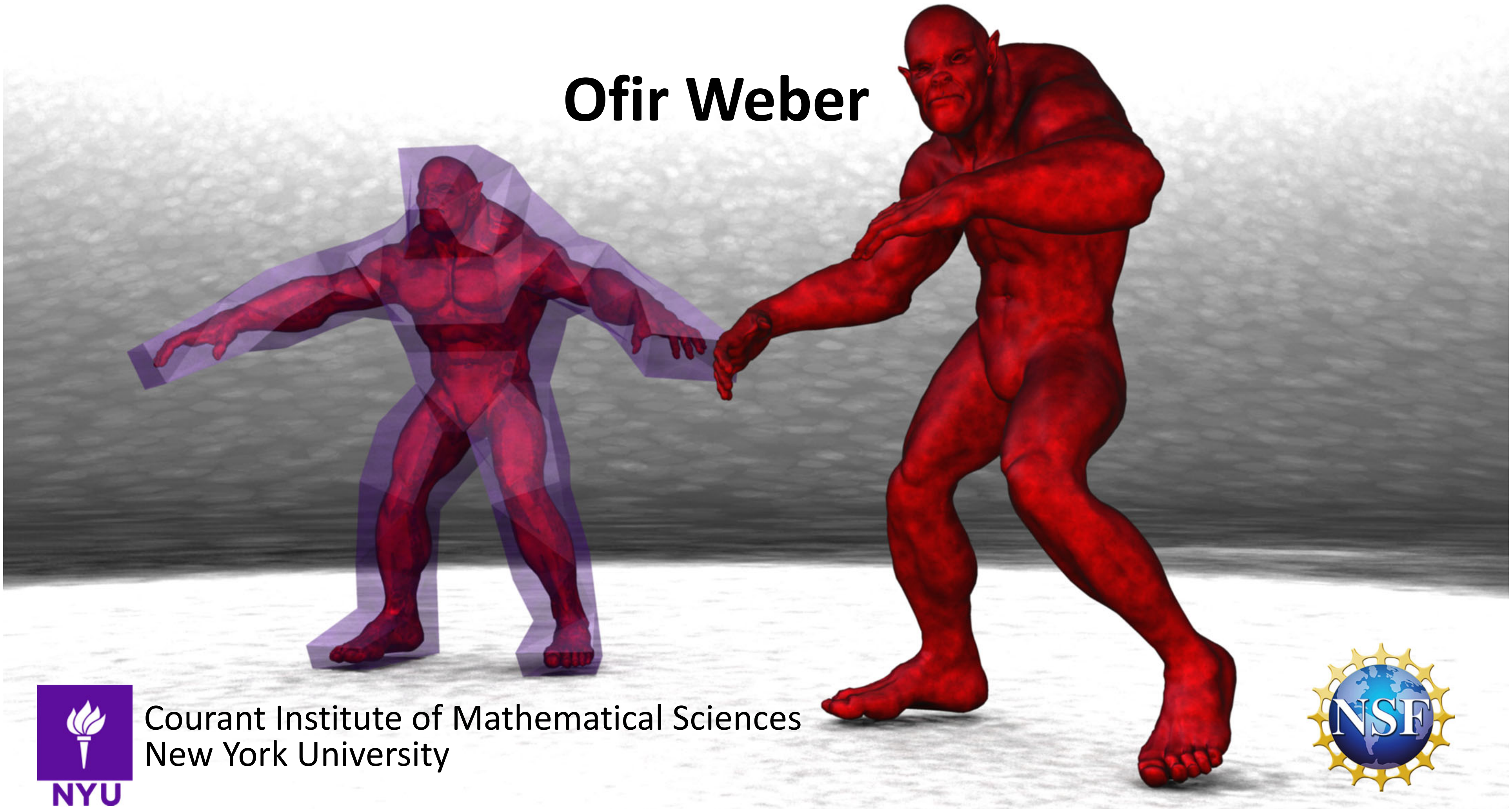
Barycentric Coordinates
as a Fundamental Tool for
Spatial Shape Deformation



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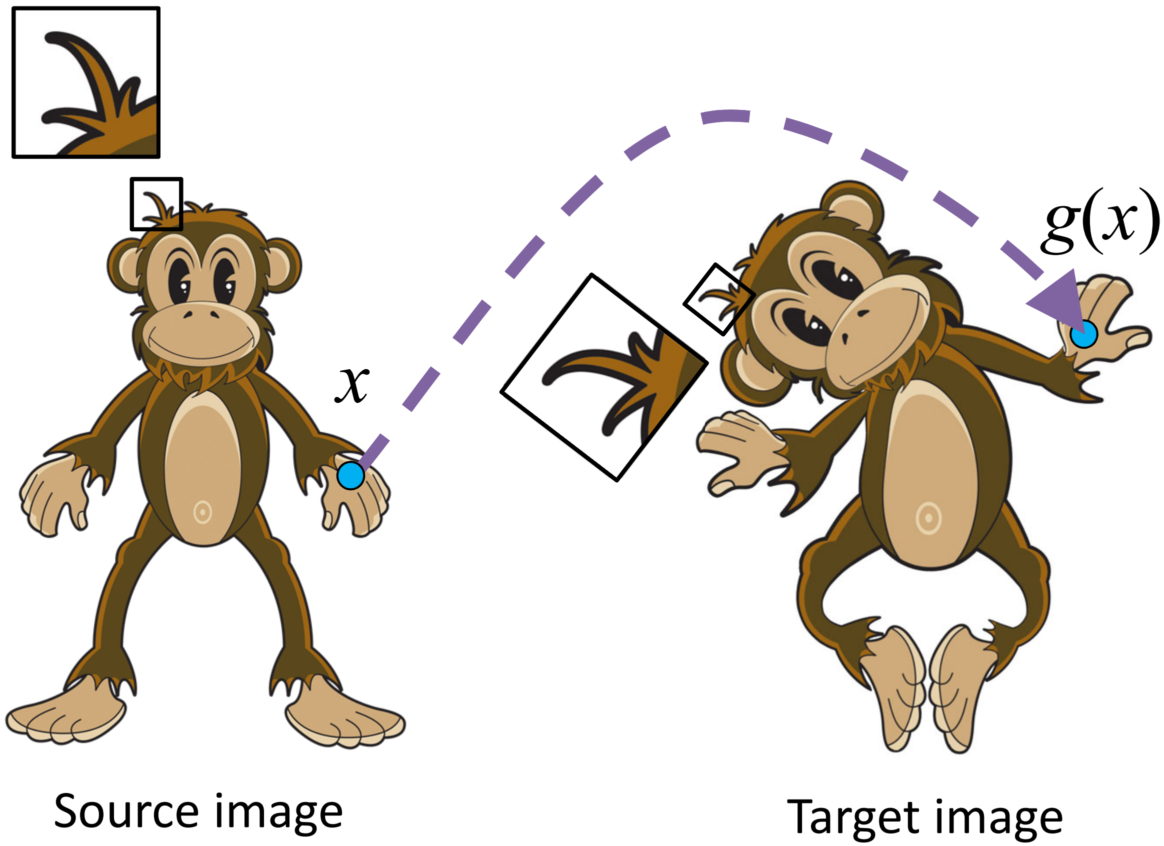
The Problem of Shape Deformation

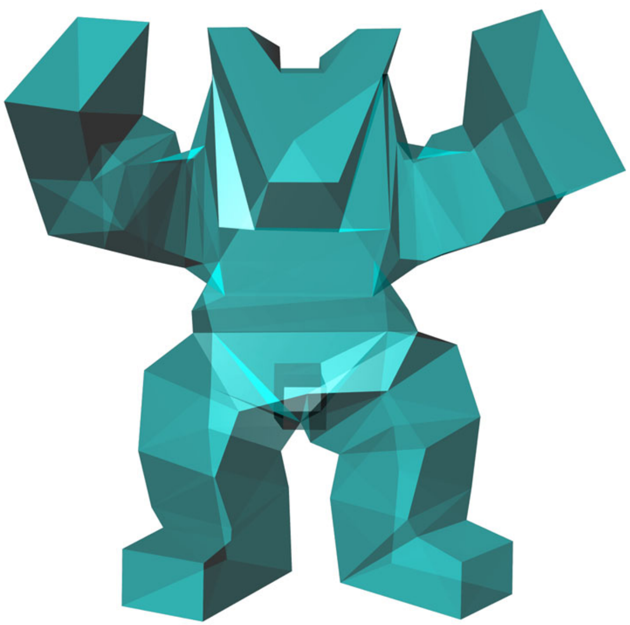


Main Challenges:

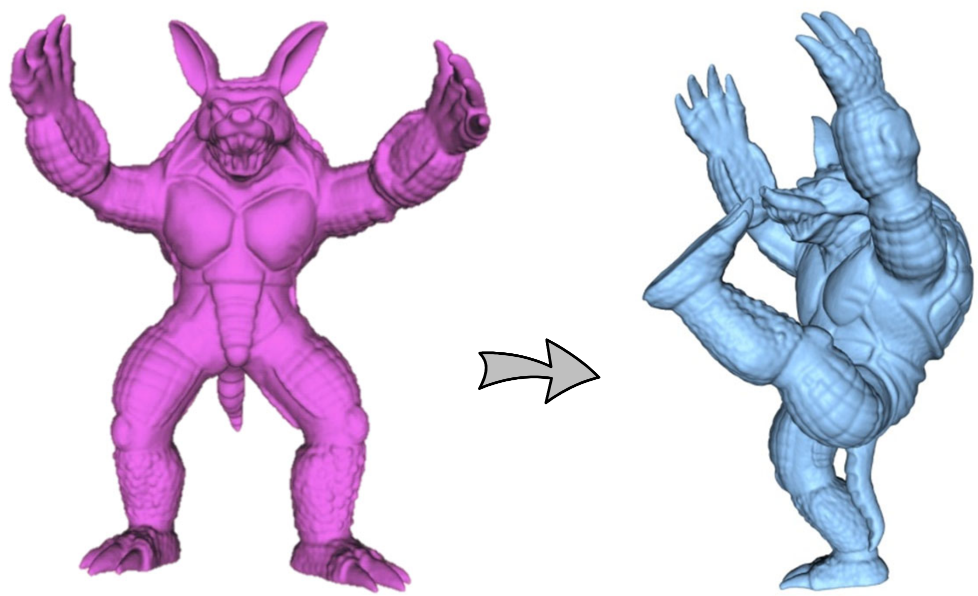
- Easy control
- Efficient, interactive
- Smooth, Shape preserving
- General geometry representation

2D/3D Spatial Deformation

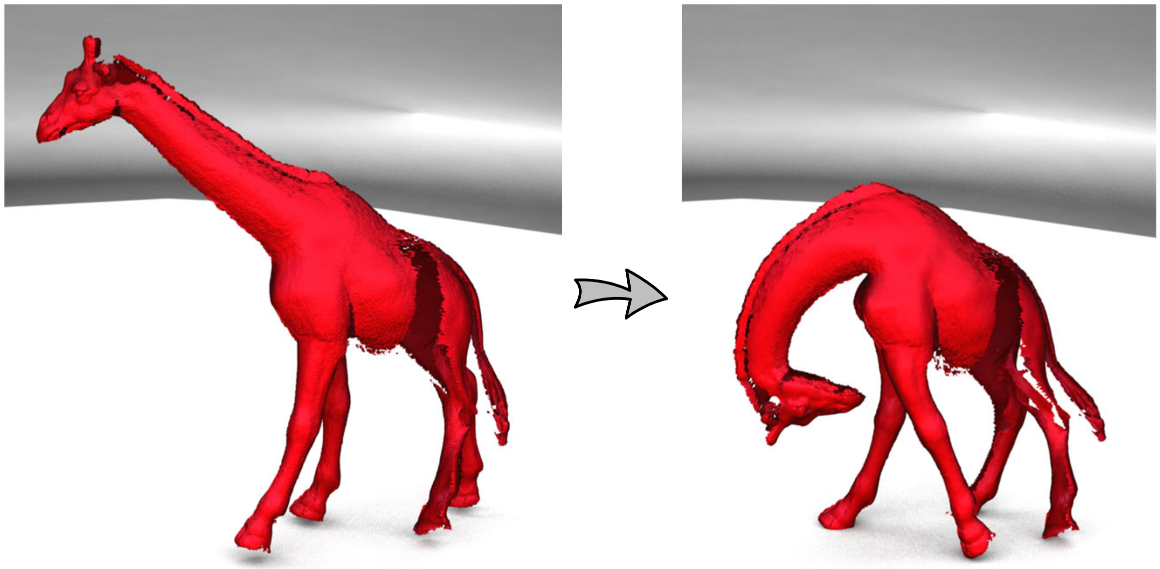
Deform $\mathbb{R}^3$ space rather than object



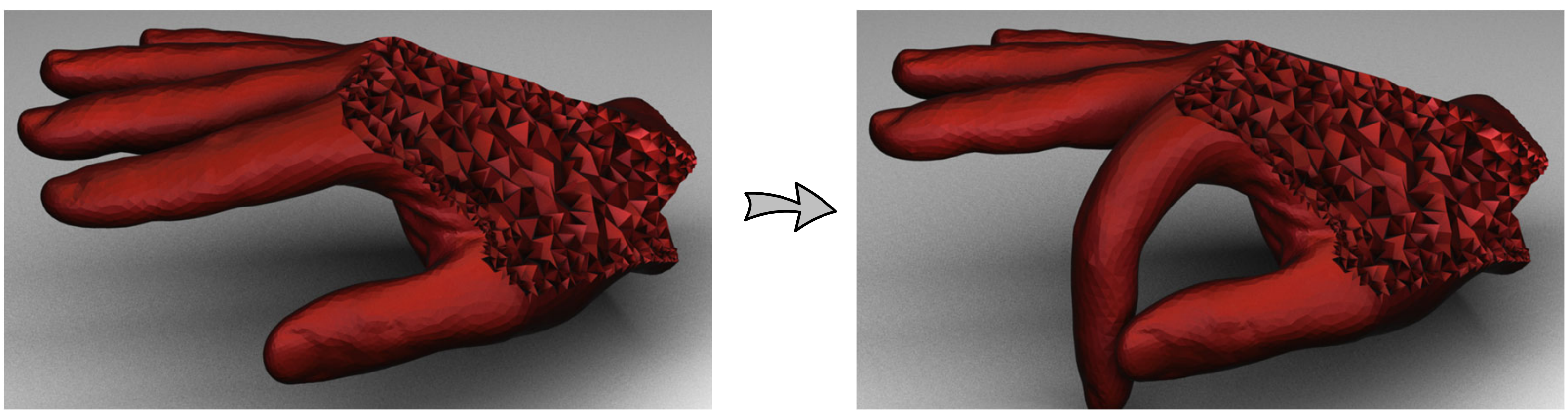
$\Omega \in \mathbb{R}^3$



Spatial Deformation - Generality



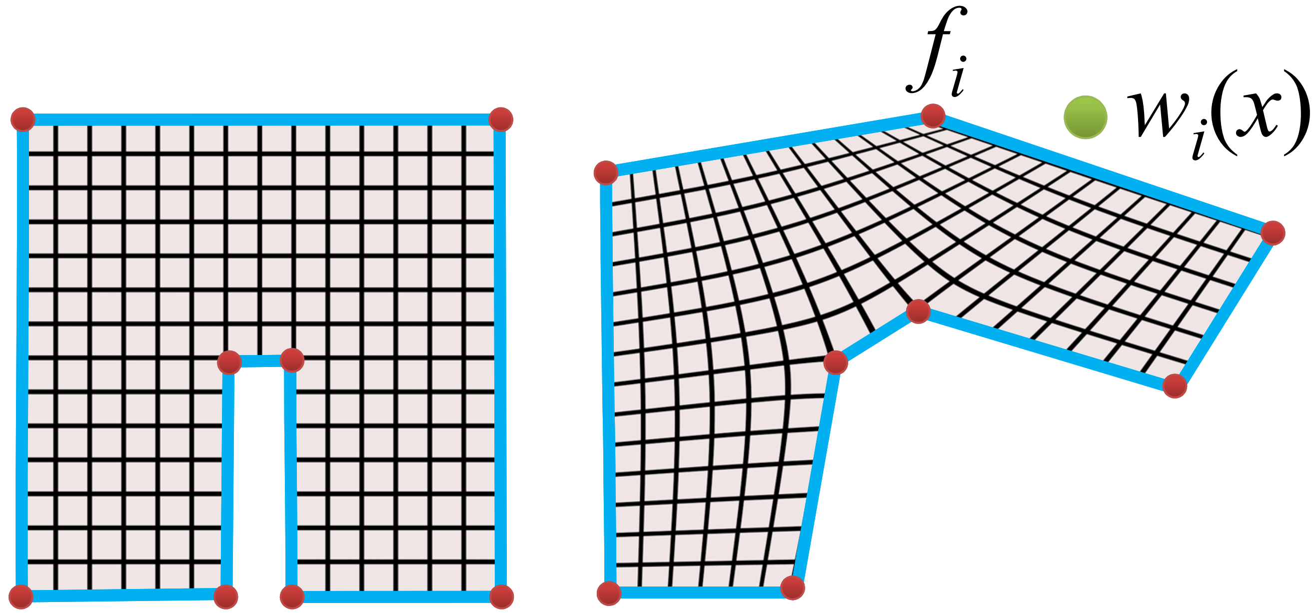
- Volumetric data
- Point clouds
- Polygon soups
- Tet-meshes



Deformation with Barycentric Coordinates

Stages:

- ✓ Source shape
- ✓ Polygonal cage
- ✓ Coordinates
- ✓ Manipulate cage
- ✓ Apply deformation

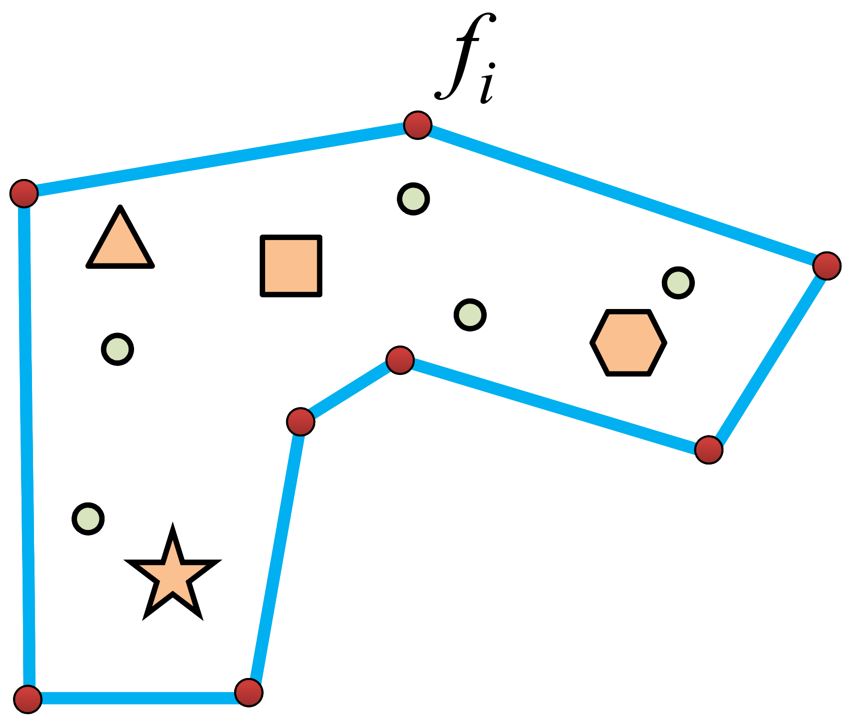


$$g(x) = \sum_{i=1}^{n} w_i(x) f_i$$

Deformation with Barycentric Coordinates

- Easy control
- Efficient, interactive, parallel, simple
- General geometry representation
- Smooth, Shape preserving

$$g(x) = \sum_{i=1}^{n} w_i(x) f_i$$



Properties of Barycentric Coordinates

Properties:

- **Constant precision**
- **Identity precision**
- Lagrange (interpolation)
- Smoothness
- Positivity
- Domain – inside/outside, convex/concave
- “Local” support
- No local critical points
- Real/complex

Some More Challenges:

- Simplify user interface even more
- Detail preservation – Conformal, As Rigid As Possible
- In 2D - avoid foldovers

Complex Cauchy Coordinates

Cauchy's Transform:

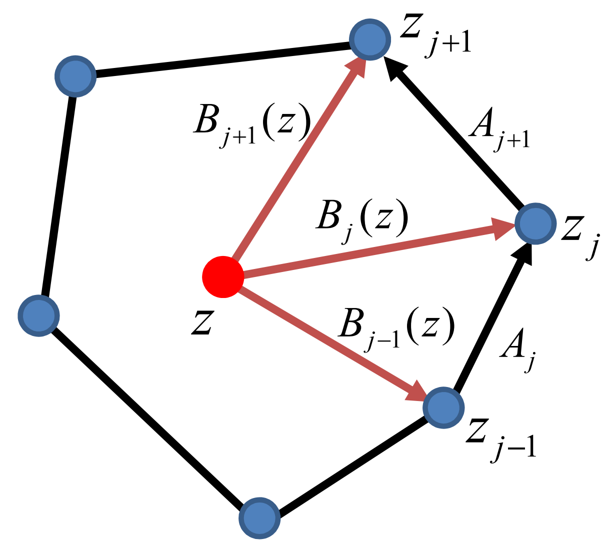
$$g(z) = \frac{1}{2\pi i} \oint_S \frac{1}{w - z} f(w)\,dw$$

Discrete Cauchy Coordinates:

$$g(z) = \sum_{j=1}^{n} C_j(z) f_j$$

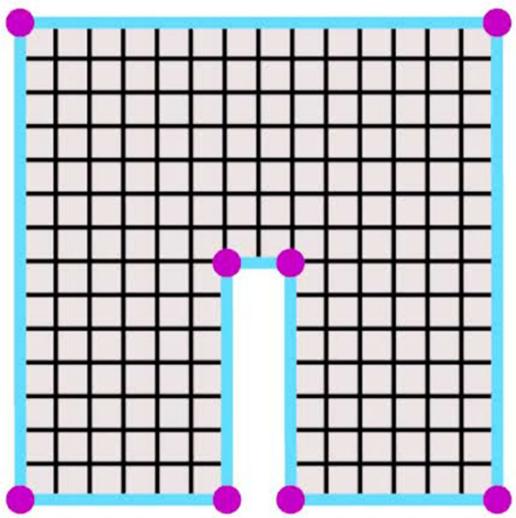
$$C_j(z) = \frac{1}{2\pi i}\left(\frac{B_{j+1}}{A_{j+1}}\log\left(\frac{B_{j+1}}{B_j}\right) - \frac{B_{j-1}}{A_j}\log\left(\frac{B_j}{B_{j-1}}\right)\right)$$

- Holomorphic
- Detail preserving – no shear
- Approximating



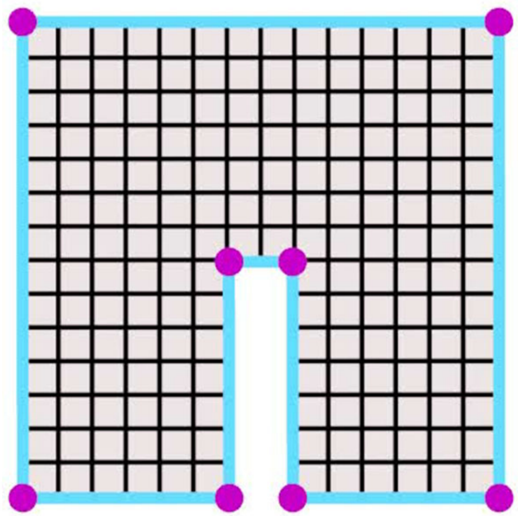
Complex Barycentric Coordinates

Real coordinates



- Affine invariant
- Interpolating boundary

Complex coordinates



- Similarity invariant
- Approximating boundary

Variational Point-2-Point Coordinates

restrict deformation to be of the form:

$$g(z) = \sum_{j=1}^{n} C_j(z) f_j$$

given:

$$r_i, g_i \quad i = 1..p$$

find:

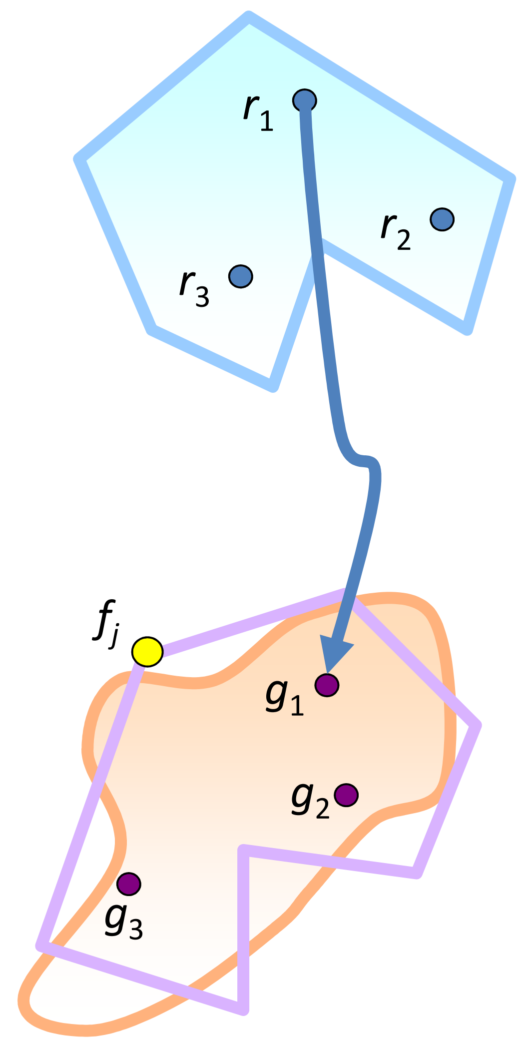
$$f_j \quad j = 1..n$$

such that:

$$g(r_i) = g_i \quad \forall i = 1..p$$

$$E(g) = \int_S \left| \quad g''(w) \quad \right|^2 ds$$

is minimized



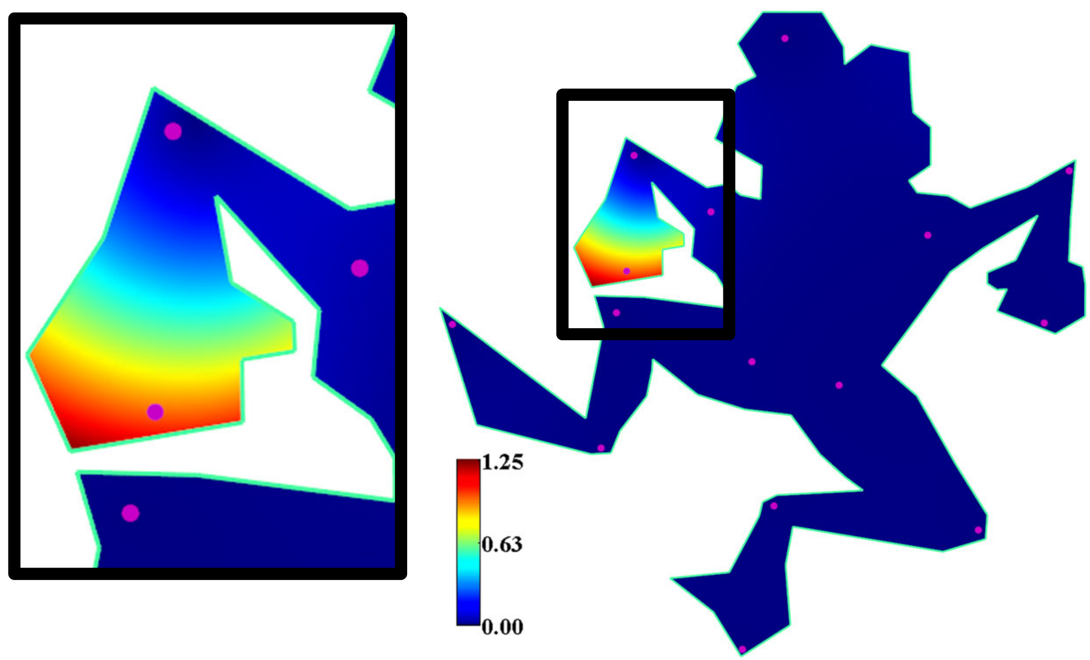
Variational Point-2-Point Coordinates

- boils down to linear system
- invert matrix in preprocessing
- P new coordinates
- p << n
- interpolating

$$f_{n\times 1} = T_{n\times p} g_{p\times 1}$$

$$C_{m\times n} f_{n\times 1} = \underbrace{C_{m\times n} T_{n\times p}}_{D_{m\times p}} g_{p\times 1}$$

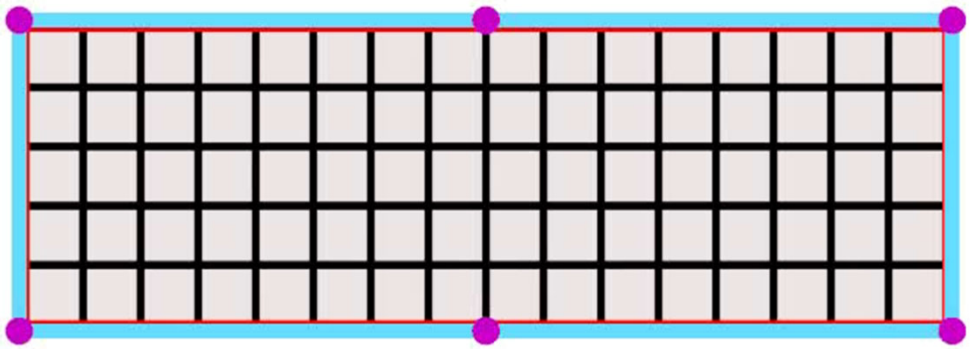
$$F_{m\times 1} = D_{m\times p} g_{p\times 1}$$



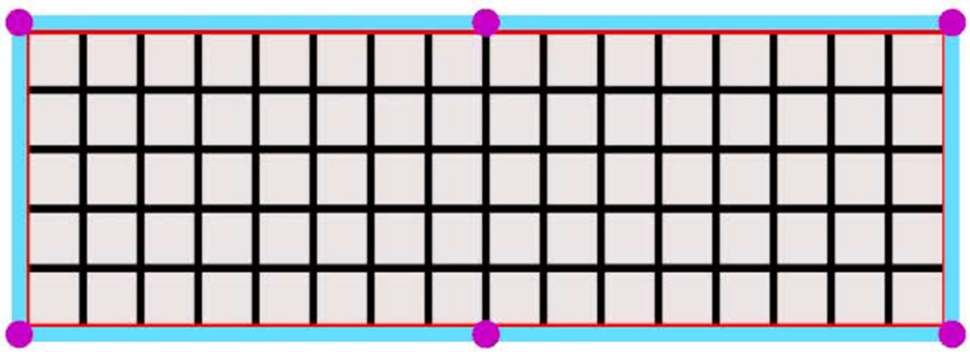
Shape aware functions

P2P Coordinates

Foldovers



Real coordinates
(harmonic)



Complex coordinates
(Cauchy)

Holomorphic Functions and Conformal Maps

$$f(z) = u(z) + iv(z)$$ Holomorphic

Treat f as a map: $f : \mathbb{R}^2 \to \mathbb{R}^2$

$$J(f) = \begin{pmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Similarity matrix
Rotation & Scale

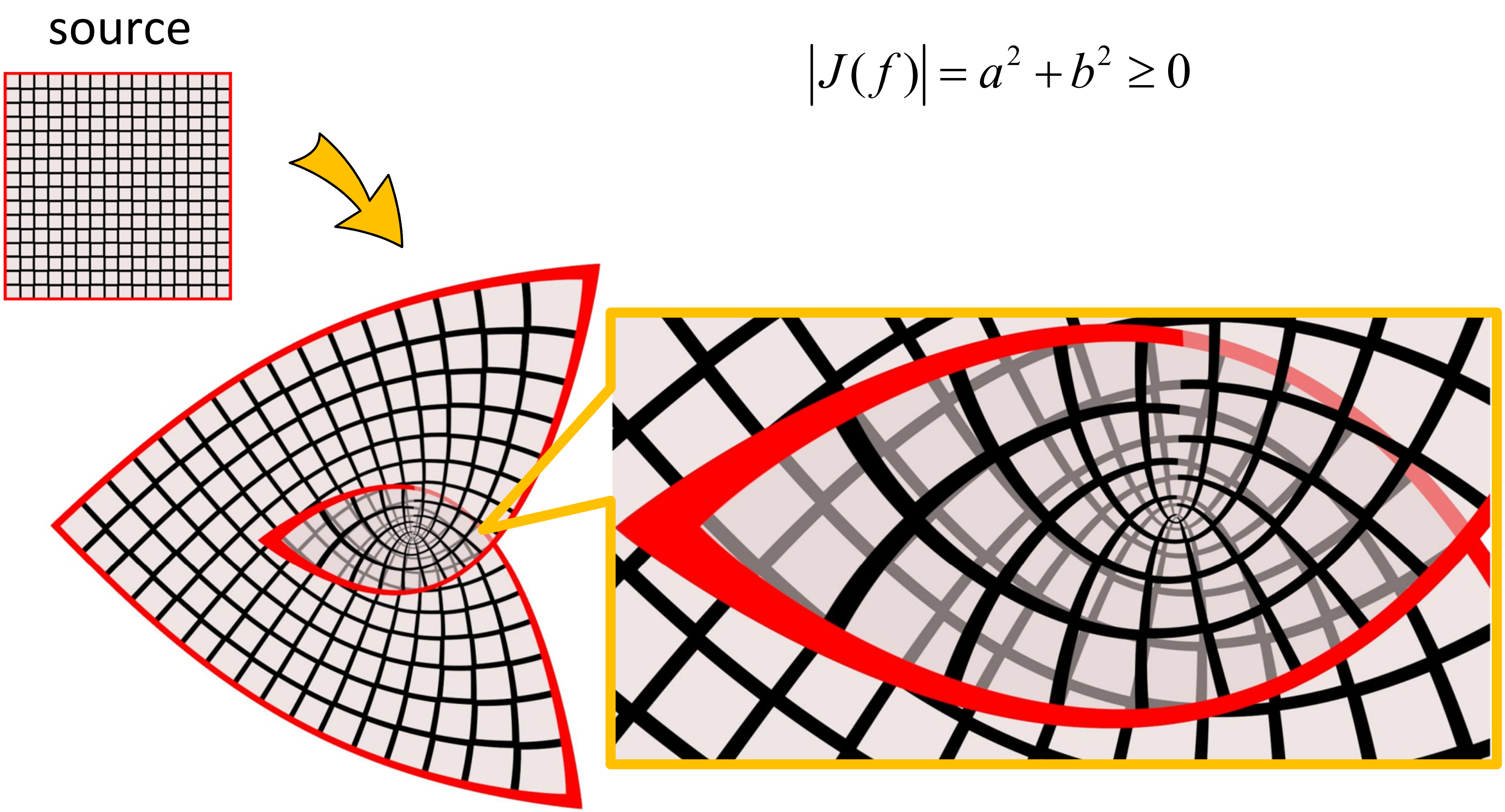
$$\left|J(f)\right| = a^2 + b^2 \geq 0$$

If Jacobian determinant strictly positive, the map is conformal

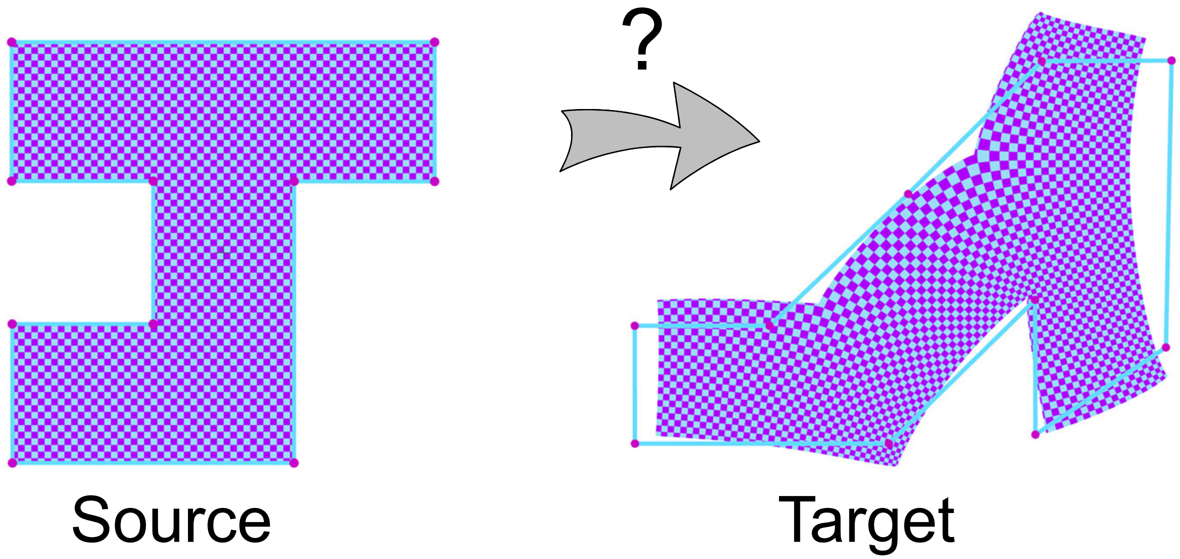
$$f'(z) \neq 0$$

Foldovers?

$$\left|J(f)\right| = a^2 + b^2 \geq 0$$

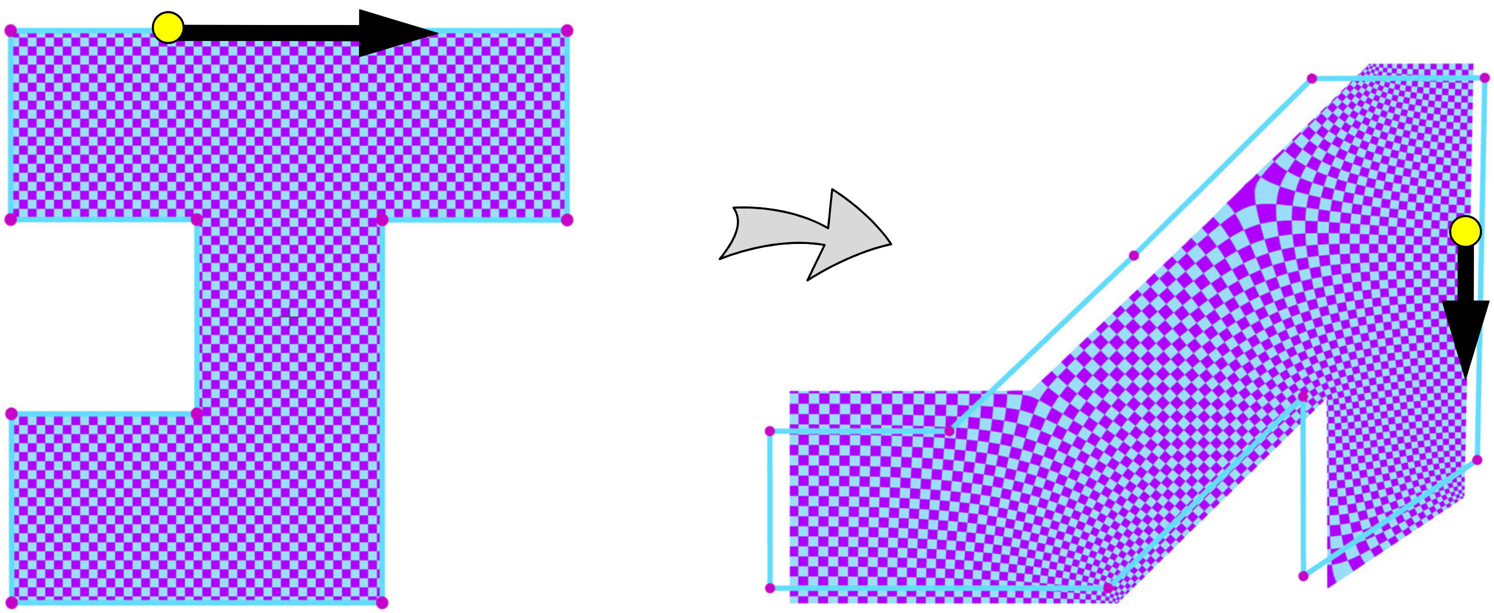


Exact Boundary Behavior?



No! ☹

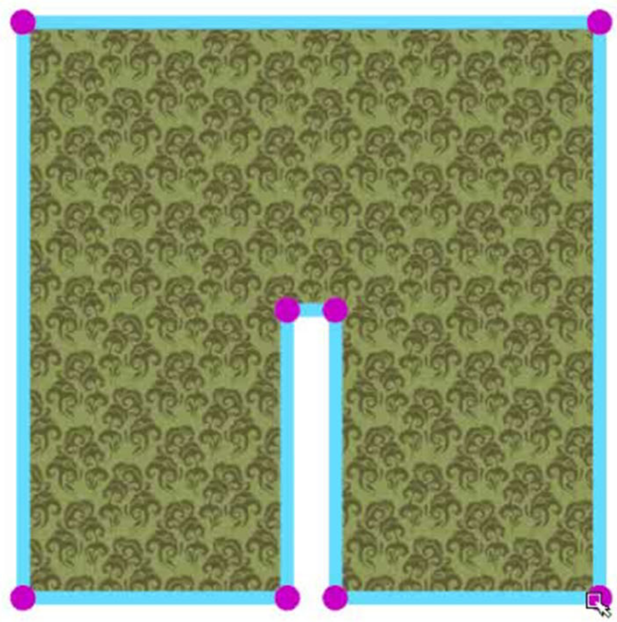
Relaxed Boundary Requirement
Angle Prescription



Source

Exact boundary angles

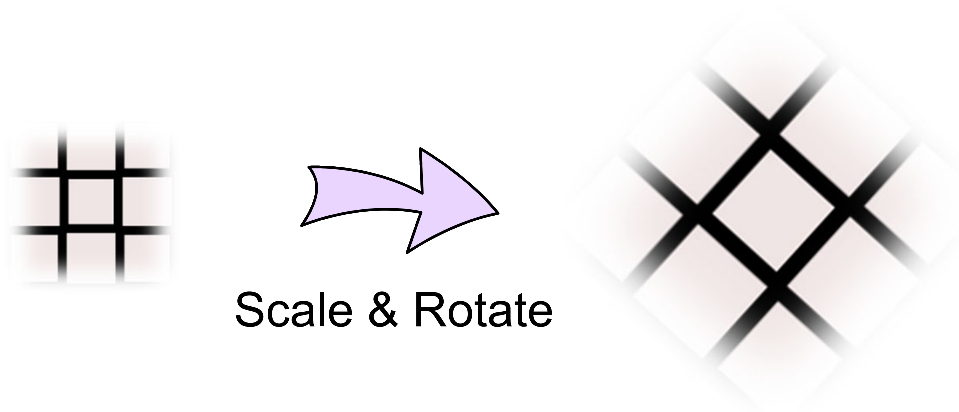
Angle Prescription



The Complex Log Derivative

$$\mathrm{Log}(z) = \mathrm{Log}(|z|) + i\,\mathrm{Arg}(z)$$

$$\mathrm{Log}(f'(z)) = \underbrace{\mathrm{Log}(|f'(z)|)}_{\substack{\phi(z) \\ \text{Conformal factor}}} + i\,\underbrace{\mathrm{Arg}(f'(z))}_{\substack{\theta(z) \\ \text{Angular factor}}}$$



The Hilbert Transform

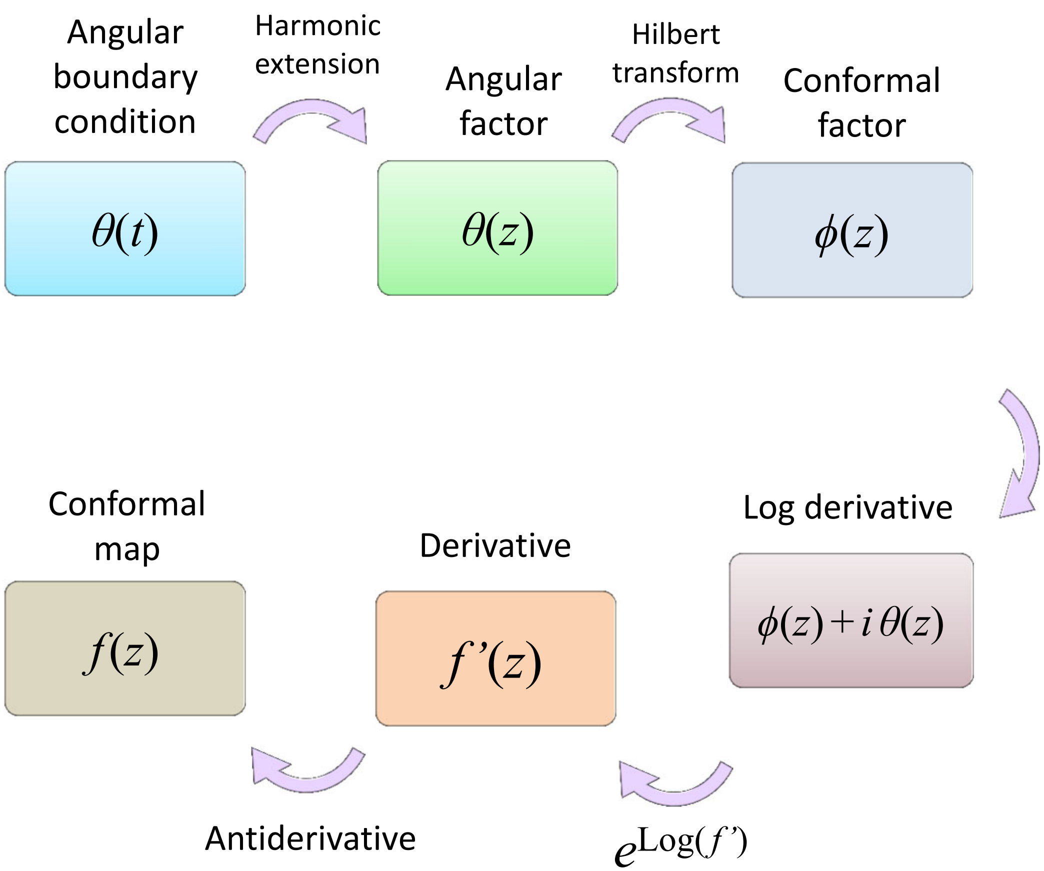
$$\mathcal{H}\left(u(z)\right)=v(z)$$ $u(z)$ and $v(z)$ harmonic

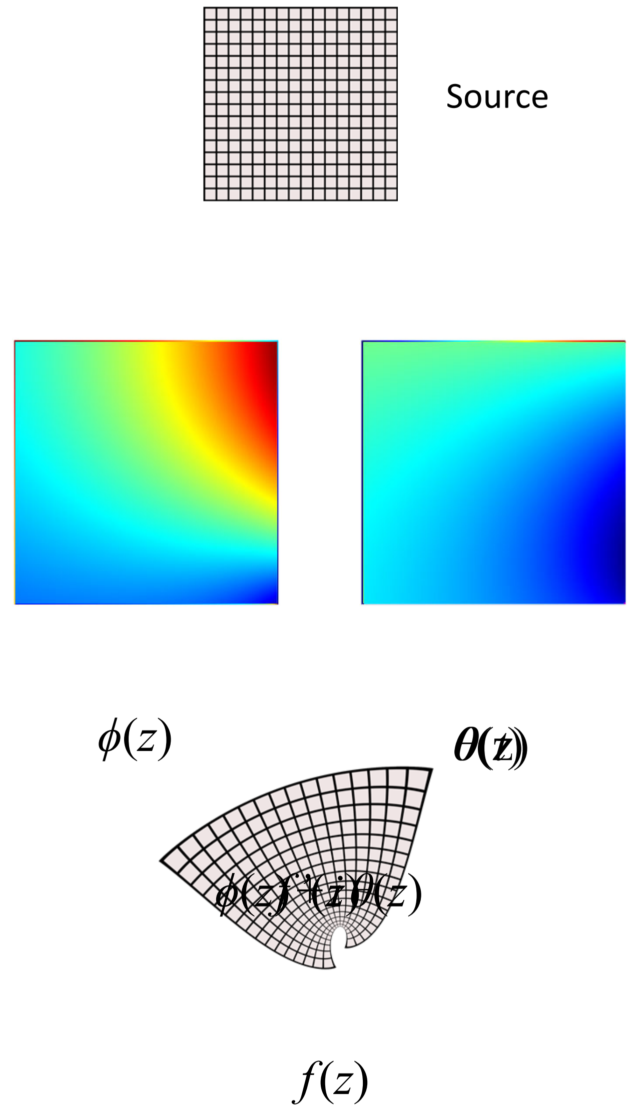
$$g(z)=u(z)+iv(z)$$

$g(z)$ holomorphic
$u(z)$ and $v(z)$ harmonic conjugate

Theorem:
Any harmonic function admits a harmonic conjugate and it is unique up to an additive constant

From Boundary Angles to Conformal Map





Linear Shape Interpolation

$A \quad \mu = 0$

$C \quad \mu = 0.5$

$B \quad \mu = 1$

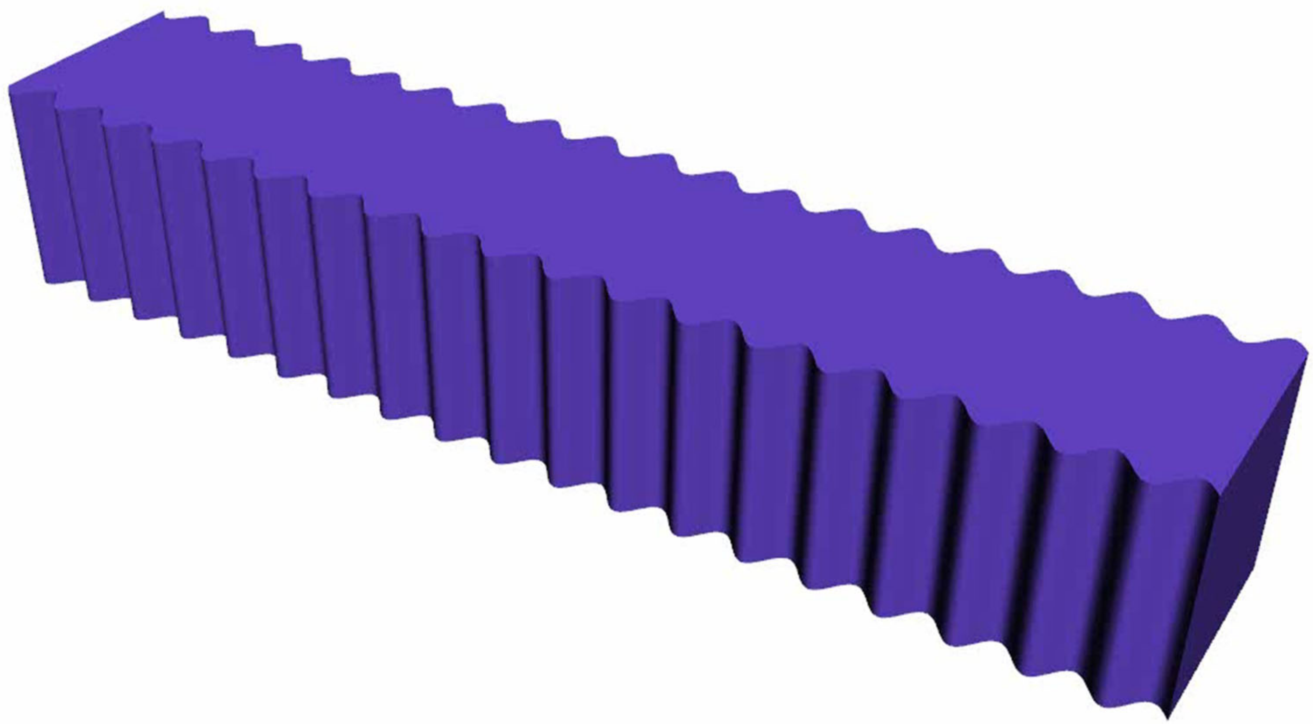
$$\theta^C(t) = (1-\mu)\theta^A(t) + \mu\theta^B(t)$$

Shape Interpolation

Interpolation of Large Rotations

Results

Cage-Based Deformation



Which Subspace?

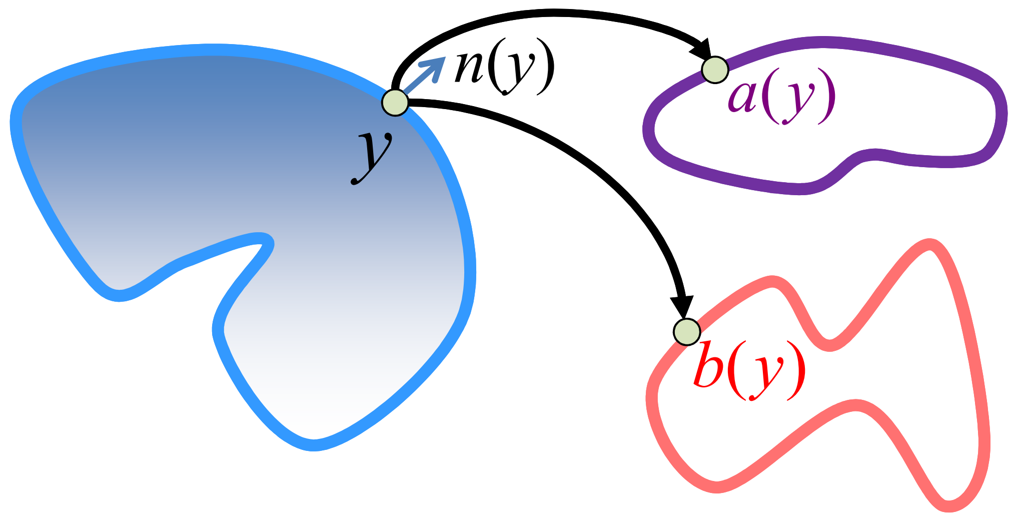
Choose f to be *harmonic* map: $f:\Omega\rightarrow\mathbb{R}^3$

$$f=\left(u(x,y,z),v(x,y,z),w(x,y,z)\right)$$

$$\nabla^2 u=0,\qquad \nabla^2 v=0,\qquad \nabla^2 w=0$$

- Smooth
- Detail preserving if derivatives are orthogonal
- Determined solely by boundary values

Generating Harmonic Maps



Projection into Harmonic Subspace

$a(y) \longrightarrow$

$b(y) \longrightarrow$

Two maps on $\partial\Omega$

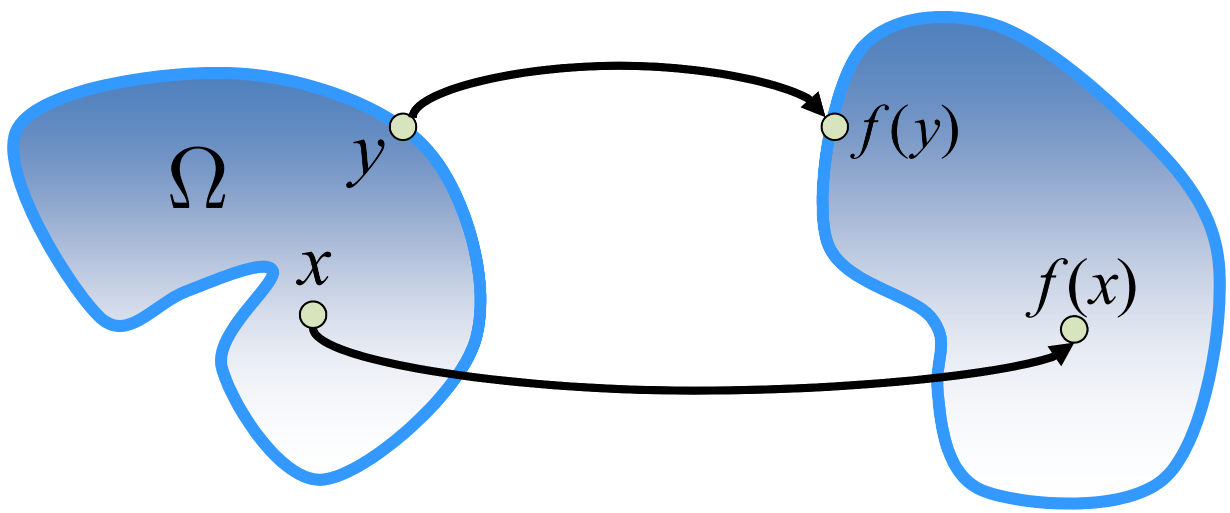
$$\oint_{y \in \partial\Omega} \begin{pmatrix} a(y)\left(\nabla G(x,y) \cdot n(y)\right) \\ -b(y)G(x,y) \end{pmatrix} dA \longrightarrow f_{a,b}(x)$$

Harmonic map on Ω

Harmonic Map – Reversed

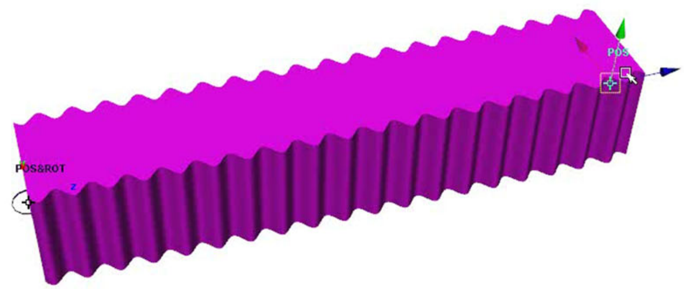
$$f(x) = \oint_{y \in \partial\Omega} \underbrace{f(y)}_{a(y)} \left(\nabla G(x,y) \cdot n(y)\right) - \underbrace{\left(\nabla f(y) \cdot n(y)\right)}_{b(y)} G(x,y)\, dA$$

Green's 3rd Identity



Finding $a(y)$ and $b(y)$

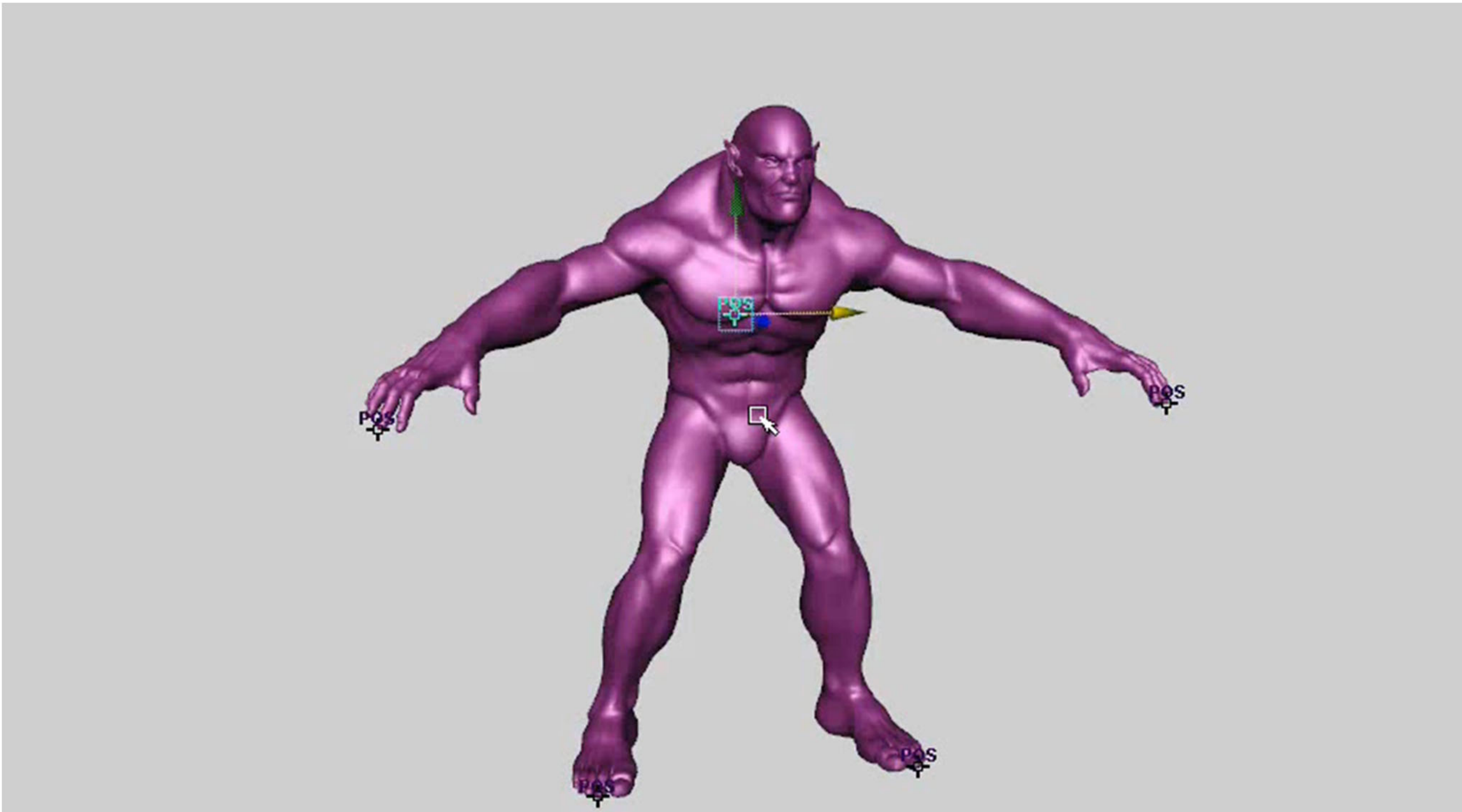
- Position constraints
- Jacobian constraints



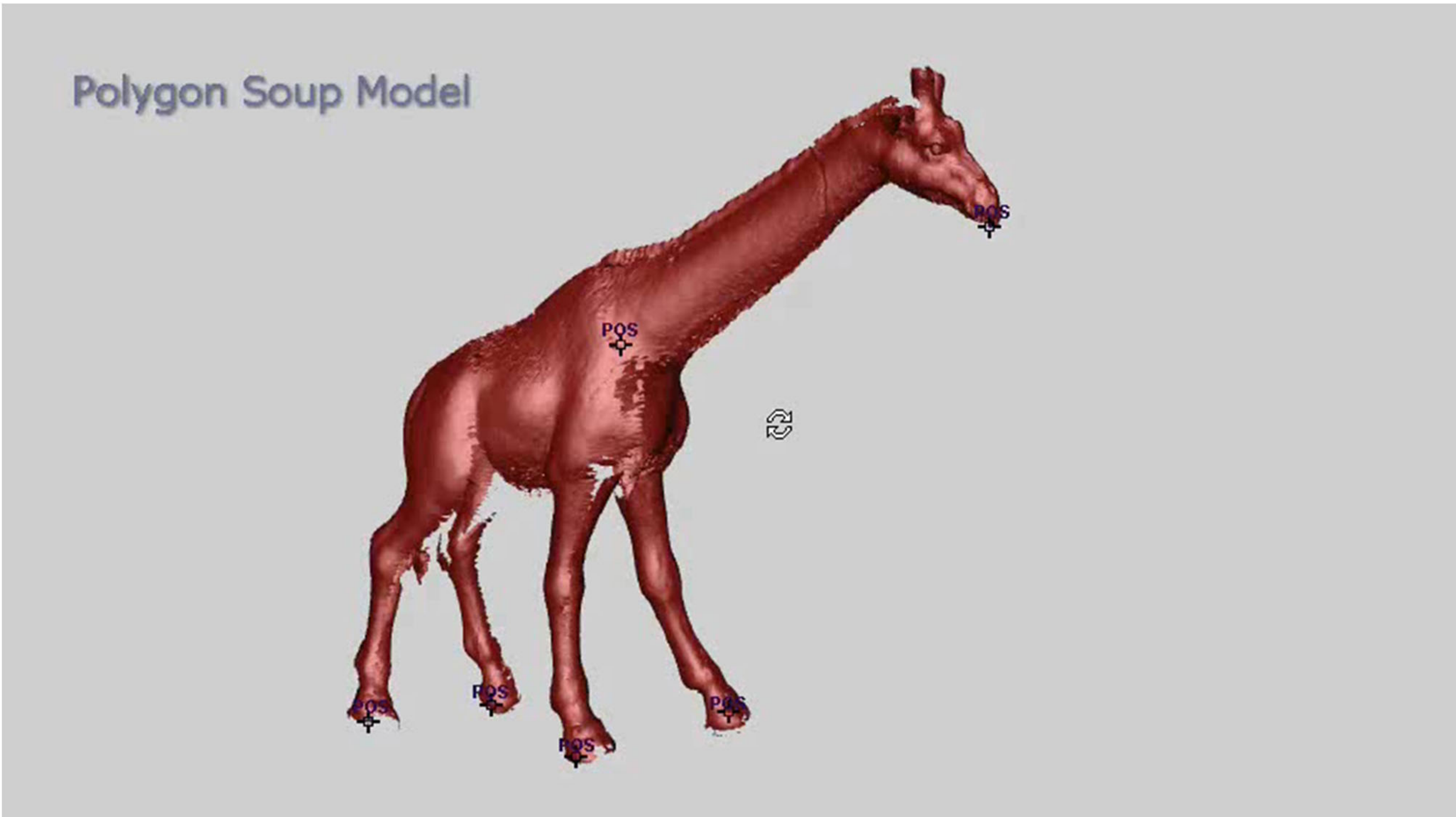
- Smoothness
- Detail preservation

$$(\hat{a},\hat{b}) = \underset{a,b}{\operatorname{argmin}}\left(E(f_{a,b})\right)$$

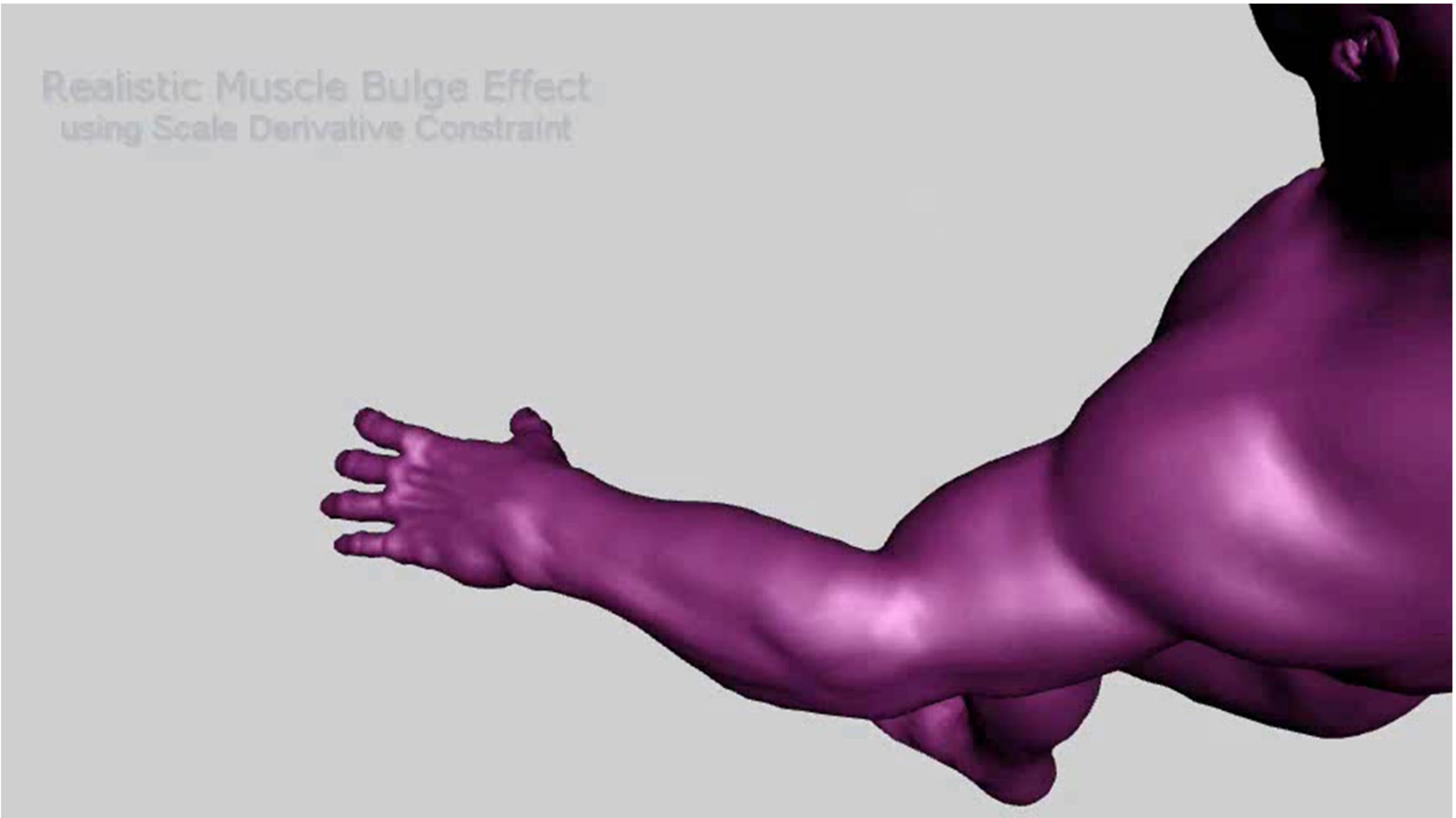
Results – As Rigid As Possible



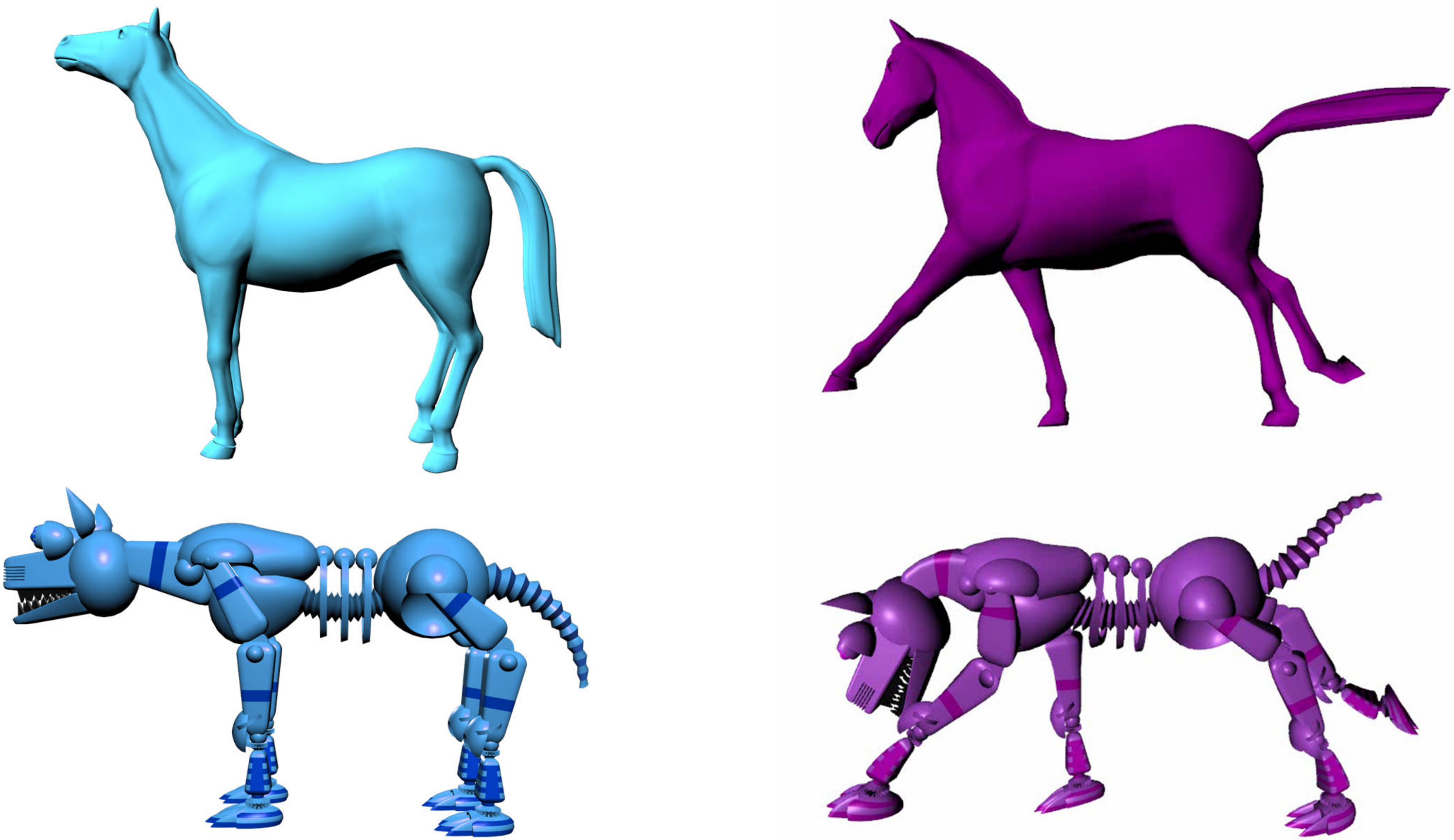
Results – Giraffe Polygon Soup



Results – Scale Constraint



Spatial Deformation Transfer



Summary

- Barycentric coordinates for shape deformation
- Efficient, interactive, parallel
- Smooth, Shape preserving
- Easy control (low dimensional subspace)
- Spatial deformation – generality
- Complex barycentric coordinates
- Variational approach takes us a step further