GENERALIZED BARYCENTRICS

RATIONAL BASES FOR

2D AND 3D ELEMENTS

WITH ALGEBRAIC SIDES

Weak vs Strong Form

- Consider second order elliptic boundary value problems with linear PDE
- L**u** – **f** = **0** is the strong statement of the differential equation
- When **f** is in H^0 , **u** is in H^2_B where B denotes boundary conditions. $||\mathbf{u}||_2 < C||\mathbf{v}||_0$
- (L**u-f**,**v**) = **0** for all **v** in C^0_E is weak form where E is "essential" boundary conditions

Finite Differences

- Finite difference equations are derived by specifying a grid and replacing the differential equation at each point of the grid by a difference equation. Finite difference approximations are thus derived from the strong form.

Finite Elements – Weak Form

- The finite element method is derived from the weak form. The region of interest is partitioned into a set of elements which cover the region without overlap. A variational principle is used to determine free parameters associated with nodes of these elements.

Minimization Problem

- The weak form is equivalent to minimizing a functional (L**v** – 2**f**,**v**) over all **v** in $C^0{}_E$
- All boundary conditions (b.c.) are B and essential b.c. are E
- Note that trial function space of **v** is less restrictive than that of the solution space $H^2{}_B$

Approximation Space

- Solution space S in H^2_B and C^0
- Approximation space R in C^0_E , $E < B$
- Seek **w** in R to minimize functional (L**w** – 2**f**,**w**)
- Characteristic of H often infinite
- Characteristic of R usually finite: parameters a_i with i = 1, 2, …, n for n dimension subspace

Patchwork Approximation

- Solution **v** over bounded region in space-time
- Cover region with finite set of non-overlapping elements
- Each element has nodes and each node has associated basis functions with a free parameter a_j for basis function W_j
- The approximation space is $R = \text{sum}_j\ a_j\ W_j$

Minimization

- Minimize by setting partials of the functional with respect to the expansion coefficients a_i to zero. When L is symmetric a factor of 2 drops out and we get the matrix equation: $G\mathbf{a} = \mathbf{s}$, where $G_{ij} = (W_i, L W_j)$ and $\mathbf{s}_i = (W_i, \mathbf{f})$
- Each basis function is non zero only over elements containing node i so integrals are evaluated element by element. The part of basis function over element e is called a "wedge" function.

Approximation Theory

- Patchwork C^0 approximation with piecewise polynomials enhances error analysis
- Polynomial interpolation of degree k to a function f with bounded derivatives through k+1 over an element with maximum chord of length h with $P(x_i , y_i) = f(x_i , y_i)$ yields $|int_{element} (P - f)| < C h^{k+3}$

Aspect Ratio

- The ratio of h to the element area is the aspect ratio, s. Let s be bounded above by S The number of elements contributing to the integral is of order S/h^2
- Thus the norm of the global approximation error as h is reduced is of order CSh^{k+1}
- As a grid is refined to decrease h, the aspect ratio must not increase significantly to retain this order of convergence.

h-P Finite Elements

- The trial function space includes the interpolating patchwork polynomial of degree k in each element and the norm of the error is bounded above by that of the interpolant. Thus, a degree one basis assures an error norm of $O(h^2)$. Error may be reduced by higher degree approximation within elements or a finer partition with smaller h.

Wedge Properties

- Wedge functions meet certain requirements
- Although functionals can be developed which admit discontinuous trial function spaces, simplest and often most efficient are those which require continuous trial function spaces
- Approximation spaces within adjacent elements must be continuous across element boundaries. The basis function associated with node i is the union of the wedge bases for the adjacent elements.

Wedge Properties – 2

- There are three types of nodes: vertex nodes where two sides of an element meet, side nodes on a side but not a vertex, and interior nodes
- A wedge must vanish on all element sides other than those which contain the node. Thus, a wedge associated with an interior node must vanish on the entire element boundary

Wedge Properties – 3

- For degree one approximation over a convex polygon element there must be a node at each vertex. The basis function normalized to unity at a vertex must vanish along opposite sides and vary linearly along each adjacent side to 0 at the adjacent vertex. This is necessary but not sufficient.
- Barycentric coordinates are positive within the element and satisfy just these conditions.

The Triangle

- A triangle is the simplest and often used element. The wedges are the barycentric coordinates. Let the triangle have vertices 1, 2 and 3. The line through points a and b is denoted as (a,b). The linear form that vanishes on this line and is normalized to unity at c is $(a;b)_c$. The three wedges are $(1;2)_3$, $(2;3)_1$ and $(3;1)_2$. Any linear combination is of degree one so these wedges are a degree one basis.

Rectangles and Parallelograms

- The other oft used element is the rectangle. It should be noted that this analysis applies to parallelograms as well.
- Let the vertices in order be 1, 2, 3, 4. Then $W_1(x,y) = (2;3)_1(3:4)_1$, and the other three similarly. The parallel opposite sides enable proof that these bilinear wedges are linear on adjacent sides.

Rectangles – 2

- The interpolant of any linear function minus the linear function vanishes on the four sides and is bilinear (of degree two) over the element. A degree two polynomial that vanishes on a boundary of order four must be the zero polynomial. Degree one approximation is thus established.

Convex Quadrilaterals

- A quadrilateral that is not a parallelogram cannot be treated this way. Parallel opposite sides are needed for linearity on adjacent sides.
- Exterior intersections of element sides are designated as EIP. It will now be shown that any element with an EIP cannot have polynomial wedges.

Failure of Polynomial Wedges

- Say sides (a,b) and (c,d) meet at EIP e.
- Interpolation of arbitrary vertex values at a, b, c and d with wedges linear on the edges leads to different values in general at e.
- Since a polynomial has no singular points over the entire plane, all the wedge basis functions cannot be polynomials.

Degree Two Wedge Basis

- Consider a triangle with vertices 1, 2 and 3. Let points 4, 5 and 6 be the midpoints of sides (1,2), (2,3) and (3,1), respectively.
- Vertex wedges are $W_1 = (2;3)_1(4;6)_1$, etc.
- Side node wedges are $W_4 = (2;3)_4(3;1)_4$, etc.
- The difference between a quadratic interpolant and a quadratic function is quadratic and vanishes on the boundary of order three. Degree two is thus established.

Degree Two – 2

- For a rectangle, the same construction applies. Now vertices are 1, 2, 3 and 4 and side nodes are 5, 6, 7 and 8. The vertex wedges are of the form $W_1 = (2;3)_1(3;4)_1(5;8)_1$.
- Side node wedges are of the form: $W_5 = (2;3)_5(3;4)_5(4;1)_5$.
- These trilinear wedges are quadratic on the sides. The interpolant is exact on the boundary of order four and must therefore be exact over the element. The basis is degree two.

Higher Degree Bases

- For higher degree bases more nodes must be added on the sides to attain the higher degree interpolation on the sides. However, interior nodes may be needed. For example, for degree three approximation over a triangle there are two nodes on each side and a tenth interior node. The interior wedge is: $W_{10} = (1;2)_{10}\,(2;3)_{10}\,(3;1)_{10}$.

Discretization

- In finite element application integrals of products of basis functions and their derivatives are computed. Numerical approximations are often used. For example, nine-point gaussian integration may be used for rectangular elements. These integrals occur in reduction of the problem to a set of equations for nodal unknowns.

Isoparametrics

- Isoparametric elements model three or four sided elements with straight or curved sides.
- We consider first an element with four sides
- Map standard square with vertices 1 = (-1,-1), 2 = (-1,1), 3 = (1,1) and 4 = (1,-1) in local coordinates p,q into element in global coordinates (x,y).
- Midpoints of sides of square are nodes 5-8.

Isoparametrics – 2

- Degree two wedge functions for the square are W_i (p,q) for 1 = 1, 2,…, 8. The eight nodes of the topologically equivalent global element have coordinates (x_i,y_i). Then the mapping is
- $x = sum_i\ x_iW_i(p,q)$, $y = sum_i\ y_iW_i(p,q)$.
- The jacobian of the transformation, J(x,y;p,q), is computed by taking derivatives with respect to p and q. Restrictions on the element geometry guarantee J nonzero over the element.

Isoparametric Integration

- Integrals are now computed from $int_{x,y}f(x,y)dxdy = int_{p,q}F(p,q)\ |J|dpdq$.
- For example, when $f(x,y) = W_i(x,y)W_j(x,y)$, $F(p,q) = W_i(p,q)W_j(p,q)$ and when $f(x,y) = \text{grad}\ W_i\ (x,y)\ .\ \text{grad}\ Wj(x,y)$, $F(p,q) = \text{grad}\ W_i(p,q)\ .\ \text{Grad}\ W_j(p,q)\ J^{-2}$.
- Quadrature weights at (p,q) points are weighted by |J|.

Finite Element Application

- All elements mapped into basic isoparametric element
- Simple quadrature rules
- Versatile
- Curved sides mapped into nearby parabolic arcs
- Only three- and four-sided elements

Limitations of Isoparametrics

- Although isoparametric bases of any degree may be used, the global approximation is only degree one.
- A phenomenon known as “locking” occuring in structural problems is due to lack of higher degree global approximation.
- True curved sides are approximated by parabolic arcs in degree two isoparametrics.

Resolutions

- Basis functions have been introduced to eliminate locking with isoparametrics. In general, knowledge of solution behavior is required.
- Alternatively, we seek basis functions in global coordinates for more general element geometry. Rational basis functions often suffice .

Quadrilateral Global Basis

- Try rational wedge functions with denominator vanishing at all EIP
- Exterior diagonal of quadrilateral is (5,6)
- Examine $W_1 = [(2;3)(3;4)/(5;6)]_1$, etc
- Establish linearity on adjacent sides.
- Degree one approximation is attained.

General Rational Basis Theorem

- A convex polygon with m sides has a rational degree-one wedge basis with denominator of maximal degree m-3 determined uniquely by the EIP. The numerators are just the products of the linear forms of the sides opposite the vertex. All the wedges are positive interior to the polygon and the "adjoint" denominator is also positive over the element, including its boundary.

Algebraic Geometry Foundations

- Before proceeding to higher degree approximation and to elements with curved sides we develop algebraic geometry foundations
- We observe immediately that we need more precise analysis on how curves meet.
- We must also determine when approximations over elements sharing a side are continuous across the side.

Plane Curves and Bezout

- Plane curves P_m and Q_n of order m and n intersect at mn points (Bezout).
- Intersection at infinity treated in projective rather than affine plane with homogeneous coordinates: line on which a+bx+cy = 0 becomes aw+bx+cy = 0 with homogeneous coordinates (w,x,y) where w = 1 is affine plane and w = 0 is line at infinity (horizon). (0,x,y) is point on w = 0 for line of slope y/x.

Intersection Sets and Divisors

- $R \cdot ST = R \cdot S + R \cdot T$ so consider only irreducible curves
- Bezout requires analysis of intersections; e.g. $P = y$, $Q = x^2$, $Q = y-x^2$, $Q = y-x^4$, etc.
- Theory of divisors distinguishes with centers and neighborhoods generated by birational transformations.
- Multiple branches and tangencies are resolved by separating centers into neighborhoods.

Max Noether’s Theorem

- Let F o G be the divisor set of F and G.
- If P o R = Q o R where P and Q do not share a component of Q , then **P** ~ **Q** mod R which means that there is an a such that **P**-a**Q** = 0 on curve R.
- Note that in the congruence theorem **P** and **Q** denote the polynomials that vanish on curves P and Q.

Noether and Convex Quadrilaterals

- Quadrilateral (1,2,3,4) has exterior diagonal (5,6) , where 5=(1,4)o(2,3) and 6=(1,2)o(3,4).
- W_1 = [(2;3)(3;4)/(5:6)]$_1$ and (3,4)o(1.2) = 6 = (5,6)o(1,2) so that (3;4) = -a(5;6) on (1,2) and W_1 = -a(2;3) = (2;3)$_1$ since W_1 is unity at 1.
- Congruences allow cancelling of factors in numerator and denominator to reduce rational functions to polynomials on element sides.

Convex Polygons

- A convex polygon with n sides is an n-gon
- An n-gon has $n(n-1)/2 - n = n(n-3)/2$ EIP
- A plane curve of order m is determined uniquely by $m(m+3)/2$ "m-independent" points. Thus, the EIP may determine a unique curve of order n-3. It can be shown that the EIP are independent.
- This curve is called the polygon "adjoint".

Pentagons and Hexagons

- A pentagon has 5 EIP which determine a unique conic. A regular pentagon has a circle adjoint.
- A 6-gon has 9 EIP which determine a unique cubic adjoint. A hexagon has three points on the absolute line (three pairs of parallel sides). The hexagon adjoint is a circle. The cubic curve is the product of this circle and the absolute line which is unity in the affine plane.

Linearity on Adjacent Sides

- The product F of sides opposite node j other than side (j+1,j+2) intersect side (j,j+1) at n-3 EIP and only at these points.
- The adjoint intersects side (j,j+1) at and only at these same EIP.
- $W_j = [(j+1;j+2)F/Q]_1$ so that $W_1 = (j+1;j+2)_1$ on side (j,j+1).

Degree-one Approximation

- Let L be any linear polynomial and consider $f(x,y) = \text{sum}_j [L(x_j, y_j) W_j(x,y)] - L(x,y)$.
- $Qf(x,y) = P_{n-2}(x,y) - L_1(x,y)Q_{n-3}(x,y)$ (where subscripts denote degrees of polynomials.)
- Linearity of the wedges on the sides guarantees $f = 0$ on the polygon boundary of order n. The only polynomial of maximum order n-2 that vanishes on this boundary of order n is the zero polynomial.

Adjoint Positivity

- The adjoint curve cannot enter an element by crossing its boundary since the EIP exhaust all such intersections. However, when $n > 5$ the adjoint could have a closed interior loop .
- For a convex polygon this can be easily proved to be false. Let the linear forms of the sides be normalized to be positive interior to the element. Let Q be normalized to unity at vertex 1. Then Q is positive at all vertices. A positive constant at each j normalizes the wedge to unity at j.
- The sum of the wedges is unity. Therefore , the adjoint is the sum of the numerators which are all positive within the element.

Genus and Rational Curves

- A curve P is rational if it has a rational parametrization: $x = f(u)/g(u)$ and $y = s(u)/t(u)$ with f, g s and t all polynomials, such that for all but a finite number of points **P**(x,y) = 0. An irreducible algebraic curve may have singular points. An irreducible conic has no singular points. A cubic can have at most one. When a curve has its maximum possible singular points it is said to be of genus zero.

Rational Curves – 2

- A rational curve is of genus zero. All irreducible conics are rational.
- A singular point of an irreducible curve is one at which there is more than one branch.
- An ordinary singular point has no branches with a common tangent.
- A birational transformation can reduce any irreducible algebraic curve into a curve with only ordinary singularities.

Rational Curves – 3

- Let r_i be the multiplicity at point i of irreducible curve Q (having only ordinary singular points) and let Q be of order n. Then the genus of Q is equal to $g = (n-1)(n-2)/2 - [\text{sum}_i\ (r_i(r_i-1))]/2$.
- If Q is the birational transform of P, then P is of genus g. (genus is invariant under birational transformations.)
- A rational curve has genus 0 and thus is equivalent to a curve with $p = \text{sum}_i\ [(r_i(r_i-1))]/2 = (n-1)(n-2)/2$

Well-Set Elements

- An element is rational if each side is an irreducible rational curve.
- The element is well-set if the curve defining a side does not extend into any point in the element and if the curve has no singular point on the side. For example, if the curve is a segment of a branch of a hyperbola, no point of the branch (except the side itself) or of the other branch can lie in the element, including its boundary.

The Adjoint of a Well-Set Element

- The order of an element is the order of its boundary which is the sum of the orders of its sides. Let n be the number of sides and m be the order.
- It has been proved that there is a unique adjoint curve of maximal order m-3 which is of order $r_j – 1$ at all multiple points of order r_j of the boundary curve (other than the vertices).
- This curve is designated as Q.

A Regular Element

- An element is regular if its adjoint curve has no point in the element, including its boundary curve.
- It has been proved for all elements of order less than six and conjectured for all elements that if an element is well-set then it is regular.
- If an element is not regular, wedges have zero denominators at points within the element and cannot be used for finite element approximation.

Computing the Adjoint

- Determining the N = m(m-3)/2 conditions satisfied by the adjoint curve, Q, can be demanding.
- Given these N conditions, one can develop a set of N linear equations for the polynomial coefficients. Resolving tangencies and other details at multiple points on Q can be tedious and programming can become difficult.
- This detracted from early implementation.

GADJ: Dasgupta's Algorithm

- Gautam Dasgupta (Columbia University) developed an algorithm for computing Q for a convex polygon which resolved early difficulties. This is the GADJ algorithm.
- The numerator for the n-gon wedge at vertex j is a product of n-2 linear forms with arbitrary normalization. Let this be $k_j N_j$ where j = n+1 is vertex j = 1 and the sum of these numerators is equal to Q. On side (j,j+1) , the wedge varies as
$L = k_j N_j /(k_j N_j + k_{j+1} N_{j+1})$. N_j and N_{j+1} have common factors which can be divided out of numerator and denominator, leaving
$L = k_j (j+1;j+2)/(k_j (j+1;j+2) + k_{j+1} (j-1;j))$

GADJ – 2

- L must be linear. The numerator is linear. Hence, the denominator must be constant. This is true if $k_{j+1} = k_j \, (j+1;j+2)_j \,/(j-1;j)_{j+1}$ so that the denominator is equal to k_j on (j,j+1).
- We can select $k_1 = 1$ and compute the remaining k_j recursively.
- Then $Q = \text{sum}_j \, (k_j \, N_j)$.
- This eliminates the EIP construction

Positivity of Q

- All the linear forms (j;j+1) may be normalized to be positive within the element. Then all the k_j are positive and Q is positive over the element, including its boundary.
- Thus, GADJ removes a major difficulty in implementation for convex polygons.

Polycons

- A polycon is a well-set element with only linear and conic sides. It has at least one conic side. Otherwise it would be a convex polygon.
- A polycon with n sides is an n-con. Let its order be m.
- We consider first degree-one approximation. For global continuity the variation along side (j,j+1) must be a unique linear function of nodal values along that side.

Degree-one Polycons

- A linear function has three degrees of freedom: $a_1 + a_2 x + a_3 y$. On a line there is a linear relationship between x and y so that a linear form has only two degrees of freedom on a linear side. Vertex nodes suffice.
- On a conic side three nodes (not all on any line) are required. Any node along the side suffices. We designate the side node on (j,j+1) as node j+1/2.
- Degree-one approximation requires m nodes.

Polycon Wedge Numerators

- The adjoint is of maximal order m-3. For degree-one approximation each numerator is of order m-2
- The numerator at side node j+1/2 is the $k_{j+1/2}$ times the product of the m-2 sides other than side (j,j+1).
- At vertex j where (j-1,j) and (j,j+1) are lines the numerator is k_j times the product of the sides opposite vertex j.
- At all other vertex nodes adjacent factors must be introduced.

Adjacent Factors

- Let the numerator at vertex j be k_j F_j R_j where F is the usual opposite factor and R is the adjacent factor.
- If a line and a conic meet at vertex j, then they also meet at EIP s. Node p is on the conic side. The linear form (s;p) is the adjacent factor R_j at vertex j.
- If both adjacent sides are conics, there are three EIP in their divisor set: (p,q,r). The adjacent side nodes are s and t. R_j is the unique conic (p,q,r,s,t).

Choice of Side Nodes

- Choice of side nodes is arbitrary. The approximation may become unstable if a side node is chosen close to a vertex. In a pilot program the side nodes were chosen to be equidistant from adjacent vertices. This was determined as the intersection of the perpendicular bisector of the vertices with the conic side.

Degree One Verification

- The approximation to a linear function L is $P_{m-2}(L) / Q_{m-3}$ and $g = P_{m-2}(L) - L\,Q_{m-3}$ is a polynomial of maximal degree m-2 that vanishes on the boundary of order m. It must be the zero polynomial. Degree-one approximation is assured.
- The wedges are no longer positive throughout the element. Positivity of the adjoint has yet to be demonstrated. When $m < 6$, Q can have no closed interior loop and positivity is ensured.

Degrees of Freedom

- A polynomial of degree r has t degrees of freedom on a curve of order s. On each side sufficient nodes must be introduced to exhaust this freedom.
- For example, $\mathbf{P}_r$ has (r+1)(r+2)/2 d.o.f. The curve P_r intersects a linear side in r points. On a conic there are 2r intersections. These include one vertex so 2r-1 side nodes suffice. When r = 1 one side node is needed.

D. O. F. – 2

- For degree two on a conic, the number of intersections is $t = 4$, including the vertex node. Thus, three side nodes suffice,
- Degree three on a conic yields $t= 6$ and five side nodes suffice.
- In general, degree r on a conic requires $2r – 1$ side nodes.

Higher Order Rational Curves

- Rational curves of order greater than two have their full complement of singular points, none of which may lie on an element side.
- Singularities on curved sides are not EIP but must be included in the adjoint curve
- Thus the singularities of curved sides of order $r > 2$ add $(r-1)(r-2)/2$ conditions on the adjoint.

Vertex Adjacent Factor – 1

- The adjacent factor R at vertex i where sides of orders f and g intersect must be considered with great care when f or g is greater than 2.
- An order r approximation on a side of order f requires the smaller of rf – 1 and r(r+3)/2 - 1 side nodes. When fewer than rf-1 nodes are required, R must contain the rf - r(r+3)/2 exterior points where R intersects f.

Vertex Adjacent Factor – 2

- The number of interior nodes required for degree r interpolation if equal to the d.o.f. of a polynomial of degree r-3 or v = (r-1)(r-2)/2.
- At a vertex where sides of order f and g intersect the adjacent factor contains the fg-1 EIP of these sides plus r(f+g) -2 nodes other than vertices on these sides plus (f-1)(f-2)/2 plus (g-1)(g-2)/2 singular points of the sides for a total of
p = fg-3 + r(f+g) + (r-1)(r-2)/2 points +(f-1)(f-2)/2 + (g-1)(g-2)/2 conditions on the adjacent curve.

Vertex Adjacent Factor – 3

- The adjacent factor for approximation of order r where sides of order f and g meet at the vertex must satisfy these p conditions.
- The adjacent factor is of order $q = f + g + r - 3$ and is determined by $s = q(q+3)/2$ conditions.
- It is easily shown that $p = s$.
- A unique adjacent factor is thus assured at all vertices.

Interior Adjacent Factors

- There are $(r-1)(r-2)/2$ interior nodes.
- The opposite factor common to all is the boundary of order m. The adjacent factors are thus all of order $r - 3$ (when $r > 3$) and require $r(r-3)/2$ conditions.
- The adjacent factor at each interior node contains the other $(r-1)(r-2)/2 - 1 = r(r-3)/2$ interior nodes. It is thus uniquely determined.

Adjacent Factors of Side Nodes

- Order r approximation on a side of order f requires fr -1 nodes on the side and its extension. The opposite factor is the boundary excluding the side and is of order m – f. The adjacent factor contains the (r-1)(r-2)/2 interior nodes and the fr-2 other nodes on the side as well as the (f-1)(f-2)/2 singular conditions of side f.
- The adjacent factor is of order f + r – 3 and thus requires s = (f+r)(f+r-3)/2 conditions. This is equal to the conditions enumerated above.

Adjoint Lemma

- Let m be the lowest order of an element for which the adjoint is not positive throughout the element. Let E be a candidate for a well-set element of order m that is not regular.
- A lemma relating adjoints of adjacent elements has been proved. This lemma may be used to relate the adjoint of E to that of a well-set element of order less than E. All candidates thus far considered have been eliminated with this lemma.

Polycon Examples

- A 3-con of order 4 is a simple example. There are two EIP on the conic which define a unique linear Q.
- A 4-con of order 6 with two adjacent conics has nine EIP which define a unique cubic Q. Three EIP are in the divisor of the two conic sides. Six more are in the divisors of the conics with the linear sides.
- The GADJ algorithm has been generalized to polypols.

Generalized GADJ

- Consider the variation along conic side (j,j+1/2,j+1). The barycentric coordinates for the triangle with the three nodes as vertices are: B_1 = $(j+1/2,j+1)_j$, B_2 = $(j;j+1)_{j+1/2}$, and B_3 = $(j;j+1/2)_{j+1}$. These three linear functions sum to unity.
- Parameters c are computed from congruences on the conic side. After dividing out common factors, the variation on the side is $k_j\, c_j\, B_1+k_2c_2B_2+k_3c_3B_3$.

Generalized GADJ – 2

- The recursion formulas for the k are now: $k_{j+1/2} = k_j c_j / c_{j+1/2}$ and $k_{j+1} = k_j c_j / c_{j+1}$.
- The GADJ algorithm is developed in the paper by Dasgupta and Wachspress in CAMWA 55 (2008) pages 1988-1997.
- EIP generated by divisors of adjacent sides only are required for determining adjacent factors for the numerators, which in turn yield the c parameters.
- The unique adjoint Q is then computed easily.

GADJ – 3

- The values of c_j and c_{j+1} may be determined by cancelling common factors in wedge numerators $\mathbf{N}_j$ and $\mathbf{N}_{j+1}$ and reducing remaining factors by congruences along side (j,j+1). This establishes validity of the construction.
- In practice, on linear side (j;j+1) one may just compute $c_j = [\mathbf{N}_j /(j+1;j+2)]_j$ and $c_{j+1} = [\mathbf{N}_{j+1} /(j-1;j)]_{j+1}$ where $k_{j+1} = c_j\, k_j /c_{j+1}$.
- On curved side (j,j+1/2,j+1), $c_{j+1/2} = [\mathbf{N}_{j+1/2} /(j;j+1)]_{j+1/2}$ and $c_j = [\mathbf{N}_j /(j+1/2;j+1)]_j$ with $k_{j+1/2} = c_j\, k_j /c_{j+1/2}$. For k_{j+1} we compute $c_{j+1} = [\mathbf{N}_{j+1} /(j:j+1/2)]_{j+1}$ with $k_{j+1} = c_j /c_{j+1}$.

Finite Element Integrations

- Structural problems which play a significant role in application often require two basic integrations over each element: $a_{jk} = W_j W_k$ and $b_{jk} = \text{grad } W_j \cdot \text{grad } W_k$ for all j and k.
- For triangles and rectangles with polynomial wedges simple quadrature formulas suffice
- For rational wedges and more general element geometry integration is more complex

Integration – 2

- Dasgupta resolved geometry problems through use of Gauss's divergence theorem to reduce area integrals to contour integrals around element boundaries.
- An integrand f is integrated with respect to y to yield a function g so that f = div(g**j**) integrated over the element reduces to the integral of g dx clockwise around the boundary.

Integration – 3

- On a linear side $y = ax + b$ and the integration is of a rational function of x
- On a conic side the rational parametrized form $x = r(t)$ and $y = s(t)$ and $dx = r'dt$ may be used. The integrands are then rational.
- Analytic integration is simplified through use of programs like Mathematica

Integration – 4

- Accuracy may be lost when integrals evaluated for each side are summed.
- Dasgupta resolved this by simplifying the analytic sums before evaluating them.
- Thus, accurate analytic integrations in global coordinates replaces numerical quadrature in local isoparametric coordinates.

Integration – 5

- Parametrization of conic and higher order sides leads to integrands with large degree polynomials in numerators and denominators as well as logarithms times these polynomials.
- Alternative Gaussian quadrature is simple.
- 9-point Gaussian quadrature was adequate for a set of test problems.
- This is efficient in a parallel environment.

Integration – 6

- Arithmetic is so rapid in modern computers that simplicity of programming with associated human effort and accuracy of results is far more crucial than number of flops.
- Relative merits of Gaussian quadrature and analytic contour integration must be addressed in this context.

Reverse Isoparametrics

- One can use the global degree-two basis functions for any n-pol to map it into a regular polygon of order n.
- This maps (x,y) into (xi,eta). The jacobian is found easily so that integrations can be performed on the n-gon.
- Simple quadrature formulas are readily found.
- The crucial stage is determining (x,y) at the quadrature nodes.

Convex Polyhedra

- A convex polyhedron is a well-set 3D element. In the initial work (cf. my 1975 book on A Rational Finite Element Basis) only vertices of order three were allowed. When a vertex is of order greater than three the element appears to be ill-set in that the adjoint contains the vertex. It is now known that the singularity there is removable. A vertex may be of any order.

3D GADJ

- We consider first n-gon elements where all vertices are of order three.
- The numerator at vertex j is k_j $\mathbf{F}_j$ where F_j is the product of the n-3 faces opposite vertex j and k_j is to be determined by GADJ.
- The recursion formula now is simply $k_{j+1} = k_j \, \mathbf{P}_j\,(j+1) \,/\, \mathbf{P}_{j+1}\,(j)$ where $\mathbf{P}_j\,(j+1)$ is the plane at j+1 not passing through j evaluated at j and $\mathbf{P}_{j+1}\,(j)$ is the plane at j not passing through j+1 evaluated at j+1.

Vertices of Order Greater than 3

- Joe Warren (Rice) recognized that the singularity is removable and developed an alternative construction of wedges.
- Based on Joe's work, I generalized my construction in [Dasgupta and Wachspress, CAMWA **55** 2008]
- At a vertex of order r one introduces an adjacent factor of order r-3. The GADJ algorithm is generalized to include such factors and generates a unique Q of maximal order n-4.

Concave Elements

- Concave 2D elements were addressed in my 1975 book
- In general these elements were projections of convex 3D elements
- Square roots appeared in these elements
- Further analysis was given more recently in Dasgupta and Wachspress, 2004
- Dasgupta has explored further.

Polytopes – 1

- A simple polytope of dimension d and order n has vertices of order d where d hyperplanes meet. Barycentric coordinates may be generated. The numerator at node j is the product of the n-d hyperplanes opposite vertex j. The denominator is determined by the GADJ algorithm.
- We consider only convex polytopes.

Polytopes – 2

- The edge connecting nodes j and j+1 is common to d-1 of the hyperplanes at j and j+1
- The one hyperplane thru j not shared with j+1 is P_(j,j+1).
- The one hyperplane not shared with j at j+1 is P_(j+1,j).
- The GADJ recursion is k_(j+1) = k_j[P_(j+1,j) at j]/[P_(j,j+1) at j+1].

Polytopes – 3

- A convex polytope of dimension d may have vertices of any order greater than d. Just as for d = 3, barycentric coordinates may be generated by introducing adjacent factors.
- At vertex j of order p with p > d, the adjacent factor is of order p – d and has a singularity of order p – d at j.