A Geometric Structure-Preserving Discretization Scheme for Incompressible Linearized Elasticity

Arash Yavari
School of Civil and Environmental Engineering &
The George W. Woodruff School of Mechanical Engineering
Georgia Institute of Technology

In collaboration with
Arzhang Angoshtari
School of Civil and Environmental Engineering
Georgia Institute of Technology

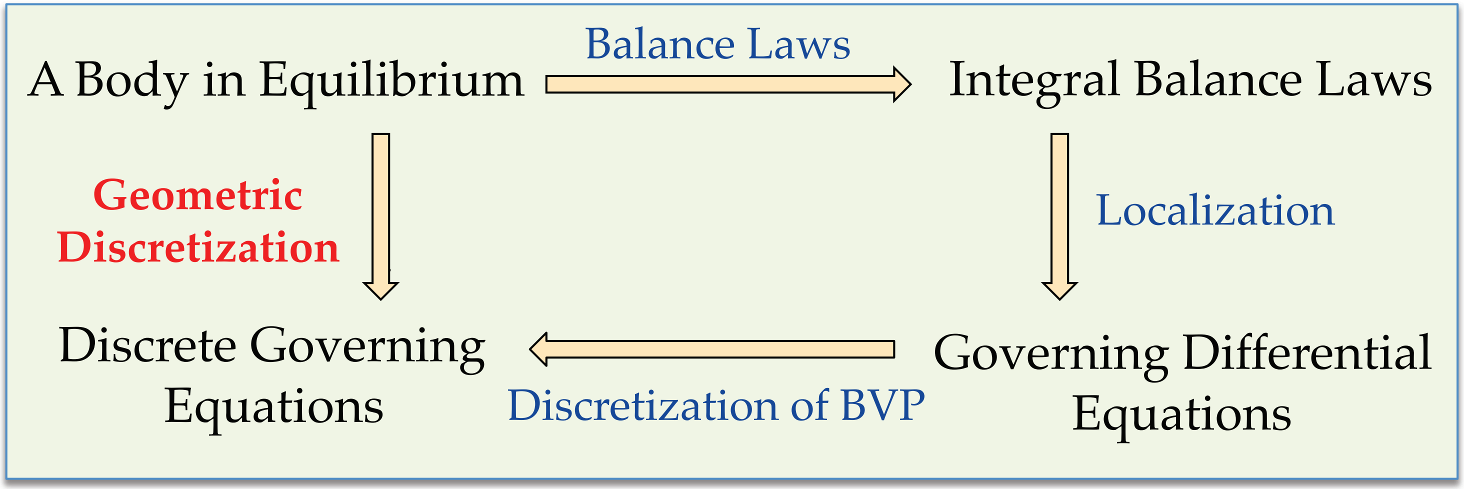
NSF Workshop on Barycentric Coordinates, Columbia University, July 27, 2012

Outline

- Motivation for Geometric Discretization of Elasticity
- Numerical Electromagnetism
- Discrete Exterior Calculus (DEC)
- Discrete Nonlinear Elasticity
- Example: Incompressible Elasticity
- Incompressible Linearized Elasticity: Continuous Case
- Discrete Configuration Manifold for Incompressible Linearized Elasticity
- Discrete Euler-Lagrange Equations
- Discrete Pressure Field
- Numerical Examples
- Conclusions and Future Directions

Structure-Preserving Discretization of Elasticity

- **Discretization of Elasticity**

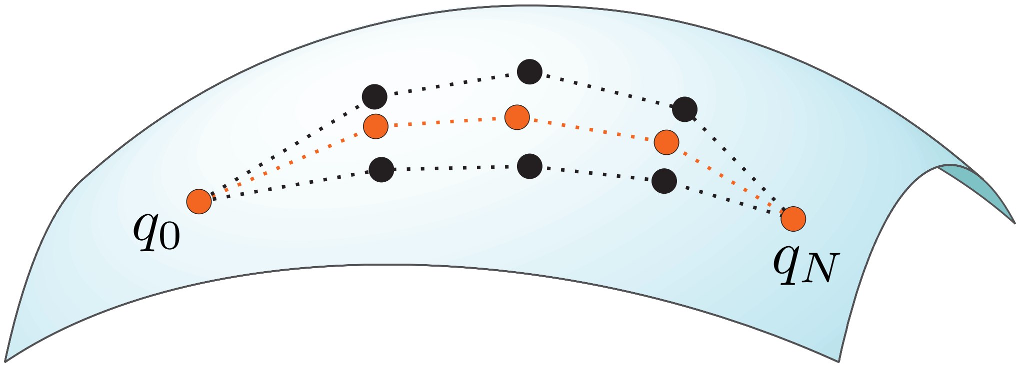


- **Goals:**
  - Rationalizing discretization schemes and putting all the existing numerical methods for solid mechanics in one abstract setting
  - **Avoiding numerical artifacts**, e.g. dissipation (for conservative systems), locking, pressure checkerboarding, etc.
  - Discretization when material manifold has a nontrivial geometry, e.g. distributed dislocations, growing bodies, etc.

Motivation: Variational Integrators

- **Discretization of mechanics** based on a **discretization of Hamilton's principle**: Moser and Veselov (1991), Veselov (1991), Marsden, et al. (1998)
- Discrete configuration space Q discrete Lagrangian $L_d : Q \times Q \to R$
- Action sum $S_d : Q^{N+1} \to R$

$$S_d = \sum_{k=0}^{N-1} L_d(q_k, q_{k+1})$$



- Hamilton's principle:

$$\delta S_d = 0$$

for fixed q_0 and q_N

- **Discrete Euler-Lagrange equations**

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = 0 \qquad \forall k = 1, ..., N-1$$

Maxwell's Equations and Numerical Electromagnetism

- Maxwell's Equations in the language of **differential forms**

$$\begin{cases} \nabla \times \mathbf{H} = \dfrac{\partial \mathbf{D}}{\partial t} + J_E, \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}, \\ \nabla \cdot \mathbf{B} = 0, \\ \nabla \cdot \mathbf{D} = \rho_E, \end{cases} \Rightarrow \begin{cases} \mathbf{d}\mathsf{H} = \dfrac{\partial \mathsf{D}}{\partial t} + J_E, \\ \mathbf{d}\mathsf{E} = -\dfrac{\partial \mathsf{B}}{\partial t}, \\ \mathbf{d}\mathsf{B} = 0, \\ \mathbf{d}\mathsf{D} = \rho_E \end{cases} \quad \textbf{Topological}$$

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} \\ \mathbf{B} = \mu_0 \mathbf{H} \end{cases} \Rightarrow \begin{cases} \mathsf{D} = *_E\ \mathsf{E} \\ \mathsf{B} = *_H\ \mathsf{H} \end{cases} \quad \textbf{Metric-dependent}$$

- $\mathbf{E}, \mathbf{D}, \mathbf{H}, \mathbf{B}$: Electric field, electric displacement, magnetic field, magnetic induction
- $J_E, \rho_R, \epsilon_0, \mu_0$: Current density, charge density, electric permittivity, and magnetic permeability
- E, H: **1-forms**, and D, B: **2-forms**
- $\mathbf{d}$: exterior derivative
- $*_E\ , *_H$: Hodge star operators

Geometric Elasticity:
Kanso, Arroyo, Tong, AY, Marsden, Desbrun, 2007

- **Elasticity** governing equations can be written in terms of **bundle-valued** differential forms.

$$\left\{\begin{array}{l} \dfrac{\partial \rho_0}{\partial t} = 0 \\ \mathrm{Div}\,\mathbf{P} + \rho_0 \mathbf{B} = \rho_0 \mathbf{A} \\ 2\rho_0 \dfrac{\partial E}{\partial \mathbf{g} \circ \varphi} = \boldsymbol{\tau} \\ \boldsymbol{\tau}^{\mathsf{T}} = \boldsymbol{\tau} \end{array}\right. \Longleftrightarrow \left\{\begin{array}{l} \dfrac{\partial \mu_0}{\partial t} = 0 \\ \mathfrak{D}\boldsymbol{\mathcal{P}} + \mathbf{B} \otimes \mu_0 = \dot{\mathbf{V}}^{\flat} \otimes \mu_0 \\ \varphi^* \alpha \wedge \langle \beta^{\sharp}, \boldsymbol{\mathcal{P}} \rangle = 2 \dfrac{\partial E}{\partial \mathbf{g}} (\alpha, \beta)\, \mu_0 \quad \forall\ \alpha, \beta \in \Omega^1(\mathcal{R}) \\ \varphi^* \alpha \wedge \langle \beta^{\sharp}, \boldsymbol{\mathcal{P}} \rangle = \varphi^* \beta \wedge \langle \alpha^{\sharp}, \boldsymbol{\mathcal{P}} \rangle \quad \forall\ \alpha, \beta \in \Omega^1(\mathcal{R}) \end{array}\right.$$

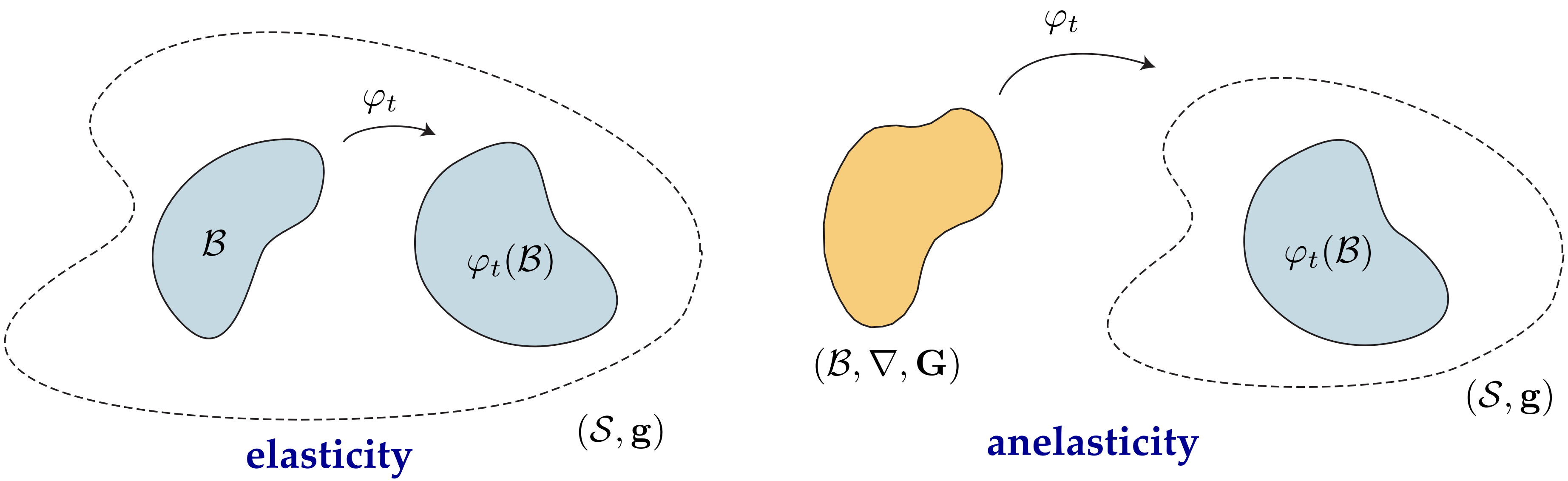
- The governing equations unlike EM **cannot be directly discretized.**
- Given a discretized body, how can one write the governing equations with no reference to continuum elasticity?

$$\frac{d}{dt} \int_{\mathcal{U}} \rho_0 \left(E + \frac{1}{2} \langle\!\langle \mathbf{V}, \mathbf{V} \rangle\!\rangle \right) dV = \int_{\mathcal{U}} \rho_0 \left(\langle\!\langle \mathbf{B}, \mathbf{V} \rangle\!\rangle + R \right) V + \int_{\partial \mathcal{U}} \left(\langle\!\langle \mathbf{T}, \mathbf{V} \rangle\!\rangle + H \right) dA$$

$$\delta \int_{t_0}^{t_1} \int_{\mathcal{B}} \mathcal{L} \left(\mathbf{X}, t, \mathbf{G}, \varphi, \dot{\varphi}, \mathbf{F}, \mathbf{g} \circ \varphi \right) dV\, dt + \int_{t_0}^{t_1} \int_{\mathcal{B}} \mathbf{F} \cdot \delta\varphi \; dV\, dt = 0$$

Geometric Elasticity & Anelasticity

- **Configuration space:** Infinite-dimensional manifold of maps between reference configuration and ambient space.
- Ambient space (a Riemannian manifold)

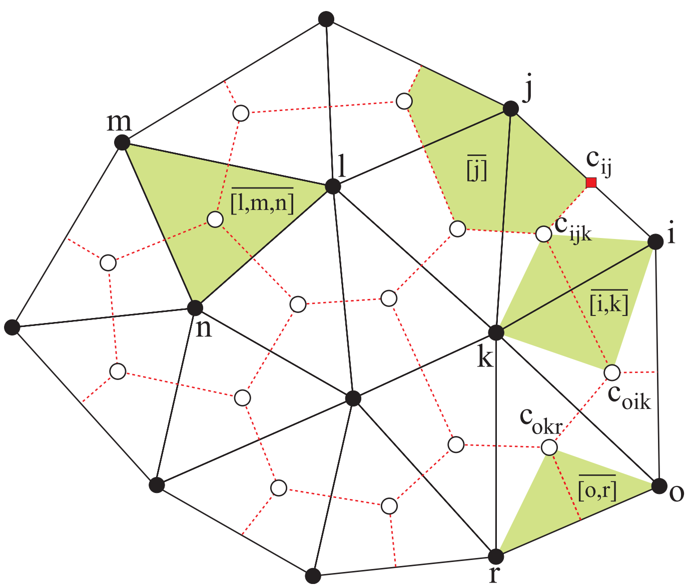


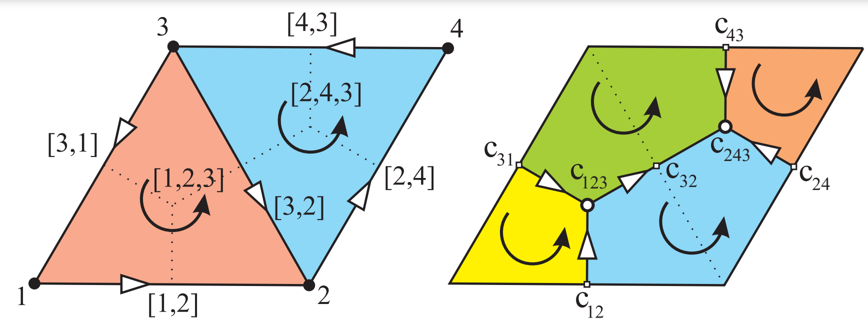
- Spatial metric **g** is a background metric and is not dynamic (unlike the metric in general relativity, which is governed by Einstein's equations).
- Deformation gradient (a two-point tensor)

$$\mathbf{F}(\mathbf{X}) : T_{\mathbf{X}}\mathcal{B} \to T_{\varphi(\mathbf{X})}\mathcal{S} \qquad \text{or} \qquad F^a{}_A(\mathbf{X}) = \frac{\partial \varphi^a}{\partial X^A}(\mathbf{X})$$

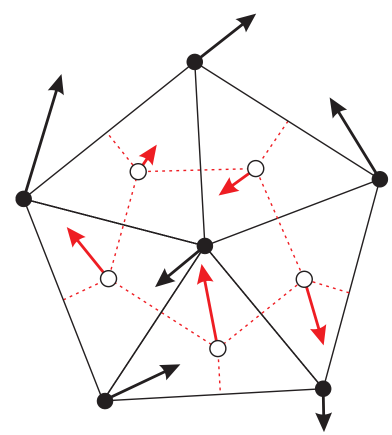
Discrete Exterior Calculus (DEC): Hirani, et al. (2005)

- **Primal and dual complexes**

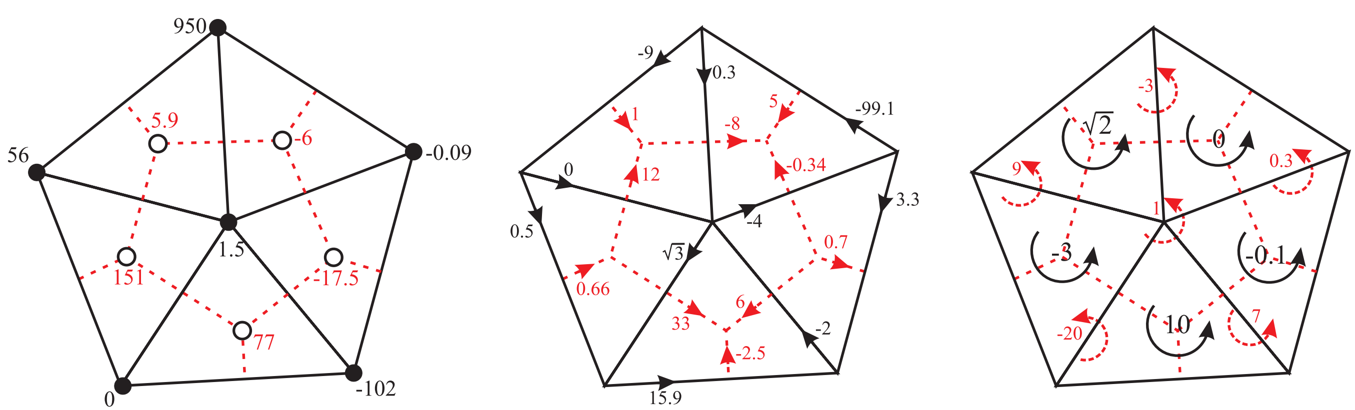




Oriented primal and dual complexes



Primal and dual vector fields



Primal and dual 0, 1 and 2- forms

Discrete Exterior Calculus (DEC)

- Continuous Hodge star $* : \Lambda^k(\mathcal{N}) \to \Lambda^{n-k}(\mathcal{N})$

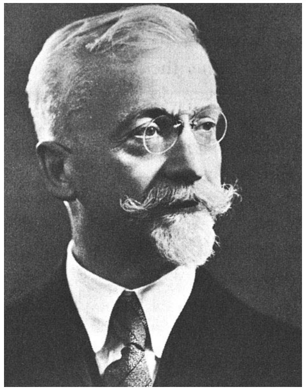
$$\boldsymbol{\alpha} \wedge *\boldsymbol{\beta} = \langle \boldsymbol{\alpha}, \boldsymbol{\beta}^\sharp \rangle \boldsymbol{\mu}$$

- Discrete Hodge star $* : \Omega_d^k(K) \to \Omega_d^{n-k}(\star K)$

$$\frac{1}{|\star\sigma^k|}\langle *\boldsymbol{\alpha}, \star\sigma^k \rangle = \frac{1}{|\sigma^k|}\langle \boldsymbol{\alpha}, \sigma^k \rangle$$

- Discrete flat operator $\flat : \mathfrak{X}_d(K) \to \Omega_d^1(\star K)$

$$\langle \mathbf{X}^\flat, \star\sigma^{n-1} \rangle = \sum_{\sigma^0 \prec \sigma^{n-1}} \mathbf{X}(\sigma^0) \cdot (\star\boldsymbol{\sigma}^{n-1})$$



Élie Cartan (1869-1951)

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

$$\int_M d\omega = \int_{\partial M} \omega$$

- Boundary $\partial_k : C_k(K) \to C_{k-1}(K)$ and coboundary operators

$$\delta^k : C^k(K) \to C^{k+1}(K) \qquad \langle \delta^k c^k, c_{k+1} \rangle = \langle c^k, \partial_{k+1} c_{k+1} \rangle$$

- Discrete exterior derivative

$$\mathbf{d} : \Omega_d^k(K) \to \Omega_d^{k+1}(K) \qquad \mathbf{d}^{k+1} \circ \mathbf{d}^k = 0$$

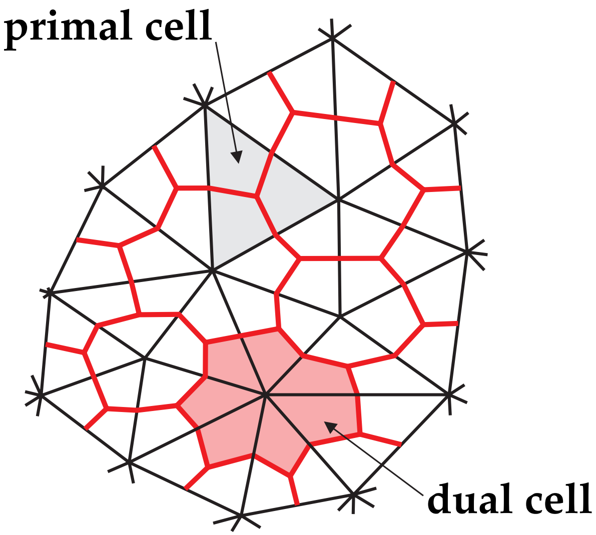
$$\left\{ \begin{array}{l} \mathrm{curl} \circ \mathrm{grad} = 0 \\ \mathrm{div} \circ \mathrm{curl} = 0 \end{array} \right.$$

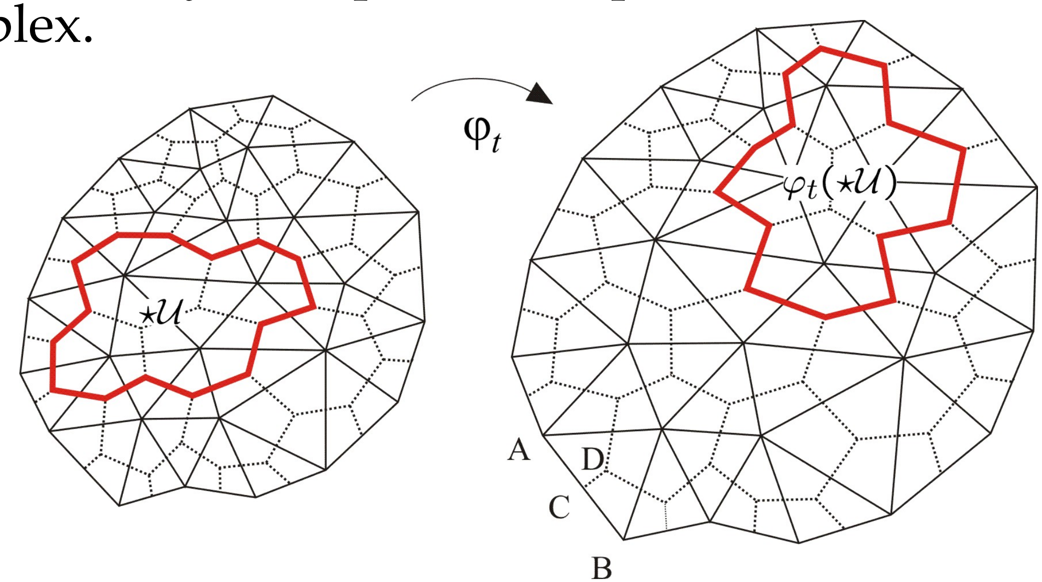
- Discrete divergence for a primal vector field

$$\langle \mathrm{div}\mathbf{X}, \star\sigma^n \rangle = *\,\mathbf{d} * \mathbf{X}^\flat \longrightarrow \text{a dual form}$$

Discrete Nonlinear Elasticity: AY, 2008

- A discretized solid is modeled by a simplicial complex. Then define a dual complex.

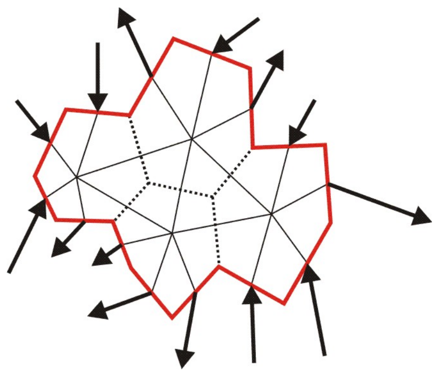


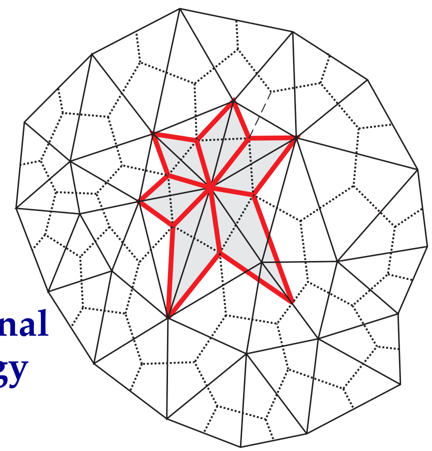


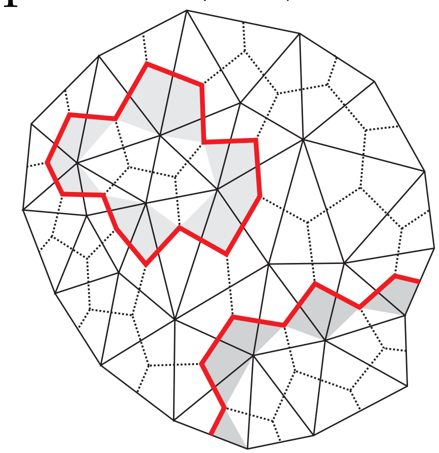
- Discrete kinematic and kinetic quantities live on different objects: discrete (vector-valued and co-vector valued) differential forms
- **Discrete deformation map:** $\varphi_t : K \longrightarrow \varphi_t(K)$
  $$\sigma_i^0(t) = \varphi_t(\sigma_i^0) \quad \forall\ \sigma_i \in K^{(0)}$$
- Discrete velocity field: $\mathbf{V}_i(t) := \langle \mathbf{V}, \sigma_i^0 \rangle = \dot{\varphi}_t(\sigma_i^0) \quad \forall\ \sigma_i^0 \in K^{(0)}$
- **Discrete strain:** A primal discrete vector-valued 1-form

Discrete Nonlinear Elasticity

- Stress is a pseudo covector-valued (n-2)-form.
- **Discrete stress:** A dual covector-valued discrete pseudo (n-2)-form







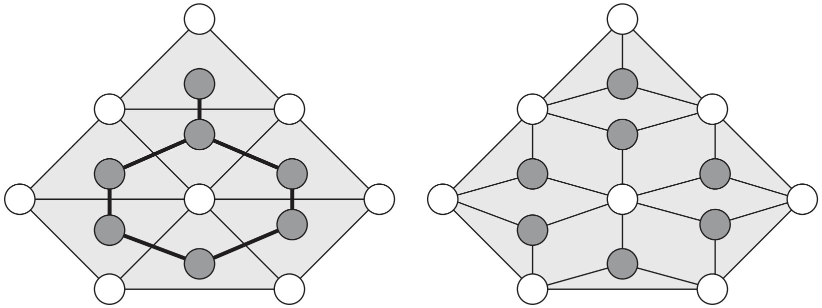
- Governing equations
  - *Energy balance and its invariance*
  - *Action principle*

Quantity	Symbol	Type
velocity	$\mathbf{v}$	vector-valued 0-form
displacement	$\mathbf{u}$	vector-valued 0-form
strain	$\mathbb{F}$	vector-valued 1-form
mass density	ρ	dual p-form
internal energy density	e	support volume-form
specific entropy	N	support volume-form
heat flux	h	dual $(p-1)$-form
heat supply	r	dual p-form
stress	$\mathbb{t}$	covector-valued $(p-1)$-form
body force	$\mathbb{b}$	covector-valued dual p-form
kinetic energy density	κ	dual p-form

Henri Poincaré
(1854-1912)
Algebraic topology
Triangulation of smooth manifolds
Independence of homology groups from triangulations

Example: Incompressible Elasticity

- The numerical solution for incompressible elasticity is usually obtained by solving near incompressible problems, i.e., solving compressible problem as the parameters tend to those of incompressible problem.
- Locking can occur in this process (Babuška and Suri,1992), i.e., loss of accuracy of solutions as the parameter(s) tend to a critical value, e.g., for linear isotropic materials $\nu \rightarrow \frac{1}{2}$.
- Mixed Methods (Arnold, 2005): Use an extension of de Rham's complex for linearized elasticity.
- Diamond Elements (Hauret, et al., 2007): Analysis for linearized elasticity. A heuristic partitioning of a simplicial complex using a dual complex.



Good convergence for incompressible nonlinear elasticity in some numerical tests.

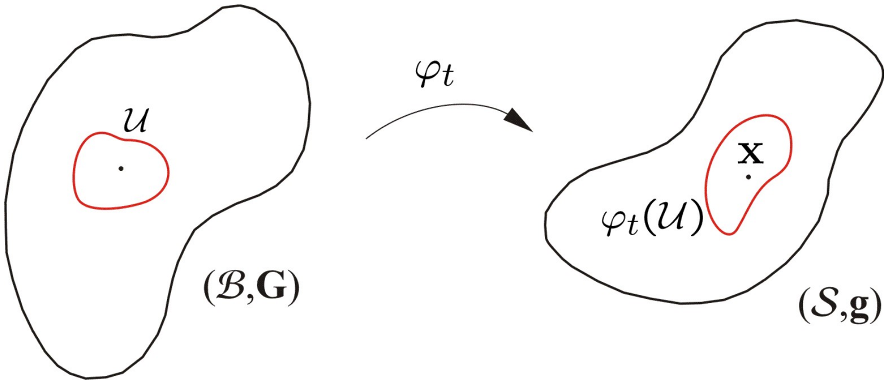
- Pavlov, et al. (2011): Geometric structure-preserving discretization for incompressible fluids

Incompressible Linearized Elasticity: Continuous Case

- **Idea:** Instead of using Lagrange multipliers work with the proper configuration manifold.

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F} = 1$$

Incompressible elasticity



Nonlinear Elasticity

$$\mathcal{C} = \{\psi : \mathcal{B} \to \mathcal{S} \mid \psi = \varphi_d \text{ on } \partial_d \mathcal{B}\}$$

$$T\mathcal{C} = \{(\psi, \mathbf{U}) \,|\psi \in \mathcal{C},\ \mathbf{U} : \mathcal{B} \to T\psi(\mathcal{B}) \text{ and } \mathbf{U}|_{\partial_d \mathcal{B}} = 0\}$$

Incompressible Nonlinear Elasticity

$$\mathcal{C}_{vol} = \{\psi \in \mathcal{C} \mid J(\psi) = 1\}$$

$$T_\psi \mathcal{C}_{vol} = \left\{\mathbf{U} \in T_\psi \mathcal{C} \mid \operatorname{div}\left(\mathbf{U} \circ \psi^{-1}\right) = 0\right\}$$

Space of variations

Ebin and Marsden (1970)

Incompressible Linearized Elasticity: Continuous Case

- Lagrangian density $L = K - V$ and **space of variations**

$$\mathfrak{U} = \{\mathbf{w} : \mathcal{B} \to \mathbb{R}^n | \operatorname{div} \mathbf{w} = 0, \mathbf{w}|_{\partial_d \mathcal{B}} = 0\}$$

$$K = \frac{1}{2}\int_{\mathcal{B}} \rho(\dot{\mathbf{u}}, \dot{\mathbf{u}})_g dv \quad V = \int_{\mathcal{B}} \mu e^{ab} e_{ab} dv - \int_{\mathcal{B}} \rho(\mathbf{b}, \mathbf{u})_g dv - \int_{\partial_\tau \mathcal{B}} (\tau, \mathbf{u})_g da$$

- **Action principle**

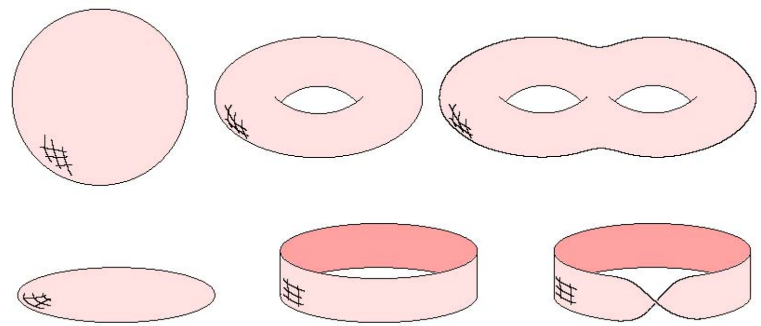
$$\delta \int_0^T L dt = 0 \Rightarrow \int_{\mathcal{B}} \left[\rho(\ddot{\mathbf{u}} - \mathbf{b}, \mathbf{w})_g + 2\mu e^{ab} \tilde{e}_{ab}\right] dv - \int_{\partial_\tau \mathcal{B}} (\tau, \mathbf{w})_g da = 0$$

- Or

$$\int_{\mathcal{B}} (\rho \ddot{\mathbf{u}} - \rho \mathbf{b} - \operatorname{div}(2\mu \mathbf{e}^\sharp), \mathbf{w})_g dv + \int_{\partial \mathcal{B}} (\langle 2\mu \mathbf{e}^\sharp, \mathbf{n}^\flat \rangle - \boldsymbol{\tau}, \mathbf{w})_g da = 0$$

Incompressible Linearized Elasticity: Continuous Case

- Let ξ be a vector field on $\mathcal{B}$. If for every $\mathbf{w} \in \mathfrak{U}$ we have $\int_{\mathcal{B}} (\xi, \mathbf{w})_g dv = 0$, then there exists a function $p : \mathcal{B} \to \mathbb{R}$ such that $\xi = -\mathrm{div}(p\mathbf{g}^\sharp)$.



- Inner product on k-form

$$(\boldsymbol{\alpha}, \boldsymbol{\beta})_g = \int_{\mathcal{M}} \boldsymbol{\alpha} \wedge (*\boldsymbol{\beta}) = \int_{\mathcal{M}} \langle \boldsymbol{\alpha}, \boldsymbol{\beta}^\sharp \rangle dv$$

- **Hodge decomposition theorem** for manifolds with boundary

$$\Omega^k(\mathcal{M}) = \mathbf{d}\big(\Omega^{k-1}(\mathcal{M})\big) \oplus \mathfrak{D}_t^k(\mathcal{M})$$

$$\mathbf{d}\big(\Omega^{k-1}(\mathcal{M})\big) = \left\{\boldsymbol{\alpha} \in \Omega^k(\mathcal{M}) \mid \exists \boldsymbol{\beta} \in \Omega^{k-1}(\mathcal{M}) \text{ such that } \boldsymbol{\alpha} = \mathbf{d}\boldsymbol{\beta}\right\}$$

$$\mathfrak{D}_t^k(\mathcal{M}) = \left\{\boldsymbol{\alpha} \in \Omega_t^k(\mathcal{M}) \mid \boldsymbol{\delta}\boldsymbol{\alpha} = 0\right\}$$

$$\Omega_t^k(\mathcal{M}) = \left\{\boldsymbol{\alpha} \in \Omega^k(\mathcal{M}) \mid \boldsymbol{\alpha} \text{ is tangent to } \partial\mathcal{M}\right\}$$

Incompressible Linearized Elasticity: Continuous Case

- **Hamilton's principle** gives us

$$\int_{\mathcal{B}} (\rho\ddot{\mathbf{u}} - \rho\mathbf{b} - \mathrm{div}(2\mu\mathbf{e}^\sharp), \mathbf{w})_g dv + \int_{\partial\mathcal{B}} (\langle 2\mu\mathbf{e}^\sharp, \mathbf{n}^\flat\rangle - \boldsymbol{\tau}, \mathbf{w})_g da = 0$$

Hodge ⇩ decomposition ⟶ A generalization of Helmholtz decomposition $\mathbf{v} = \mathrm{grad}\,\phi + \mathrm{curl}\,\mathbf{A}$

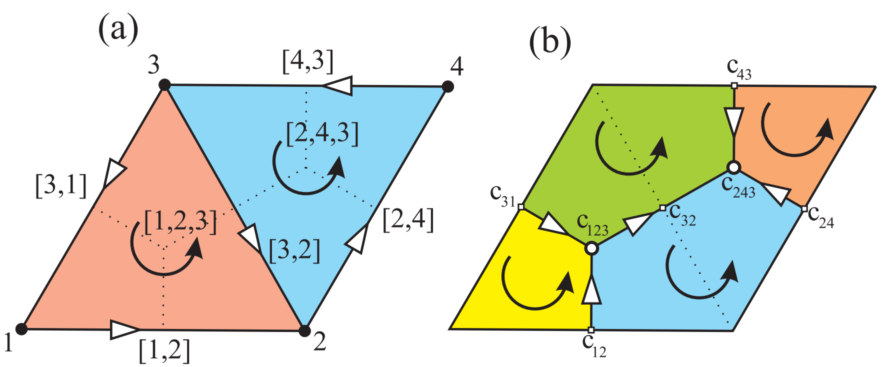
$$\begin{aligned} \rho\ddot{\mathbf{u}} &= \rho\mathbf{b} + \mathrm{div}(2\mu\mathbf{e}^\sharp - p\mathbf{g}^\sharp), && \text{in } \mathcal{B} \\ \tau &= \langle 2\mu\mathbf{e}^\sharp - p\mathbf{g}^\sharp, \mathbf{n}^\flat\rangle && \text{on } \partial_\tau\mathcal{B} \end{aligned}$$

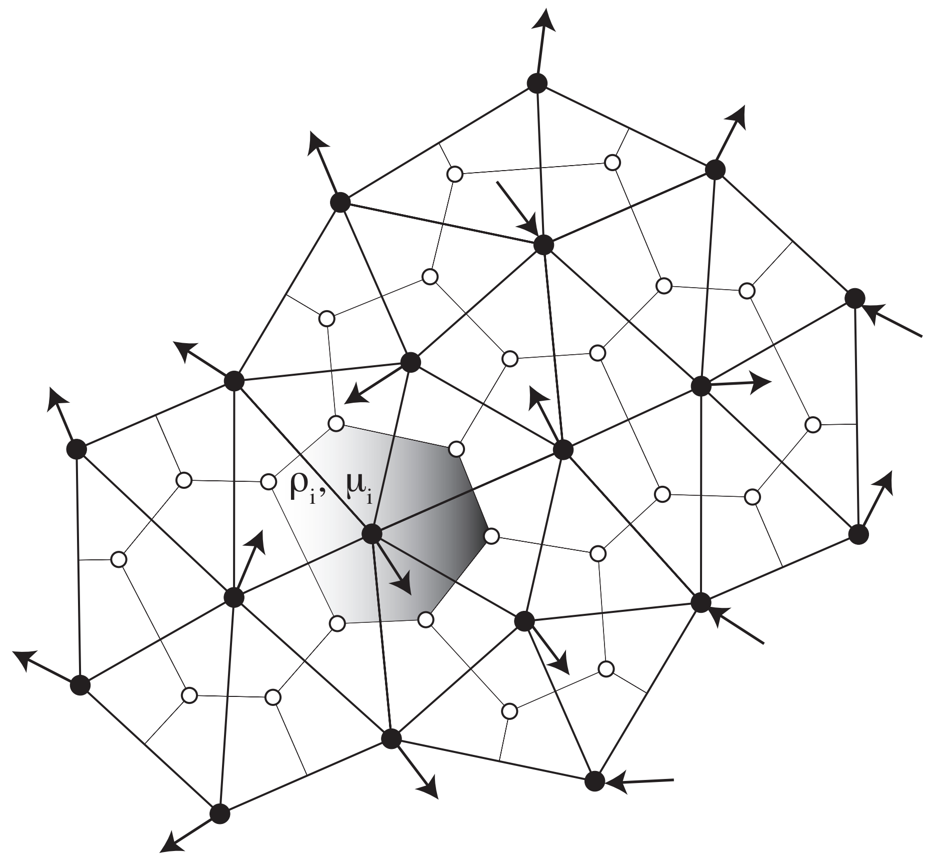
- **Remark:** The solution space for incompressible fluids is similar to that of incompressible linearized elasticity. Variation of the velocity field has a non-standard form (Lin constraint).

Discrete Incompressible Linearized Elasticity

- **Primary unknowns**: displacement field

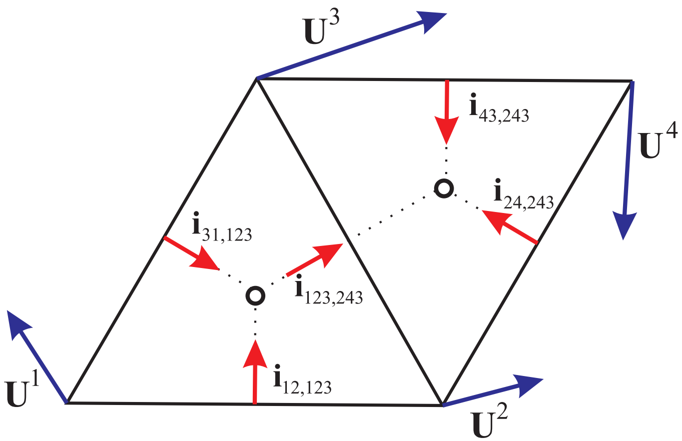
$$\boldsymbol{X}_{n\bar{\mathsf{P}}_h \times 1} = \left\{ \begin{array}{c} \mathbf{U}^{1}_{n\times 1} \\ \vdots \\ \mathbf{U}^{\bar{\mathsf{P}}_h}_{n\times 1} \end{array} \right\}$$





oriented primal complex → oriented dual complex

Discrete Configuration Manifold for Incompressible Linearized Elasticity



$$\langle \mathrm{div}\mathbf{U}, \star[1,2,3]\rangle = \frac{1}{|[1,2,3]|}\sum_{i=1}^{4}\mathbf{c}_{1i}\cdot\mathbf{U}^i$$

$$\langle \mathrm{div}\mathbf{U}, \star[2,4,3]\rangle = \frac{1}{|[2,4,3]|}\sum_{i=1}^{4}\mathbf{c}_{2i}\cdot\mathbf{U}^i$$

$$\begin{aligned}
\mathbf{c}_{11} &= |[3,1]|\mathbf{i}_{31,123} + |[1,2]|\mathbf{i}_{12,123}, & \mathbf{c}_{12} &= |[1,2]|\mathbf{i}_{12,123} - |[3,2]|\mathbf{i}_{123,243}\\
\mathbf{c}_{13} &= |[3,1]|\mathbf{i}_{31,123} - |[3,2]|\mathbf{i}_{123,243}, & \mathbf{c}_{14} &= \mathbf{0}\\
\mathbf{c}_{21} &= \mathbf{0}, & \mathbf{c}_{22} &= |[2,4]|\mathbf{i}_{24,243} + |[3,2]|\mathbf{i}_{123,243}\\
\mathbf{c}_{23} &= |[4,3]|\mathbf{i}_{43,243} + |[3,2]|\mathbf{i}_{123,243}, & \mathbf{c}_{24} &= |[2,4]|\mathbf{i}_{24,243} + |[4,3]|\mathbf{i}_{43,243}
\end{aligned}$$

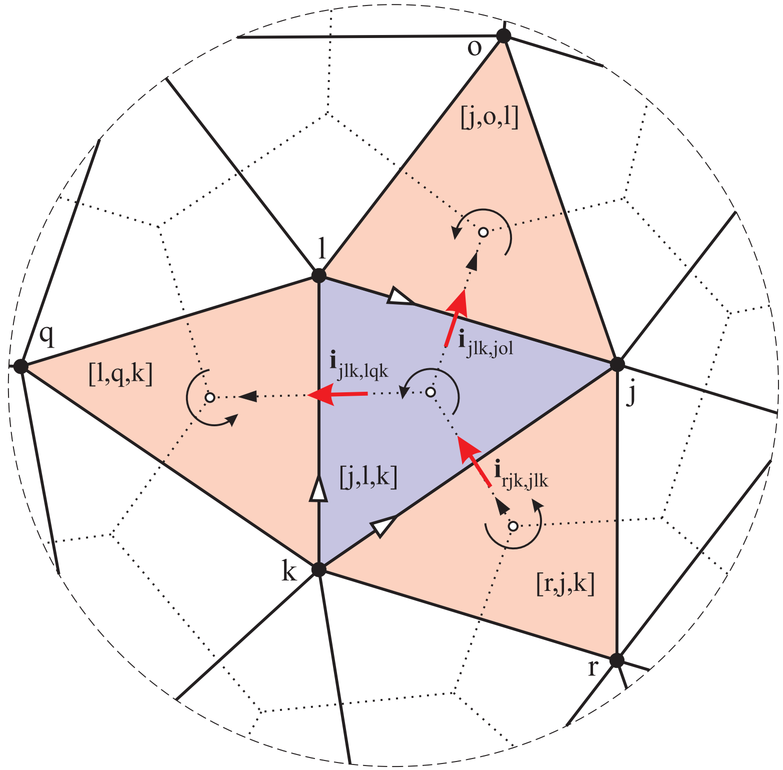
$$\mathbb{I}_{2\times 8}\boldsymbol{\Upsilon}_{8\times 1} = \mathbf{0} \qquad \mathbb{I}_{2\times 8} = \begin{bmatrix} \mathbf{c}_{11}^{\mathsf{T}} & \mathbf{c}_{12}^{\mathsf{T}} & \mathbf{c}_{13}^{\mathsf{T}} & \mathbf{c}_{14}^{\mathsf{T}} \\ \mathbf{c}_{21}^{\mathsf{T}} & \mathbf{c}_{22}^{\mathsf{T}} & \mathbf{c}_{23}^{\mathsf{T}} & \mathbf{c}_{24}^{\mathsf{T}} \end{bmatrix} \qquad \boldsymbol{\Upsilon}_{8\times 1} = \left\{\mathbf{U}^1 \ldots \mathbf{U}^4\right\}^{\mathsf{T}}$$

Discrete Configuration Manifold for Incompressible Linearized Elasticity

- **Incompressibility**

$$\bar{\mathbb{I}}^h_{\mathsf{D}_h \times (2\mathsf{P}_h)} \bar{\boldsymbol{X}}_{(2\mathsf{P}_h)\times 1} = \mathbf{0}$$

- The case n=2:



$$\bar{\mathbb{I}}^h = \begin{bmatrix} \mathbf{c}_{11}^{\mathsf{T}} & \cdots & \mathbf{c}_{1\mathsf{P}_h}^{\mathsf{T}} \\ \vdots & \ddots & \vdots \\ \mathbf{c}_{\mathsf{D}_h 1}^{\mathsf{T}} & \cdots & \mathbf{c}_{\mathsf{D}_h \mathsf{P}_h}^{\mathsf{T}} \end{bmatrix}_{\mathsf{D}_h \times (2\mathsf{P}_h)}$$

$$\mathbf{c}_{ij} = |[k,j]|\mathbf{i}_{rjk,jlk} - |[l,j]|\mathbf{i}_{jlk,jol}$$

$$\mathbf{c}_{il} = -|[l,j]|\mathbf{i}_{jlk,jol} - |[k,l]|\mathbf{i}_{jlk,lqk}$$

$$\mathbf{c}_{ik} = -|[k,l]|\mathbf{i}_{jlk,lqk} + |[k,j]|\mathbf{i}_{rjk,jlk}$$

Discrete Configuration Manifold for Incompressible Linearized Elasticity

$$\mathrm{div}\mathbf{U} = 0 \sim \bar{\mathbb{I}}^h_{\mathsf{D}_h \times (2\mathsf{P}_h)} \bar{\mathbf{X}}_{(2\mathsf{P}_h)\times 1} = \mathbf{0}$$

$$\begin{aligned} \chi(|K_h|) &= \#(0\text{-simplices}) - \#(1\text{-simplices}) + \#(2\text{-simplices}) \\ &= \mathsf{P}_h - \mathsf{E}_h + \mathsf{D}_h = 1 \end{aligned}$$

$$3\mathsf{D}_h = \mathsf{E}^b_h + 2\mathsf{E}^i_h, \quad \mathsf{P}^b_h = \mathsf{E}^b_h$$

$$\mathsf{D}_h = 2\mathsf{P}^i_h + \mathsf{P}^b_h - 2 = 2\mathsf{P}_h - \mathsf{P}^b_h - 2 \implies \mathsf{D}_h < 2\mathsf{P}_h$$

- **Theorem:** Let K_h be a 2-dimensional well-centered primal mesh such that $|K_h|$ is a simply-connected set. Then the associated incompressibility matrix $\bar{\mathbb{I}}^h$ is full-ranked.

$$\mathrm{rank}(\bar{\mathbb{I}}^h) = \mathsf{D}_h \implies \mathrm{nullity}(\bar{\mathbb{I}}^h) = 2\mathsf{P}_h - \mathrm{rank}(\bar{\mathbb{I}}^h) = 2\mathsf{P}_h - \mathsf{D}_h$$

$\{\bar{\mathbf{w}}_i \in \mathbb{R}^{2\mathsf{P}_h}\}_{i=1}^{2\mathsf{P}_h - \mathsf{D}_h}$ a basis for the null space

Discrete Kinetic Energy for Incompressible Linearized Elasticity

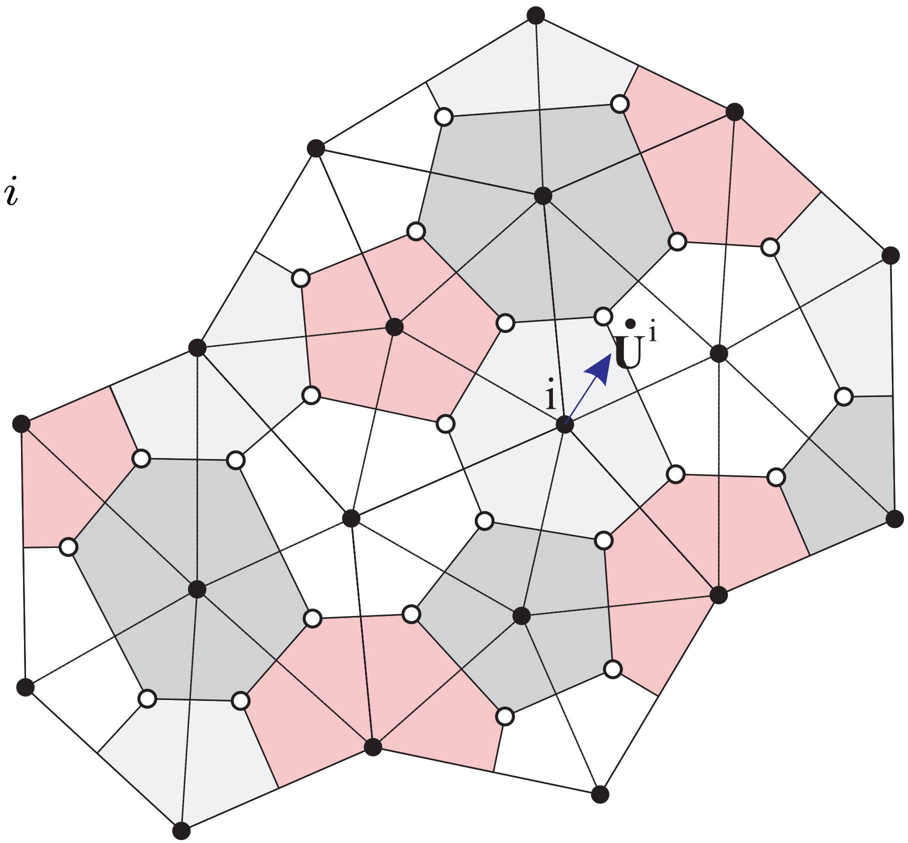
- **Kinetic energy**

$$K^d = \frac{1}{2}\sum_{i=1}^{\bar{\mathsf{P}}_h} \rho_i |\star\sigma_i^0|\ \dot{\mathbf{U}}^i \cdot \dot{\mathbf{U}}^i$$

$$K^d = \frac{1}{2}\dot{\boldsymbol{X}}^{\mathsf{T}}\mathbf{M}\dot{\boldsymbol{X}}$$

- $\mathbf{M} \in \mathbb{R}^{(n\bar{\mathsf{P}}_h)\times(n\bar{\mathsf{P}}_h)}$

  a diagonal matrix



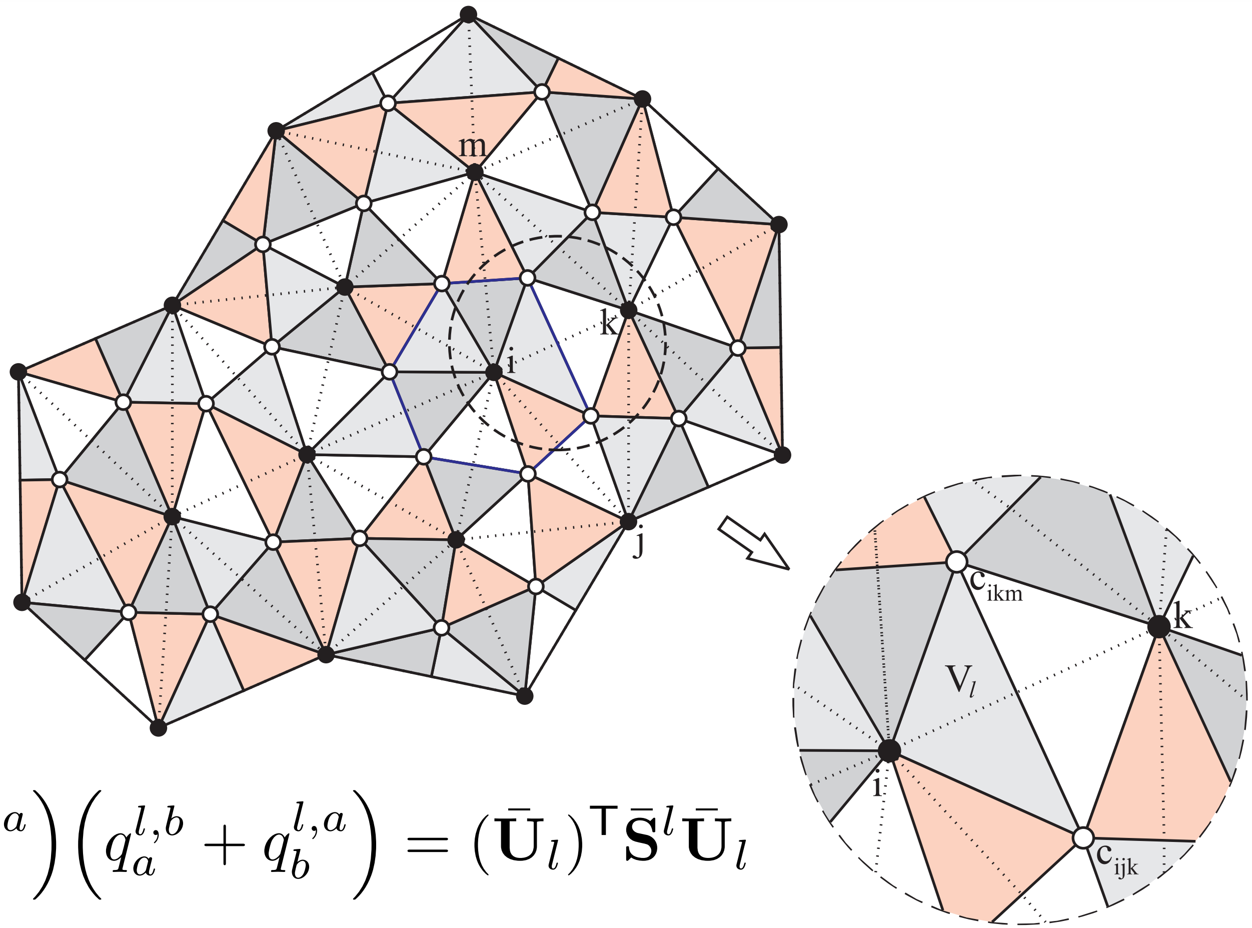
$$\mathbf{M}_{jk} = \begin{cases} \rho_i|\star\sigma_i^0|, & \text{if } j = k = n(i-1)+s \text{ with } 0 \le s < n,\ 1 \le i \le \bar{\mathsf{P}}_h, \\ 0, & \text{if } j \neq k. \end{cases}$$

Discrete Elastic Energy for Incompressible Linearized Elasticity

$$E^d = \sum_l^{2\mathsf{E}_h} \mathcal{E}^l$$

$$\mathcal{E}^l = \int_{V_l} \mu e^{ab} e_{ab} dv$$

$$V_l = \left| \overline{[i,k]} \bigcap (\star \sigma_i^0) \right|$$



$$\mathcal{E}^l = \frac{1}{4} \mu_l V_l \left(q_a^{l,b} + q_b^{l,a} \right) \left(q_a^{l,b} + q_b^{l,a} \right) = (\bar{\mathbf{U}}_l)^\mathsf{T} \bar{\mathbf{S}}^l \bar{\mathbf{U}}_l$$

$$\bar{\mathbf{U}}_l = \{\mathbf{U}^i, \mathbf{U}^j, \mathbf{U}^k, \mathbf{U}^m\}^\mathsf{T} \in \mathbb{R}^8$$

$$E^d = \frac{1}{2} \mathbf{X}^\mathsf{T} \mathbf{S} \mathbf{X}$$

Discrete Action for Incompressible Linearized Elasticity

- **Discrete Lagrangian** $L^d = K^d - V^d$, where $V^d = E^d - B^d - T^d$

$$B^d = \sum_{i=1}^{\mathsf{P}_h} m^i \mathbf{B}^i \cdot \mathbf{U}^i = \mathbf{b} \cdot \boldsymbol{X} + B_e^d$$

$$\begin{cases} m^i = \rho_i |\star \sigma_i^0| \\ \mathbf{b} = \left\{ \mathbf{b}^1, \ldots, \mathbf{b}^{\bar{\mathsf{P}}_h} \right\}^{\mathsf{T}} \\ \mathbf{b}^i = m^i \mathbf{B}^i \\ B_e^d = \displaystyle\sum_{i=\bar{\mathsf{P}}_h+1}^{\mathsf{P}_h} m^i \mathbf{B}^i \cdot \mathbf{U}^i \end{cases}$$

$$\begin{cases} T^d = \mathbf{t} \cdot \boldsymbol{X} \\ \mathbf{t} = \{ \mathbf{t}^1, \ldots, \mathbf{t}^{\bar{\mathsf{P}}_h} \} \end{cases}$$

$\{\sigma_i^0\}_{i=1}^{\bar{\mathsf{P}}_h} \longrightarrow$ Primal vertices without essential BC

$\{\sigma_i^0\}_{i=\bar{\mathsf{P}}_h+1}^{\mathsf{P}_h} \longrightarrow$ Primal vertices with essential BC

Discrete Action for Incompressible Linearized Elasticity

- **Discrete Lagrangian**

$$L^d = \frac{1}{2}\dot{\boldsymbol{X}}^\top \mathbf{M}\dot{\boldsymbol{X}} - \boldsymbol{X}^\top \mathbf{S}\boldsymbol{X} + \mathbf{F}\cdot\boldsymbol{X} + L_e^d$$

$$\mathbf{F} = -\mathbf{s} + \mathbf{b} + \mathbf{t}, \quad L_e^d = K_e^d - E_e^d + B_e^d$$

- Variation field $\mathbf{X}_\epsilon$

$$\boldsymbol{X}_0 = \boldsymbol{X}, \qquad \frac{d}{d\epsilon}\Big|_{\epsilon=0}\boldsymbol{X}_\epsilon = \delta\boldsymbol{X}$$

- **Incompressibility**

$$\mathbb{I}^h\boldsymbol{X}_\epsilon = \mathbf{u}^h \implies \mathbb{I}^h\,\delta\boldsymbol{X} = \mathbf{0} \implies \delta\boldsymbol{X} \in \mathsf{Ker}(\mathbb{I}^h)$$

Discrete Action for Incompressible Linearized Elasticity

- **Hamilton's principle**

$$\delta \int_{t_1}^{t_2} L^d dt = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \int_{t_1}^{t_2} \left(\frac{1}{2} \dot{\boldsymbol{X}}_\epsilon^\mathsf{T} \mathbf{M} \dot{\boldsymbol{X}}_\epsilon - \boldsymbol{X}_\epsilon^\mathsf{T} \mathbf{S} \boldsymbol{X}_\epsilon + \mathbf{F} \cdot \boldsymbol{X}_\epsilon + L_e^d \right) dt$$

$$= \int_{t_1}^{t_2} \left[\dot{\boldsymbol{X}}^\mathsf{T} \mathbf{M} \left(\frac{d}{dt} \delta \boldsymbol{X} \right) - \boldsymbol{X}^\mathsf{T} (\mathbf{S} + \mathbf{S}^\mathsf{T}) \delta \boldsymbol{X} + \mathbf{F} \cdot \delta \boldsymbol{X} \right] dt$$

$$= -\int_{t_1}^{t_2} \left[\mathbf{M} \ddot{\boldsymbol{X}} + (\mathbf{S} + \mathbf{S}^\mathsf{T}) \boldsymbol{X} - \mathbf{F} \right] \cdot \delta \boldsymbol{X} dt = 0$$

$$\left[\mathbf{M} \ddot{\boldsymbol{X}} + (\mathbf{S} + \mathbf{S}^\mathsf{T}) \boldsymbol{X} - \mathbf{F} \right] \cdot \delta \boldsymbol{X} = 0$$

- Orthogonal decomposition of the space of displacements

$$\mathbb{R}^{n\bar{\mathsf{P}}_h} = \mathsf{Ker}(\mathbb{I}^h) \oplus \mathsf{Ker}(\mathbb{I}^h)^\perp$$

$$\delta \boldsymbol{X} \in \mathsf{Ker}(\mathbb{I}^h) \Longrightarrow \mathbf{M} \ddot{\boldsymbol{X}} + (\mathbf{S} + \mathbf{S}^\mathsf{T}) \boldsymbol{X} - \mathbf{F} = \boldsymbol{\Lambda}$$

$$\boldsymbol{\Lambda} = \left\{ \boldsymbol{\Lambda}^1, \ldots, \boldsymbol{\Lambda}^{\bar{\mathsf{P}}_h} \right\}^\mathsf{T}$$

Discrete pressure gradient

Discrete Action for Incompressible Linearized Elasticity

- From rank-nullity theorem

$$\dim(\mathsf{Ker}(\mathbb{I}^h)^\perp) = n\bar{\mathsf{P}}_h - \mathsf{nullity}(\mathbb{I}^h) = \mathsf{rank}(\mathbb{I}^h) = \mathsf{R}$$

$\{\mathbf{z}^1, \ldots, \mathbf{z}^{\mathsf{R}}\}$ a basis for $\mathsf{Ker}(\mathbb{I}^h)^\perp$

$$\boldsymbol{\Lambda}(t) = \sum_{i=1}^{\mathsf{R}} \Lambda_i(t)\,\mathbf{z}^i$$

- **Discrete Euler-Lagrange** equations

$$\begin{cases} \mathbf{M}\ddot{\boldsymbol{X}} + \left(\mathbf{S} + \mathbf{S}^{\mathsf{T}}\right)\boldsymbol{X} - \mathbf{F} = \sum_{i=1}^{\mathsf{R}} \Lambda_i \mathbf{z}^i \\ \mathbb{I}^h \boldsymbol{X} = \mathbf{u}^h \end{cases}$$

Discrete Pressure Field

- Cauchy stress $\sigma^{ab} = 2\mu\, e^{ab} - p\, g^{ab}$
- **Pressure gradient** $\nabla p = (\mathbf{d}p)^{\sharp}$
- If "flat" is dual to primal, then "sharp" has to be primal to dual for them to be inverse operations.
- Pressure gradient is a primal vector field. So, this means that $(\mathbf{d}p)^{\sharp}$ is primal. Therefore, $\mathbf{d}p$ is a dual 1-form. This means that p is a dual 0-form.
- Laplace-Beltrami operator

$$\Delta : \Omega^0(M) \to \Omega^0(M) \qquad \Delta f = *\mathbf{d} * \mathbf{d} f = *\mathbf{d} * \left[(\mathbf{d} f)^{\sharp} \right]^{\flat}$$

$$\Delta : \Omega^0_d(K) \to \Omega^0_d(K) \quad \text{or} \quad \Delta : \Omega^0_d(\star K) \to \Omega^0_d(\star K)$$

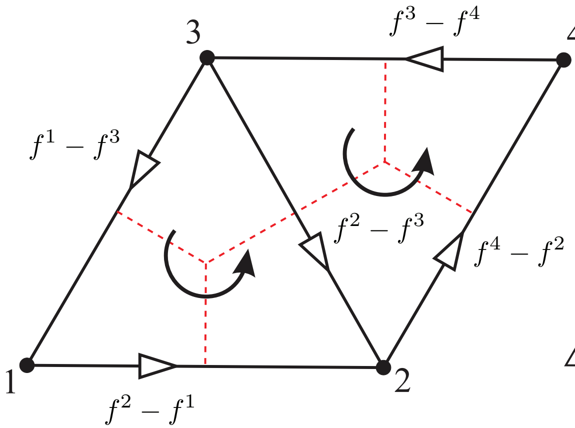
- The smooth Δ operator is not injective. The same is true for the primal discrete Δ operator. However, the dual discrete Δ operator is bijective.

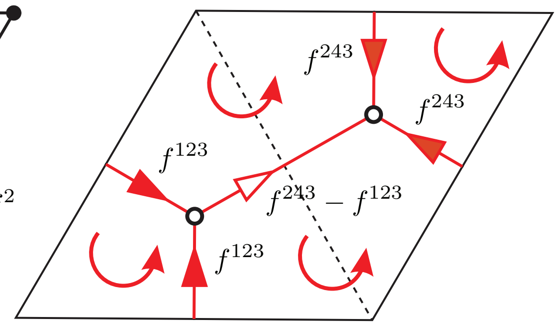
Discrete Pressure Field

- For any $a \in \mathbb{R}$, for both the smooth and primal discrete exterior derivative $\mathbf{d}(f + a) = \mathbf{d}f$
- But for the dual discrete exterior derivative $\mathbf{d}(f + a) \neq \mathbf{d}f$

$\{f^1, f^2, f^3, f^4\}$ primal

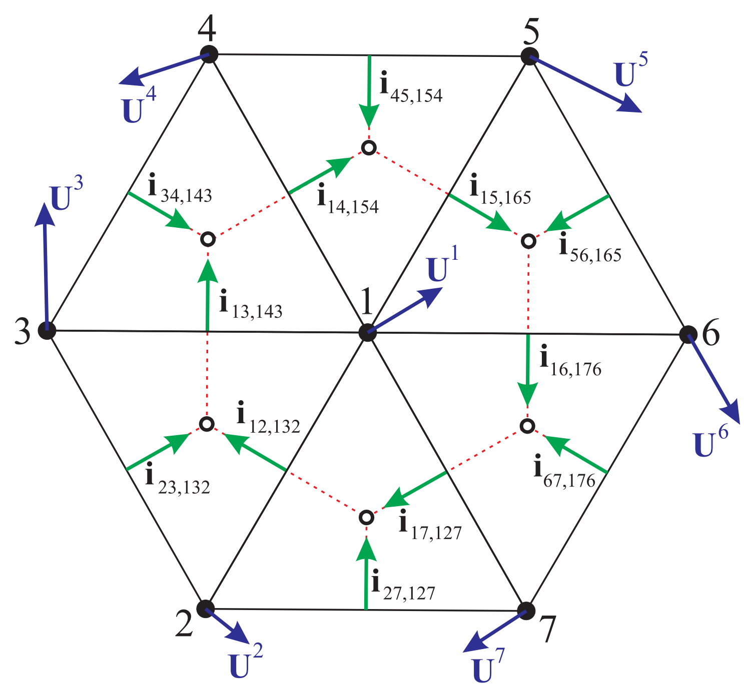
$\{f^{123}, f^{243}\}$ dual





- Example

$$\mathbf{p} = \{p^{132}, p^{143}, p^{154}, p^{165}, p^{176}, p^{127}\}^{\mathsf{T}}$$



Discrete Pressure Field

- Pressure Laplacian $\Delta \mathbf{p} = \mathbb{L}_{6\times 6}\, \mathbf{p}$ (★)
- Theorem. Let K_h be a planar well-centered primal mesh such that $|K_h|$ is a simply-connected set. Then, the matrix $\mathbb{L}^h \in \mathbb{R}^{D_h \times D_h}$ is non-singular.
- $\mathbf{\Lambda}$ is the pressure gradient $\nabla p = (\mathbf{d}p)^\sharp$ and hence

$$\Delta p = *\mathbf{d} * \mathbf{\Lambda}^\flat \qquad (\star\star)$$

- From (★) and (★★)

$$\mathbb{L}_{6\times 6}\, \mathbf{p} = \bar{\mathbb{I}}_{6\times 14} \mathbb{G}_p \longrightarrow$$

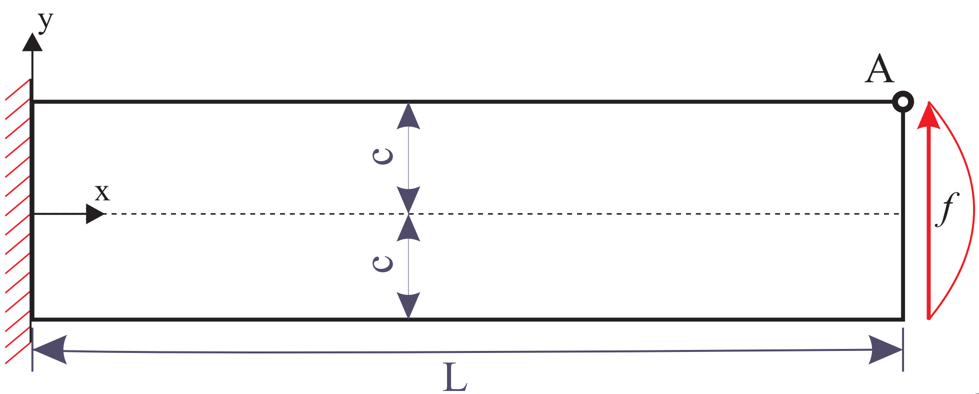
The same as $\mathbf{\Lambda}$ but with values at primal vertices with essential B.C.
(average value of the closest vertices)

$$\mathbb{L} = \begin{bmatrix} r_{1,2}+r_{1,3}+r_{2,3} & -r_{1,3} & 0 & 0 & 0 & -r_{1,2} \\ -r_{1,3} & r_{1,3}+r_{1,4}+r_{3,4} & -r_{1,4} & 0 & 0 & 0 \\ 0 & -r_{1,4} & r_{1,4}+r_{1,5}+r_{4,5} & -r_{1,5} & 0 & 0 \\ 0 & 0 & -r_{1,5} & r_{1,5}+r_{1,6}+r_{5,6} & -r_{1,6} & 0 \\ 0 & 0 & 0 & -r_{1,6} & r_{1,6}+r_{1,7}+r_{6,7} & -r_{1,7} \\ -r_{1,2} & 0 & 0 & 0 & -r_{1,7} & r_{1,7}+r_{1,2}+r_{2,7} \end{bmatrix}$$

$$r_{i,j} = \frac{|[i,j]|}{|\star[i,j]|}$$

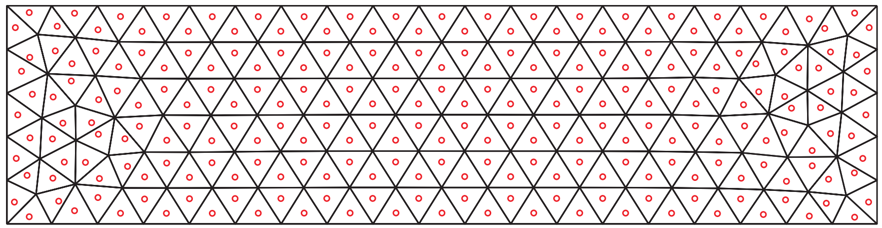
Example 1: Cantilever Beam

- **Cantilever beam** under a parabolic distributed shear force at its free end



$$f(y) = \frac{3F}{4c^3}(c^2 - y^2)$$

$$\operatorname{div}\mathbf{u} = \frac{1}{EI}(1+\nu)(1-2\nu)F(x-L)y$$



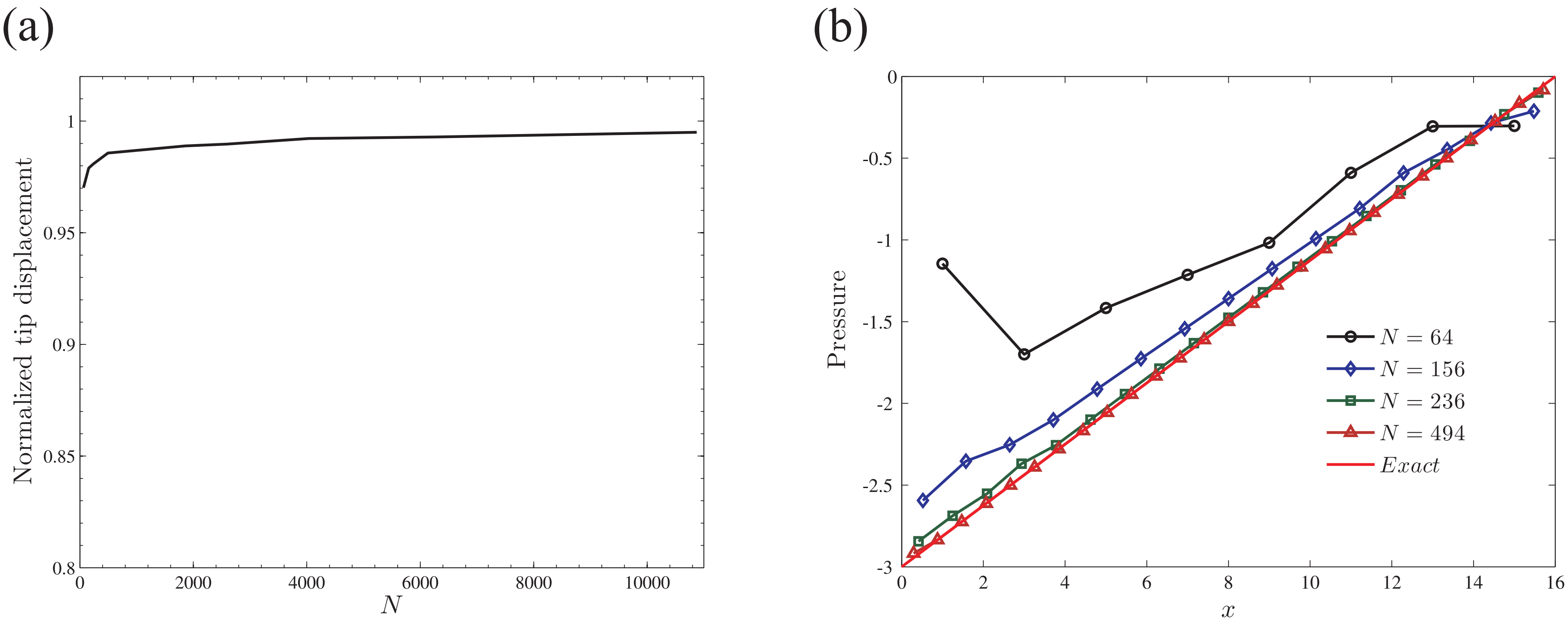
$$p(x,y) = -\frac{F(x-L)y}{2I}$$

$$u_x = \frac{(1-\nu^2)Fy}{6EI}\left(3x^2 - 6Lx + \frac{\nu\, y^2}{1-\nu}\right) - \frac{Fy}{6I\mu}\left(y^2 - 3c^2\right)$$

$$u_y = \frac{(1-\nu^2)F}{6EI}\left[\frac{3\nu(L-x)y^2}{1-\nu} + 3Lx^2 - x^3\right]$$

Example 1: Cantilever Beam

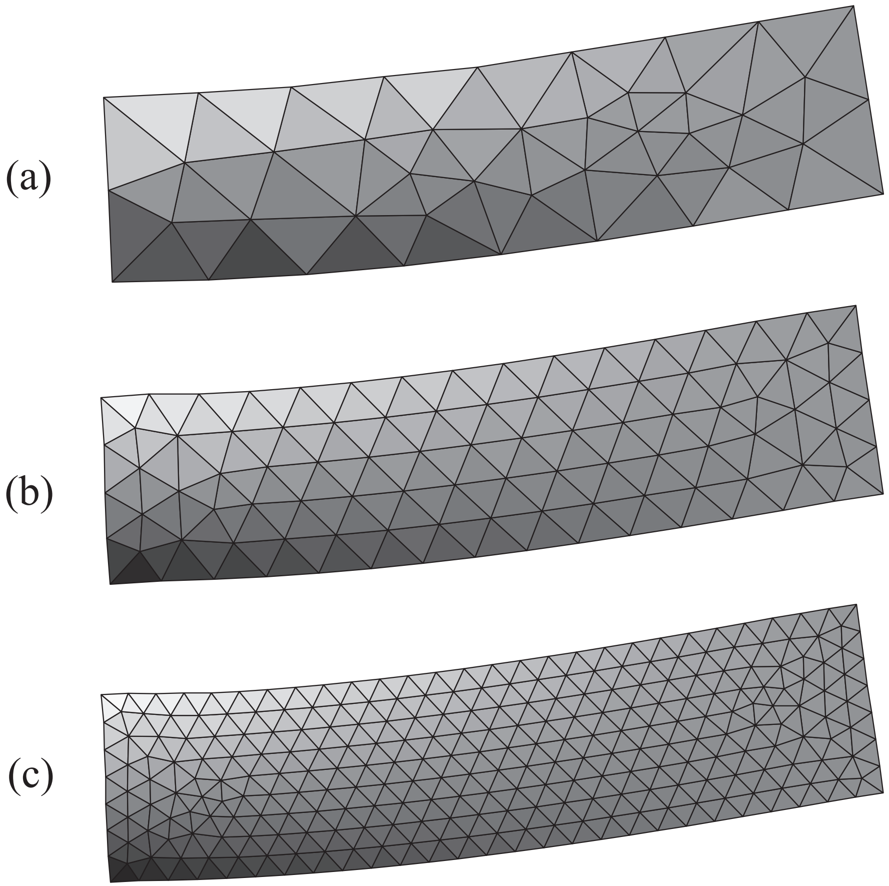
- **Tip displacement and pressure**



$$\text{Normalized diaplacement} = \frac{U_y^A}{u_y^A}$$

Example 1: Cantilever Beam

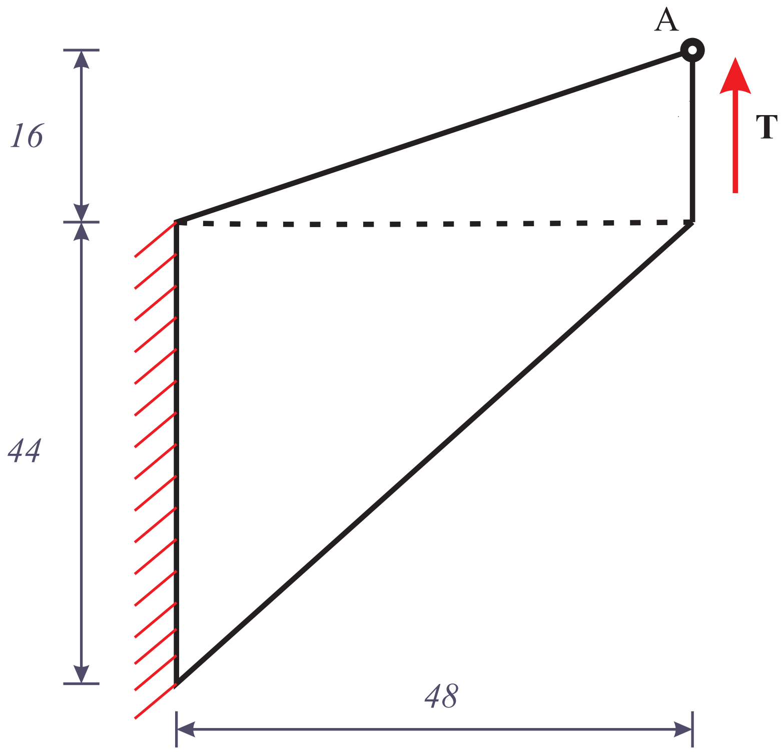
- **Pressure field** for the beam problem for meshes with (a) N=64, (b) N=156, (c) N=494, where N is the number of primal 2-cells of the mesh.

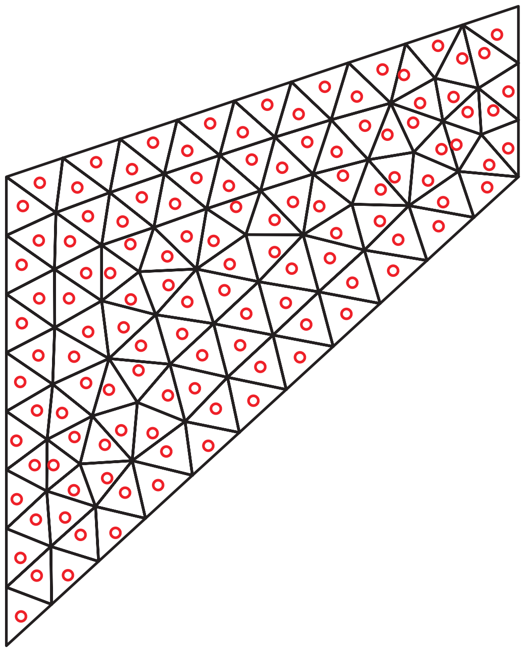


No pressure checkerboarding!

Example 2: Cook's Membrane

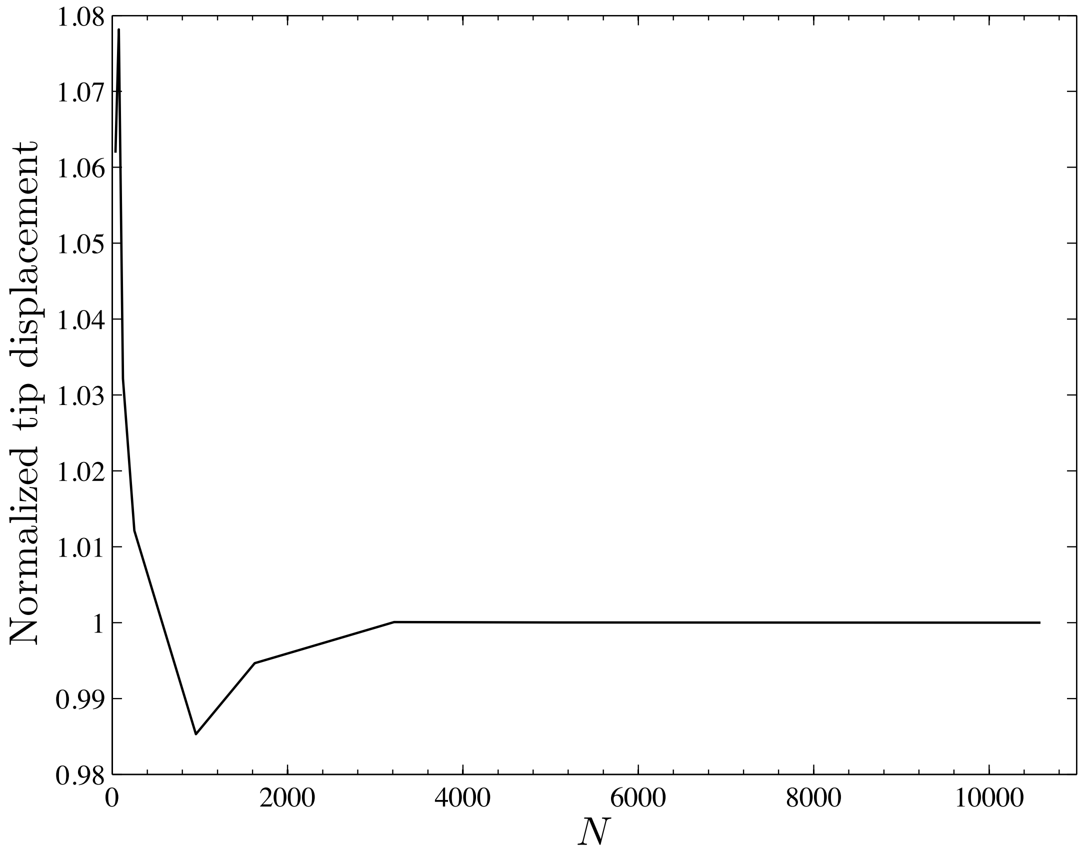
- **Boundary conditions and loading**

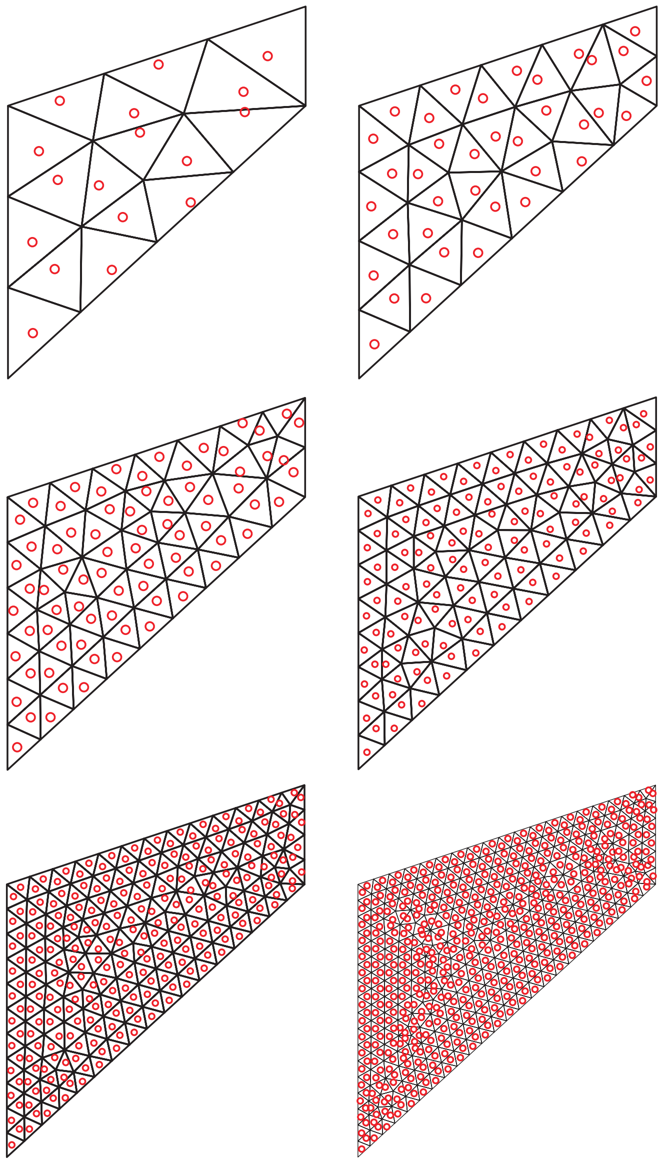




Example 2: Cook's Membrane

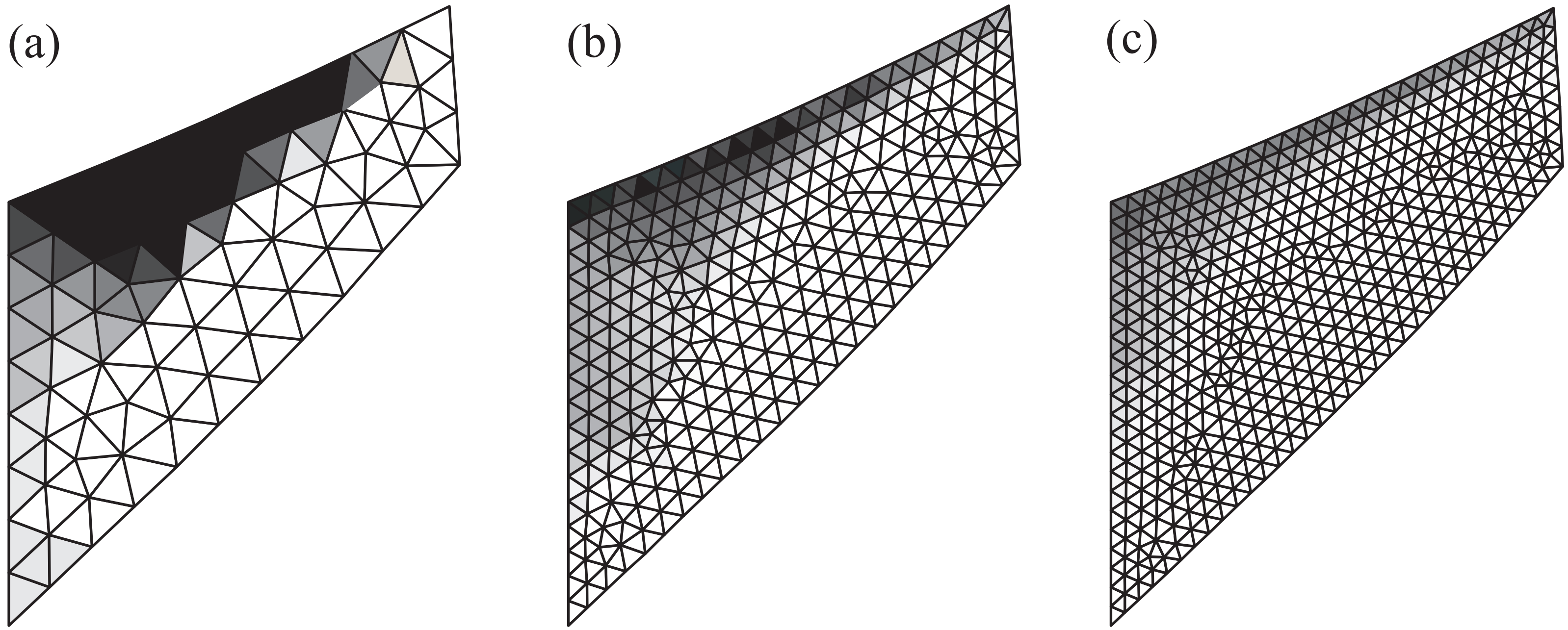
- Convergence of displacements





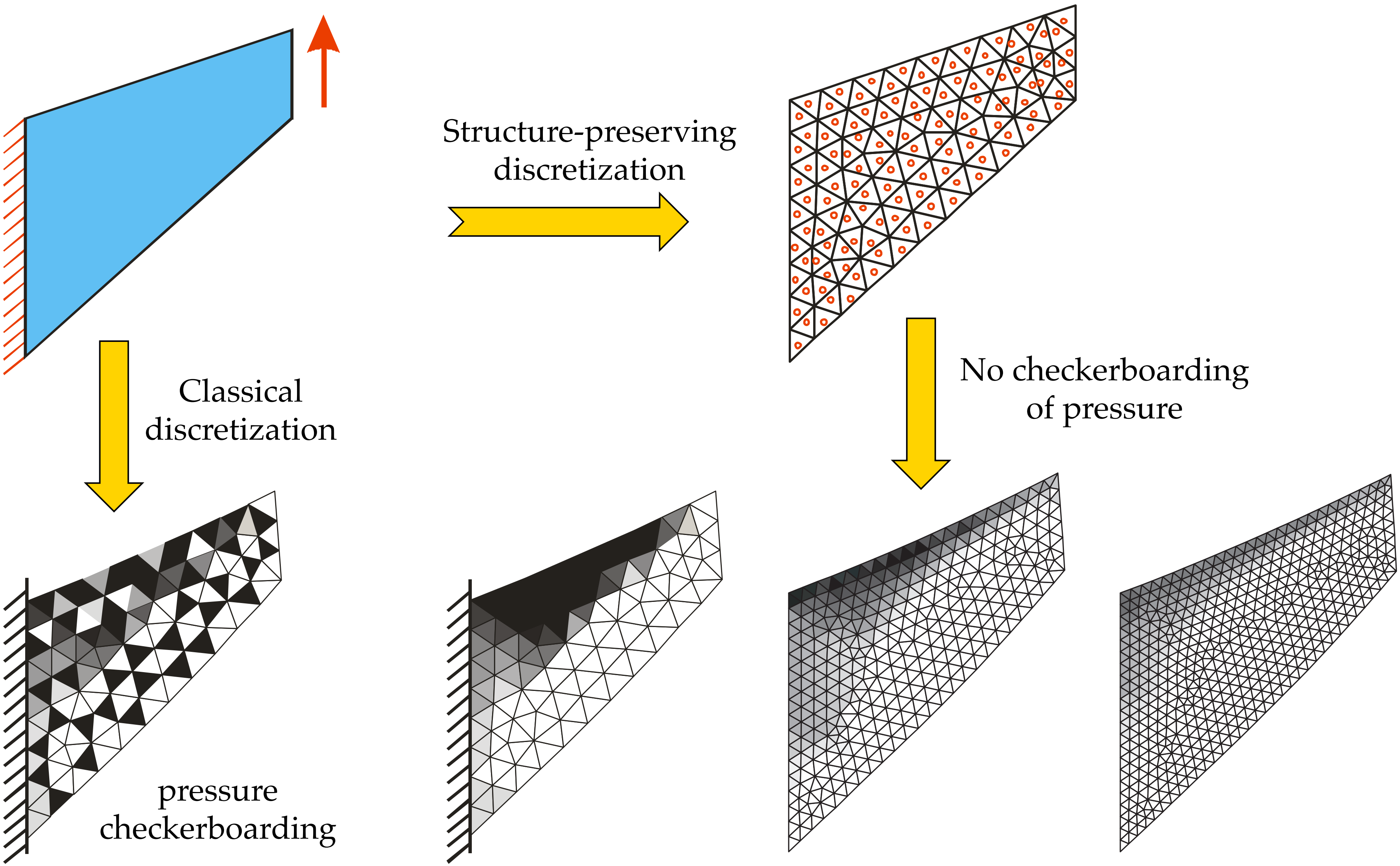
Example 2: Cook’s Membrane

- The pressure field for the Cook's membrane for meshes with (a) N=123, (b) N=530, (c) N=955, where N is the number of primal 2-cells of the mesh.



No pressure checkerboarding!

Example 2: Cook's Membrane



Current and Future Work

- Generalization to 3D problems
- Fluid mechanics (fixed mesh)
- Nonlinear elasticity: Requires deforming meshes. For formulation with circumcentric duals requires remeshing a deforming domain.
- Convergent analysis: What is the proper topology on cochains?
- Differential complex of nonlinear elasticity
- Discretization when the rest configuration is evolving?
- …