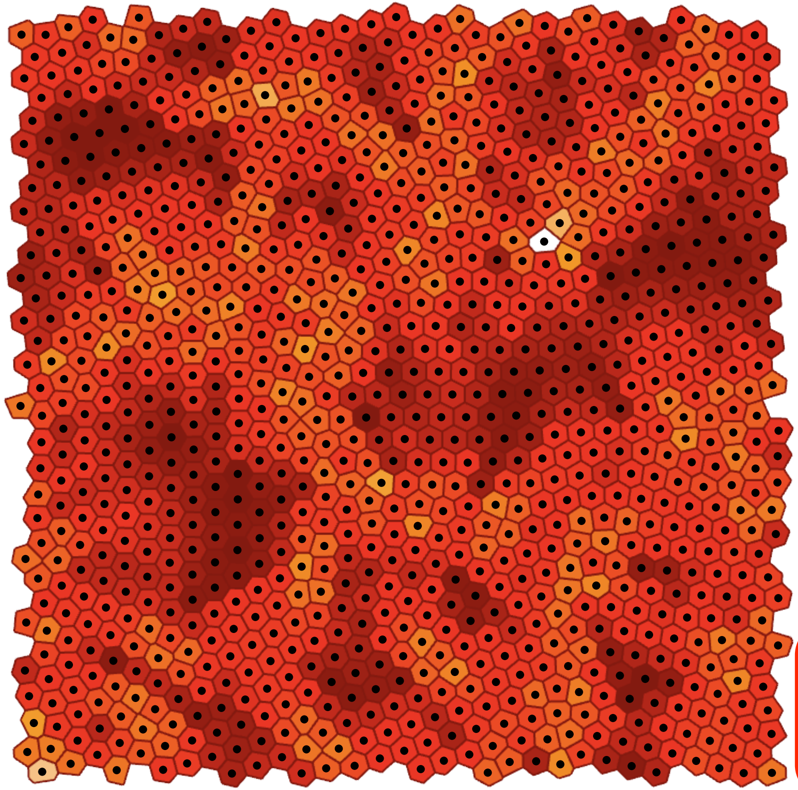
Lloyd iteration:

$$\mathbf{x}_i \leftarrow \mathbf{b}_i^w \ \forall\, i$$

Gradient Descent:

1. $\mathbf{d} = \nabla_{\mathbf{X}} \mathcal{F}$
2. Find time-step β along $\mathbf{d}$
3. $\mathbf{X} \leftarrow \mathbf{X} - \beta \mathbf{d}$

Avoid Structure



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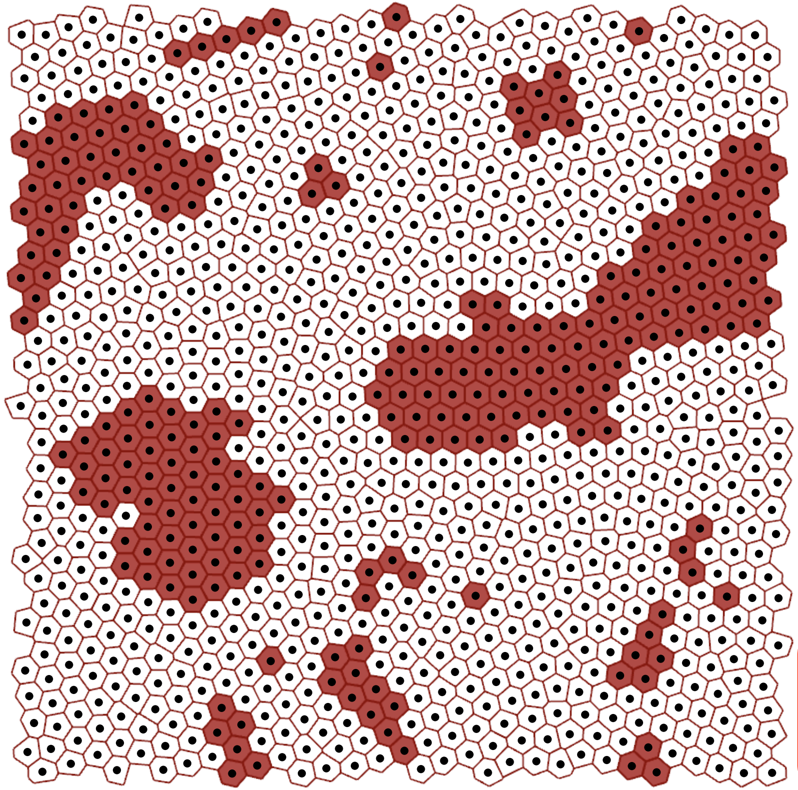
Avoid Structure



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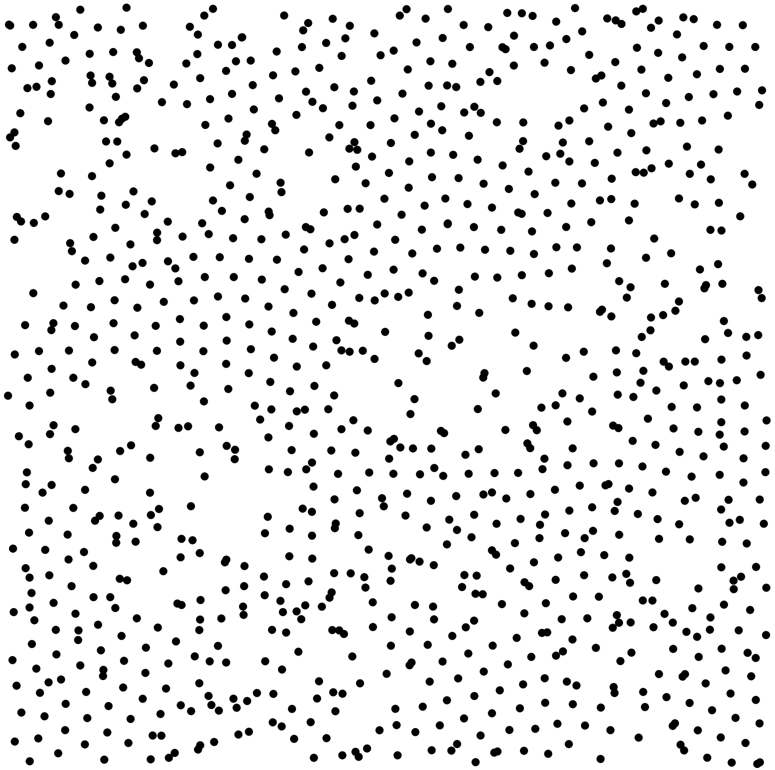

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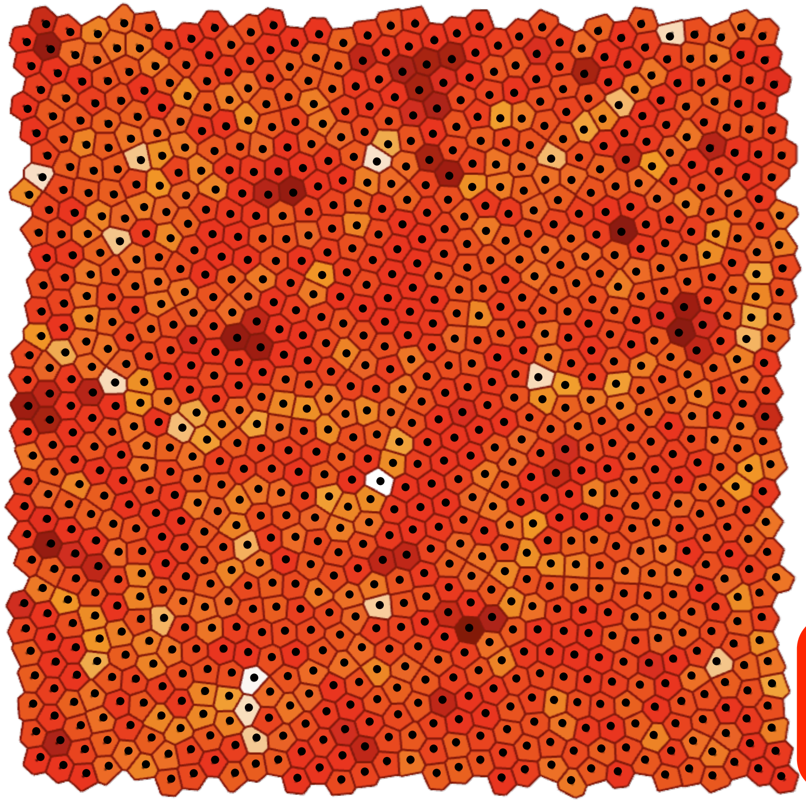
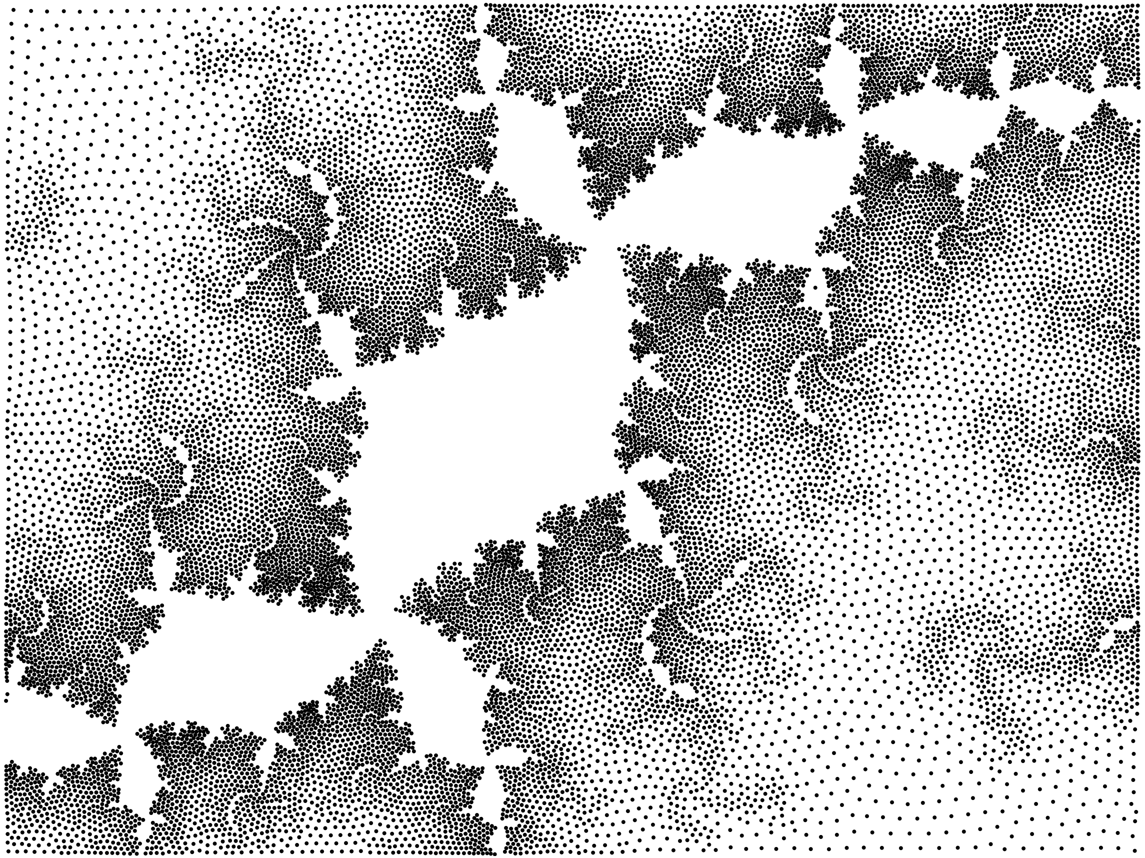


$$r_i = \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} |\mathcal{E}_i - \mathcal{E}_j|$$

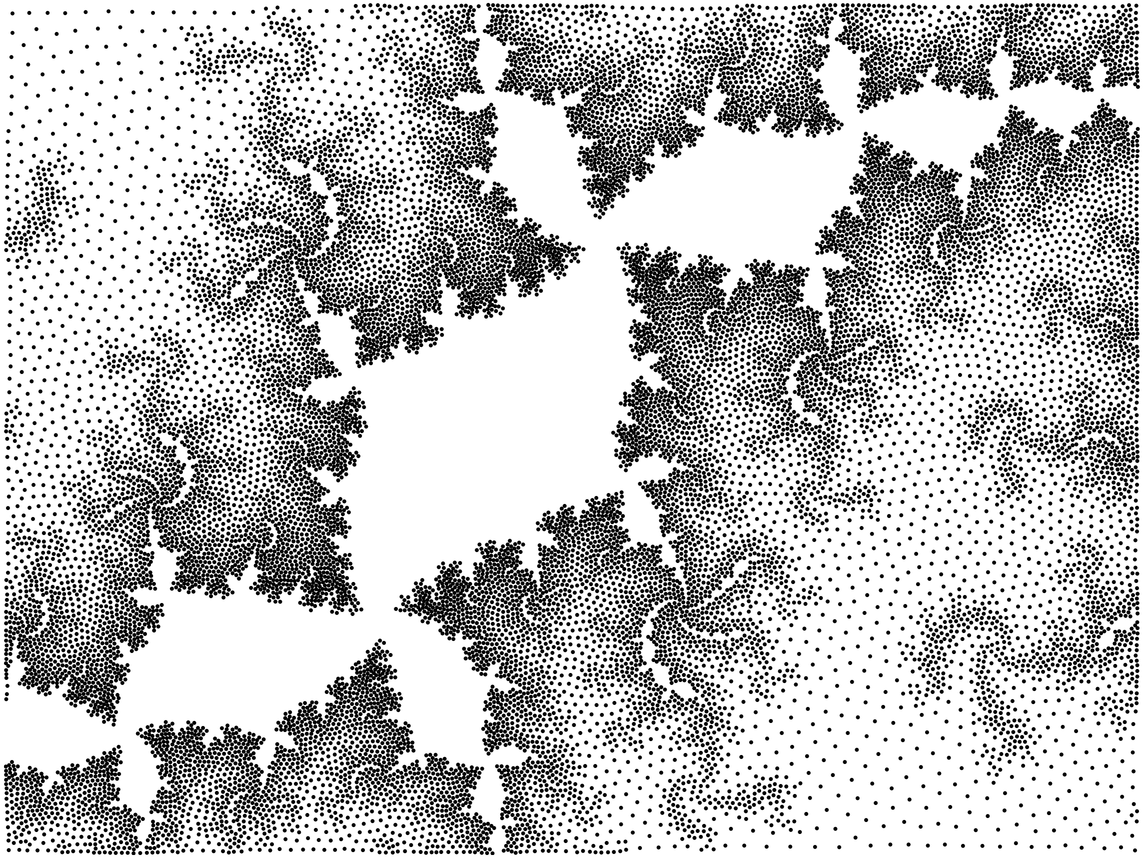
Results

CVT vs Blue Noise



[CVT]

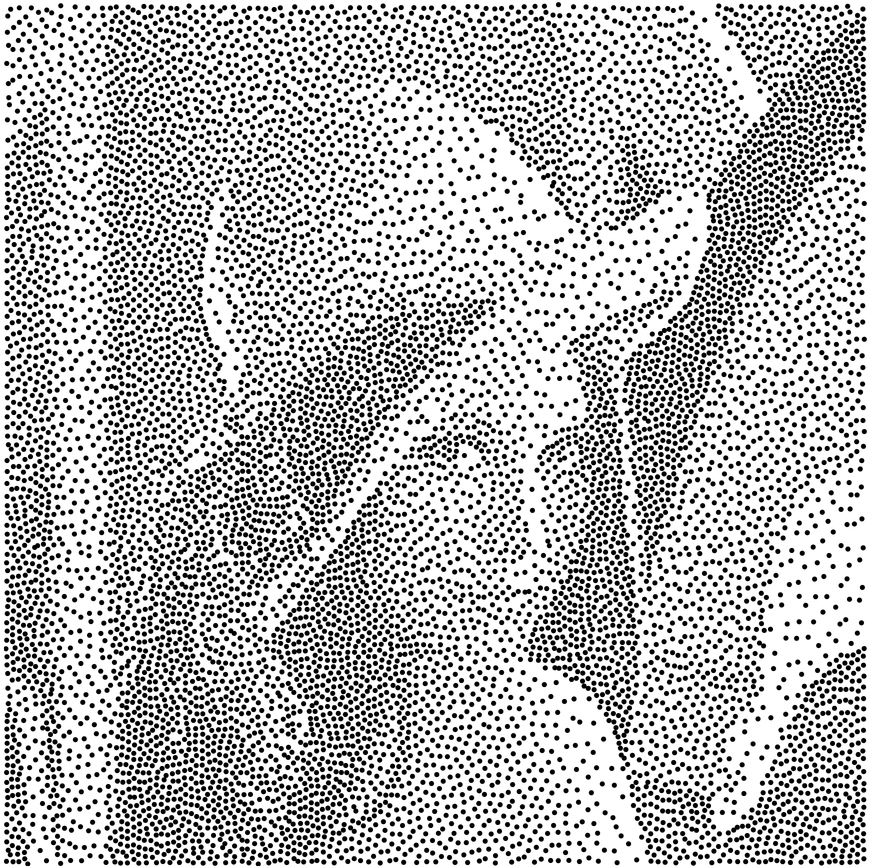
CVT vs Blue Noise



[Ours]

COMPARISON

[Chen et al. 2012]
[Schlomer et al 2011]
[Xu et al. 2011]
[Fattal 2011]
[Balzer et al. 2009]
[Lagae and Dutre 2006]



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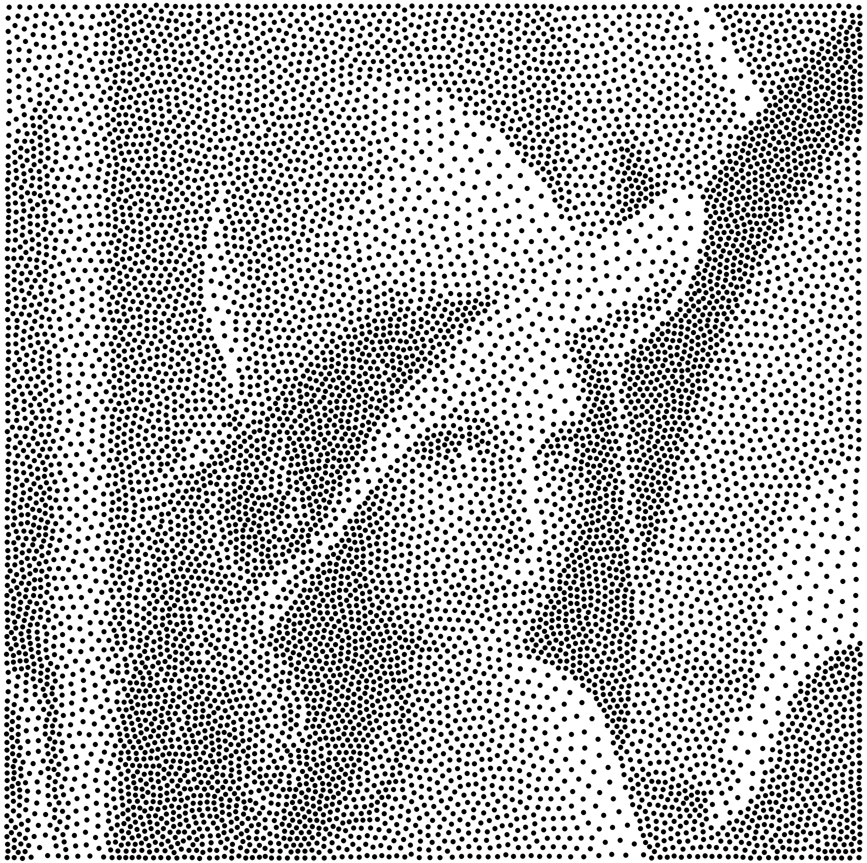
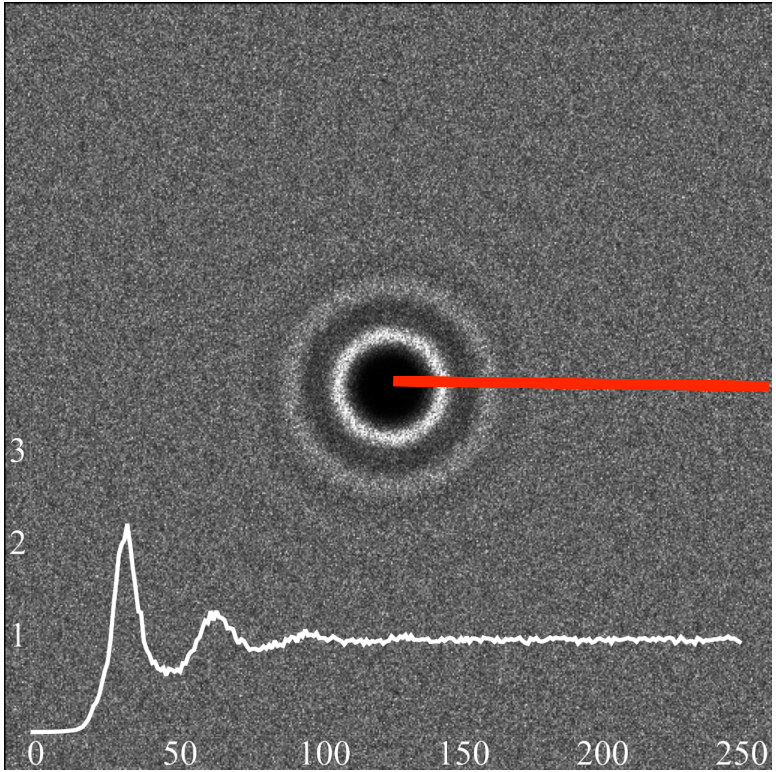


[Ours]

SPECTRUM



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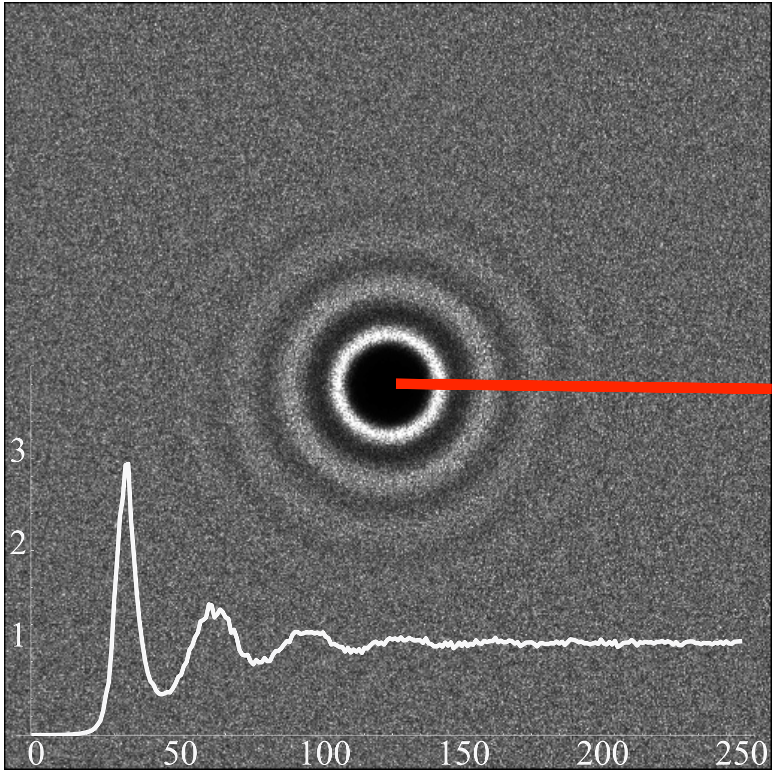
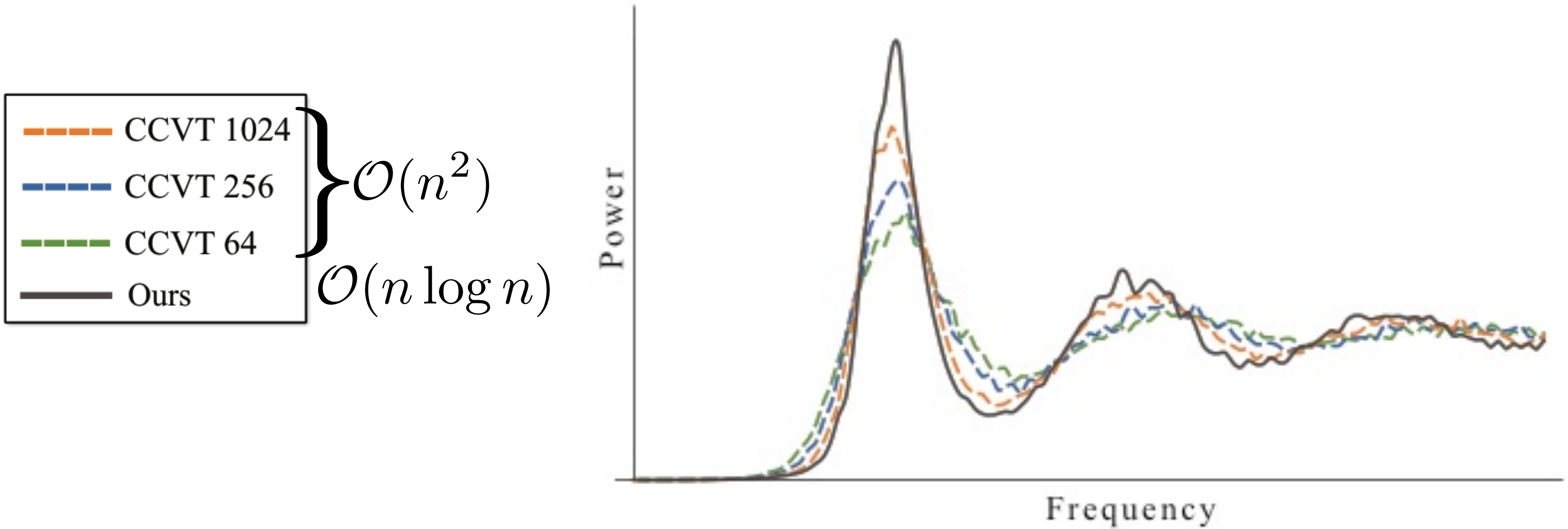


[Ours]

PERFORMANCE

$$\min_{\mathbf{X},W} \mathcal{E}(\mathbf{X}, W) \text{ s.t. } m = m_i$$

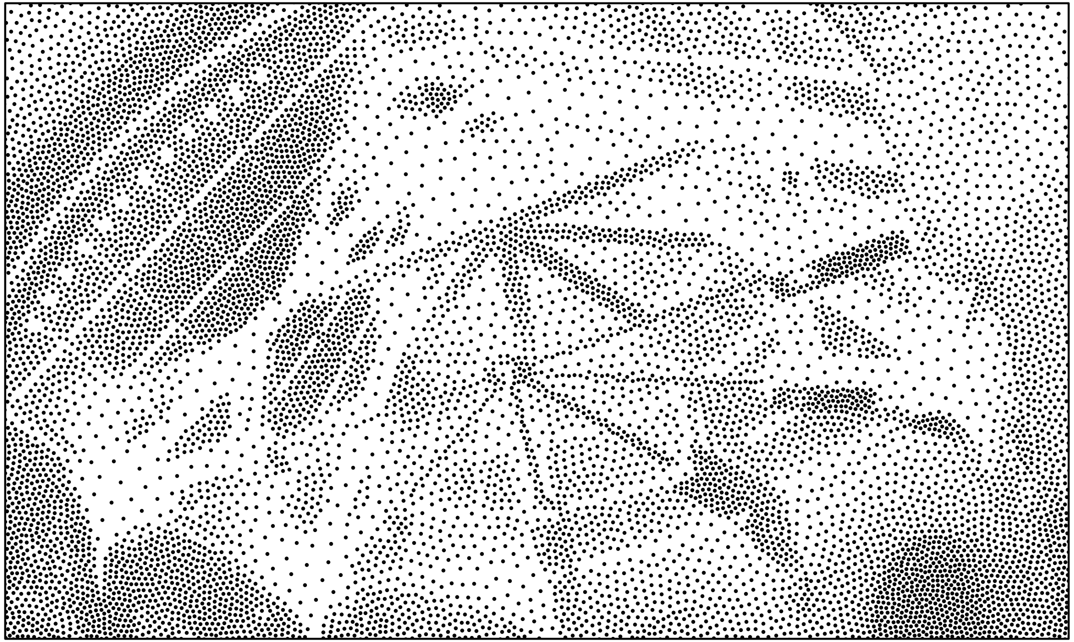
CCVT [Balzer et al. 2009]

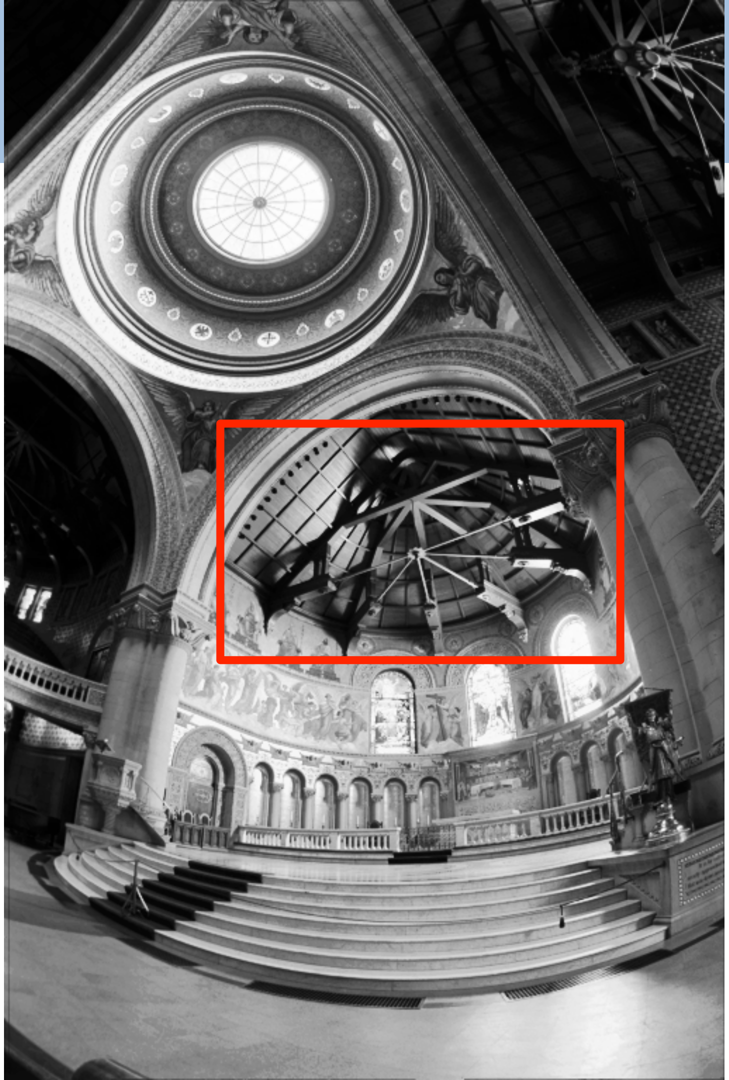


POINTILLISM

40k points

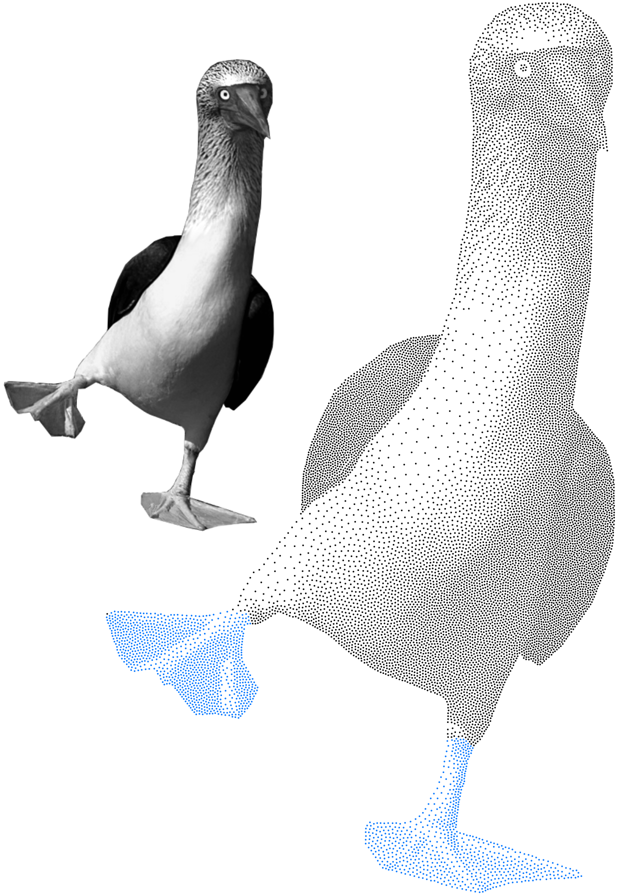
A LOT OF POINTS …





Conclusion

- Blue Noise as meshing with constraints
  - Diagonal Hodge-star for 0-forms
- Power-DEC
  - Better numerics
  - Discrete/Smooth counterparts



Thank you!