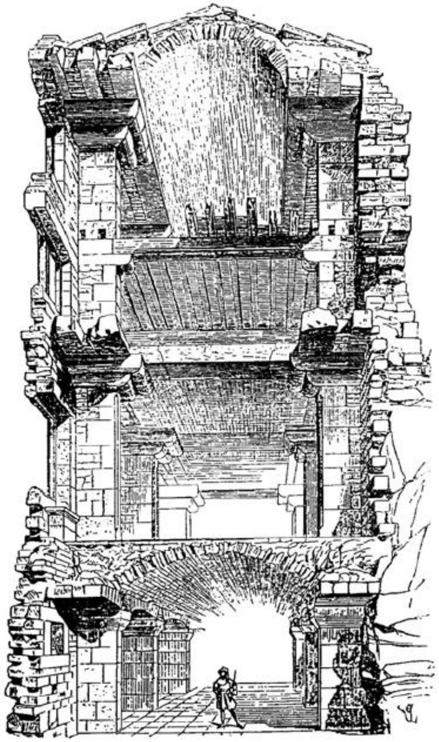
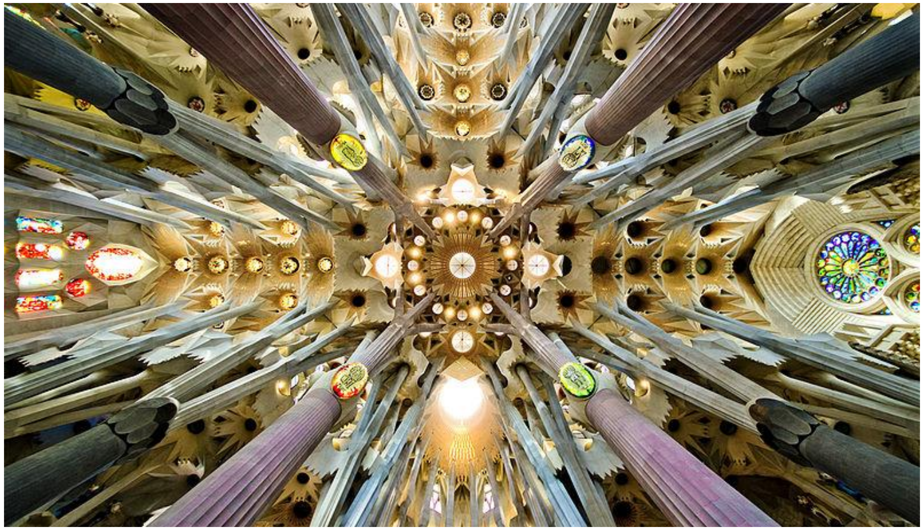
Design of Self-supporting Surfaces

Etienne Vouga	Columbia University
Mathias Höbinger	Evolute / TU Wien
Johannes Wallner	TU Graz / TU Wien
Helmut Pottmann	KAUST



[Viollet-le-Duc 1858]



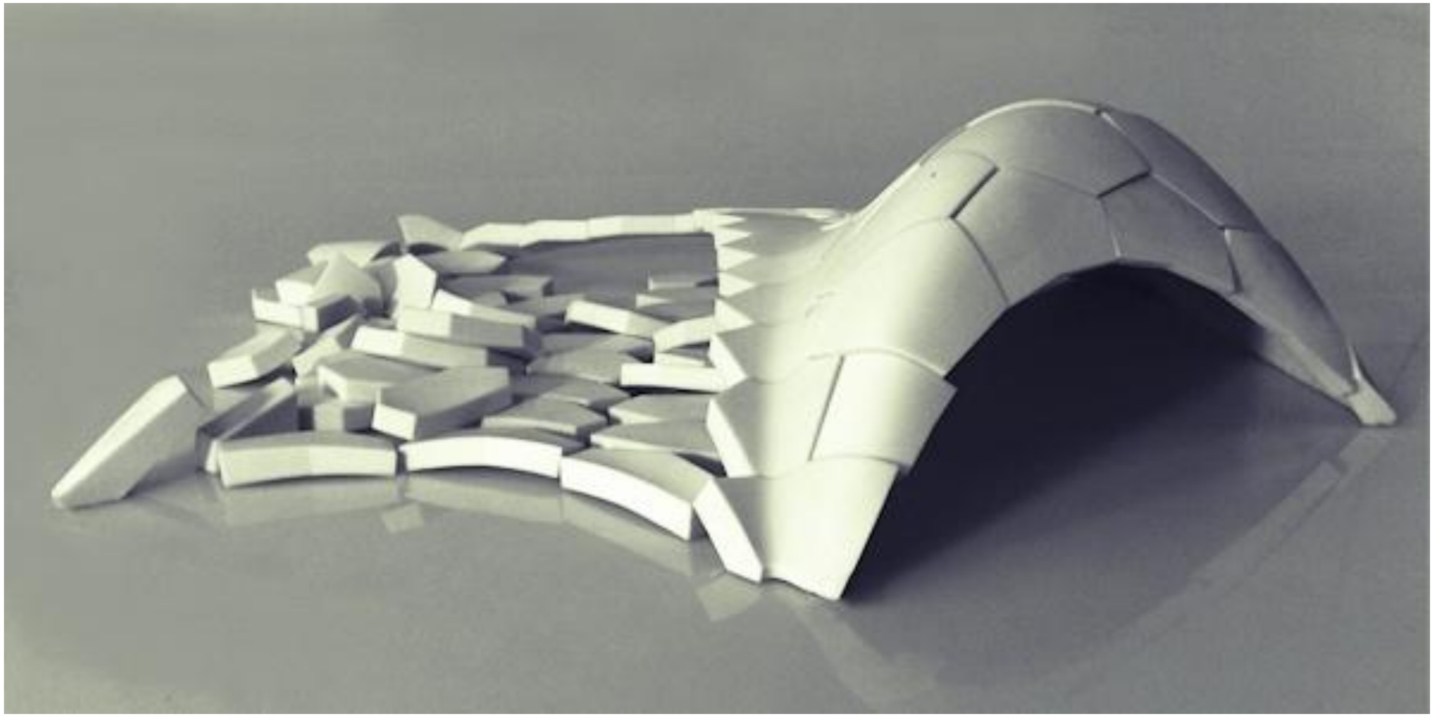
[Gaudí. Photo by Flikr user “SBA73”]

[Ramage et al 2008]

[Sobek 2008]

Why Masonry?

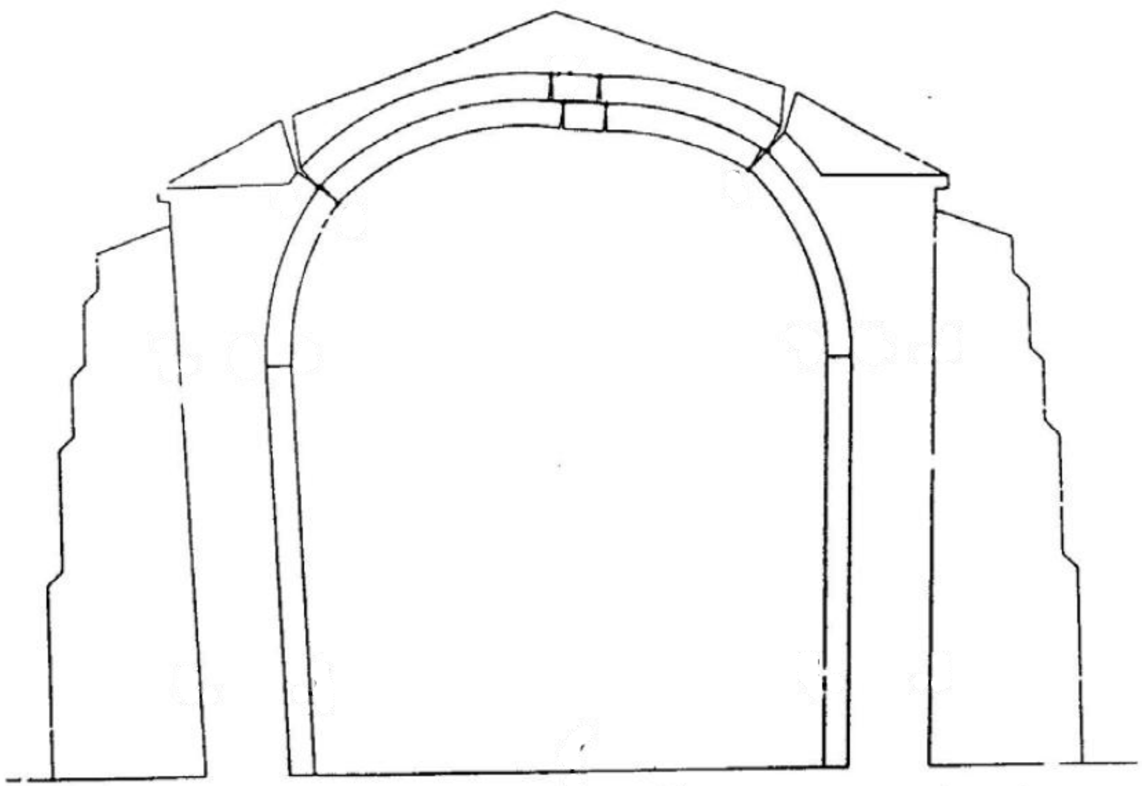
Material failure is unimportant: masonry structures fail due to *bad geometry*



[Philippe Block]

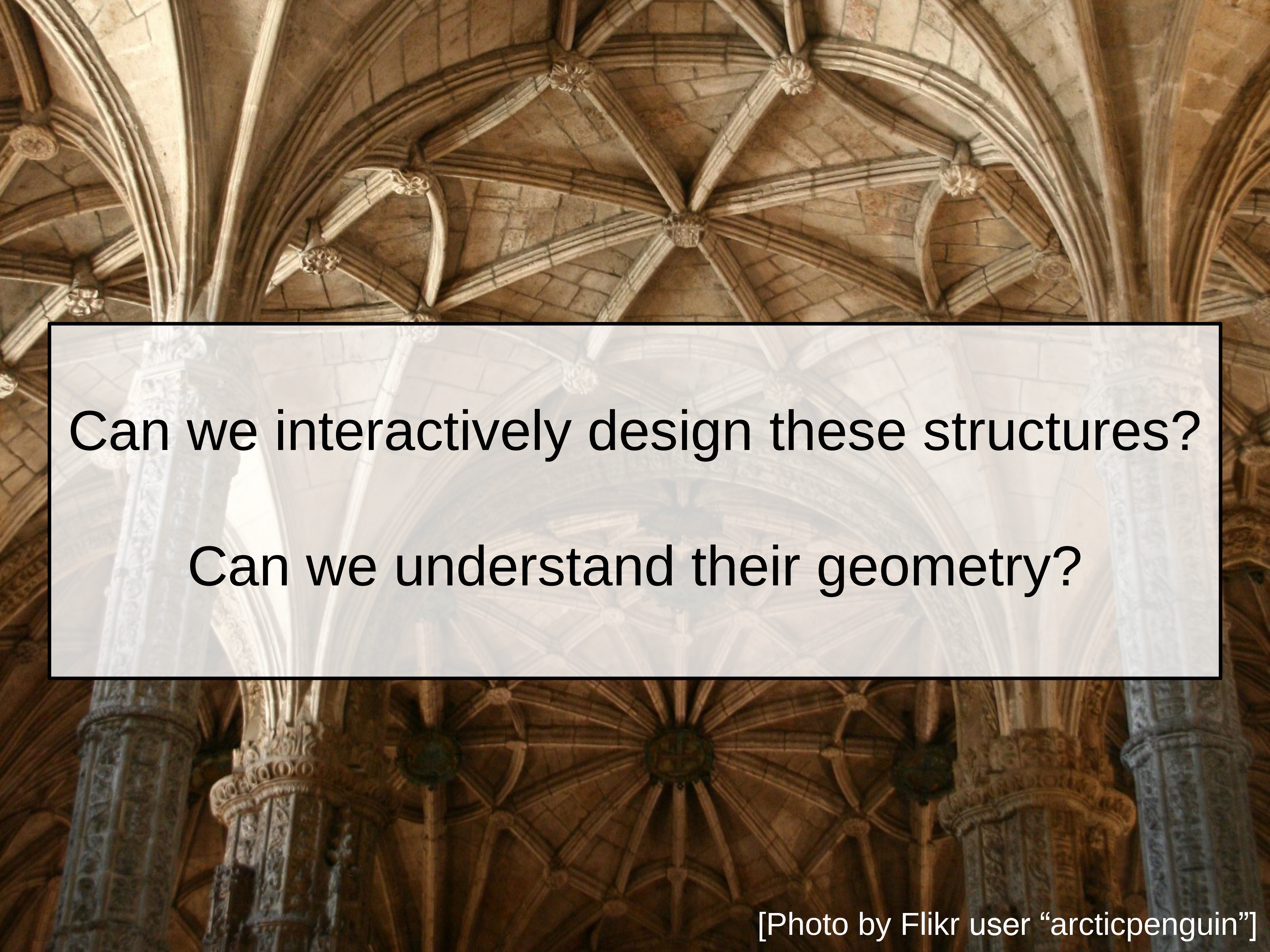
Why Masonry?

"Small elastic displacements" incorrect deformation model



[Huerta and López 1997]

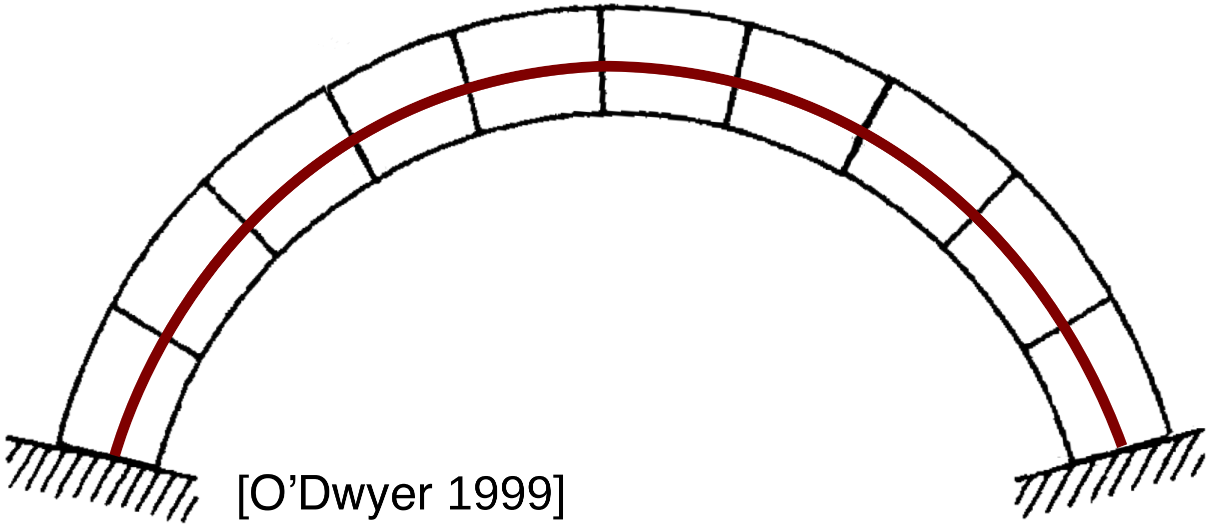
[Photo by Flikr user “arcticpenguin”]



[Photo by Flikr user “arcticpenguin”]

Safe Theorem [Heyman]

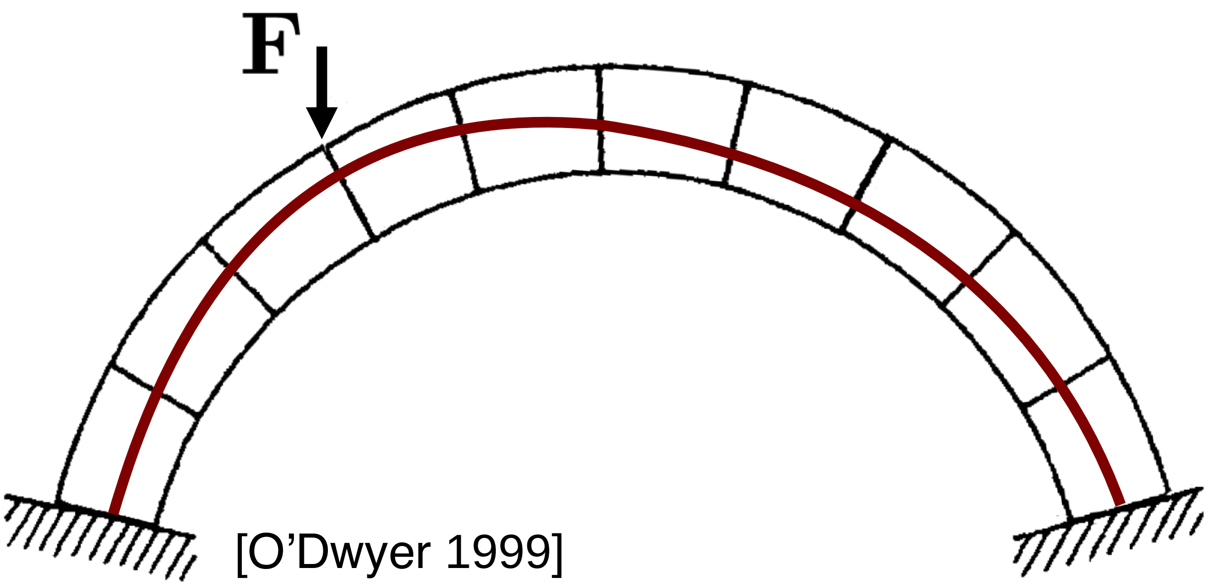
If embedded membrane
is stable, so is structure



[O'Dwyer 1999]

Safe Theorem [Heyman]

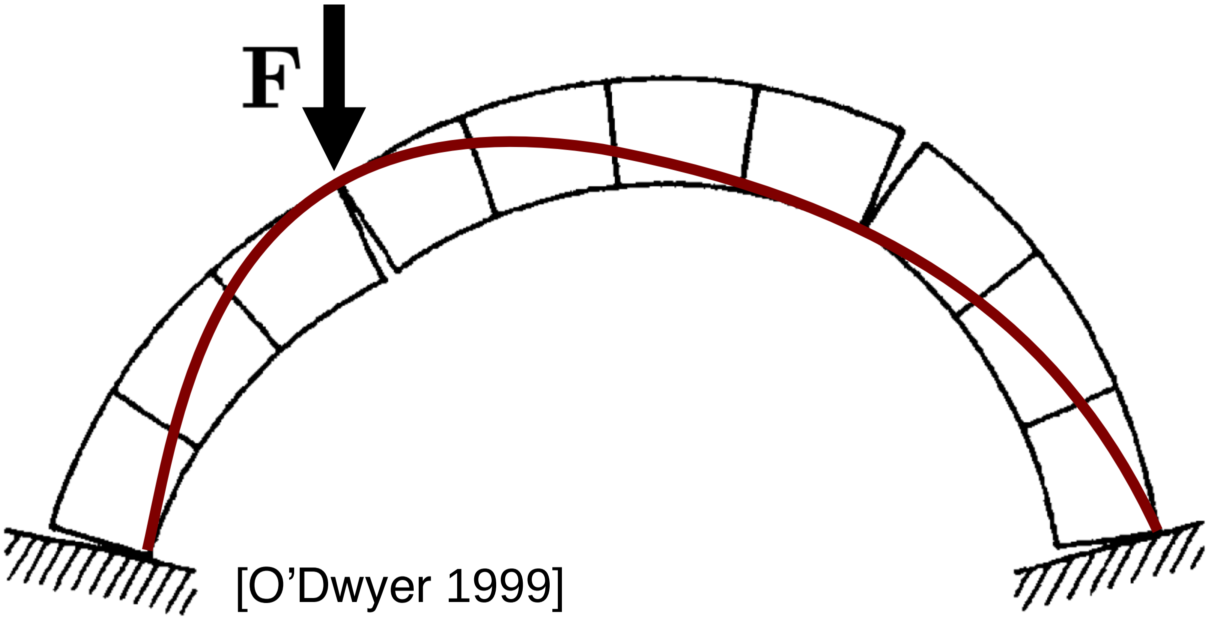
If embedded membrane
is stable, so is structure



[O'Dwyer 1999]

Safe Theorem [Heyman]

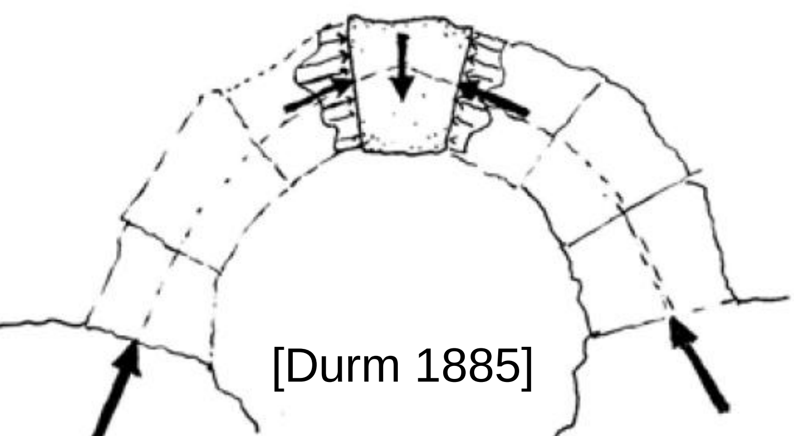
If embedded membrane
is stable, so is structure

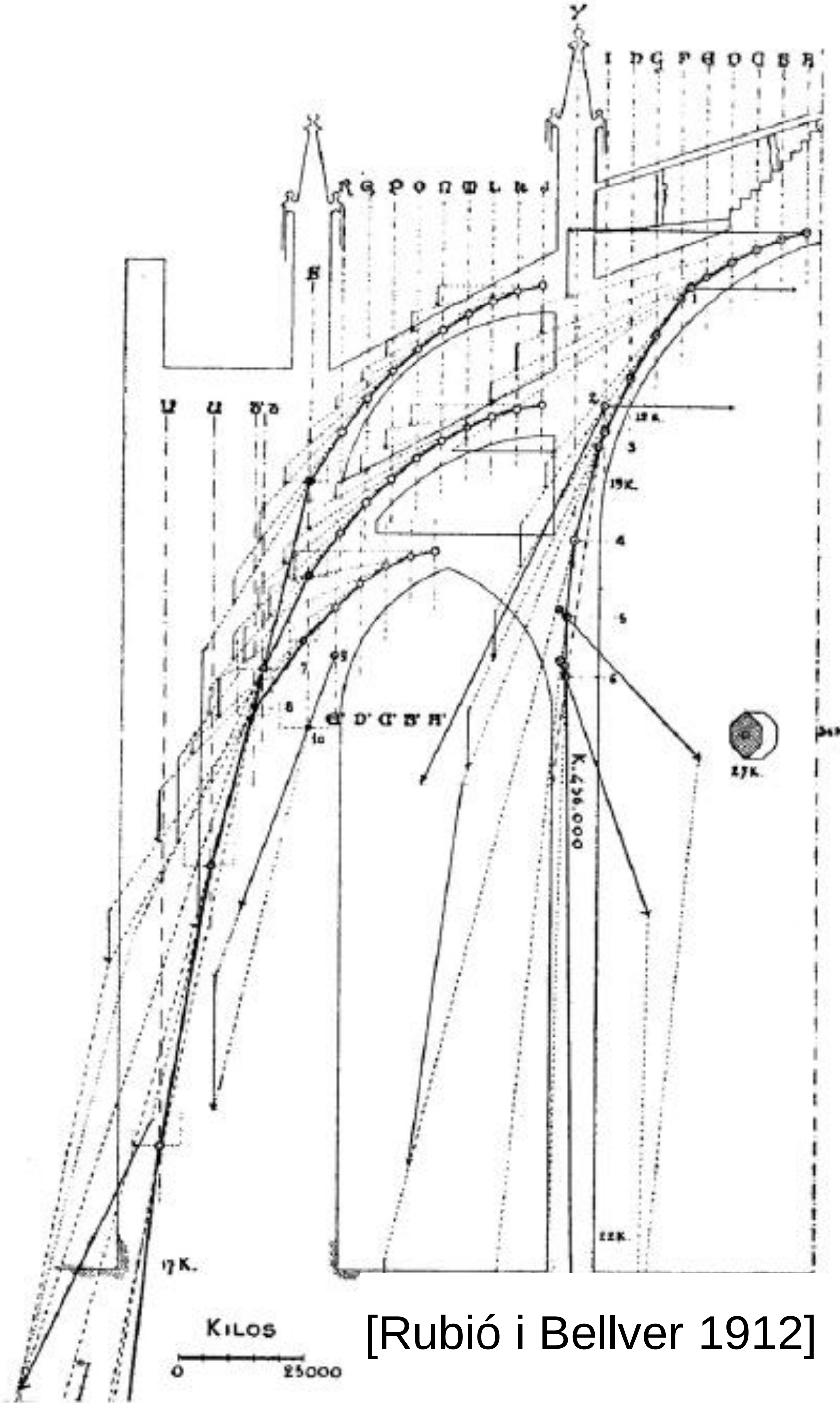


[O’Dwyer 1999]

Safe Theorem [Heyman]

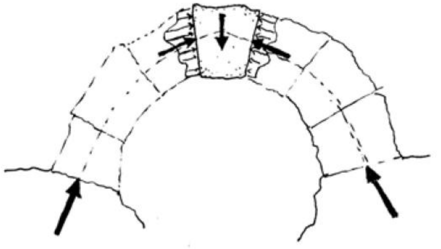
If embedded membrane is stable, so is structure


[Durm 1885]


[Rubió i Bellver 1912]

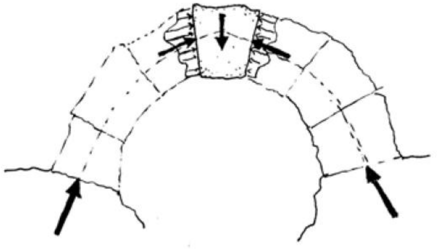
Modeling Assumptions

Safe Theorem



Modeling Assumptions

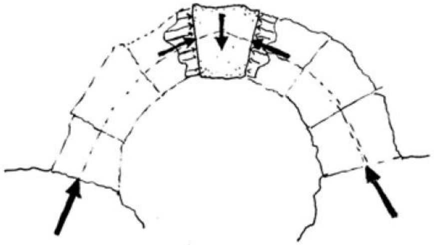
Safe Theorem



Infinite compressive, no tensile strength

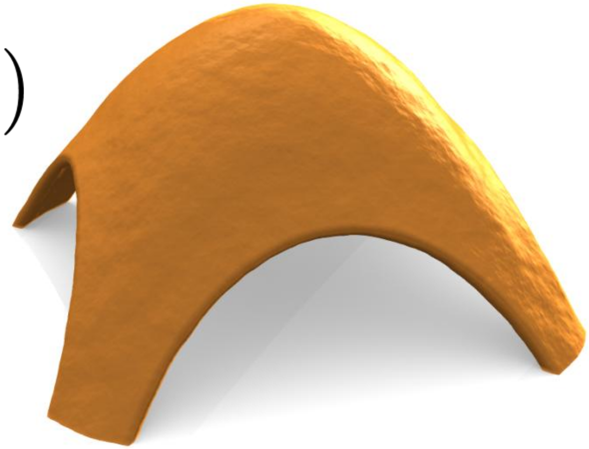
Modeling Assumptions

Safe Theorem



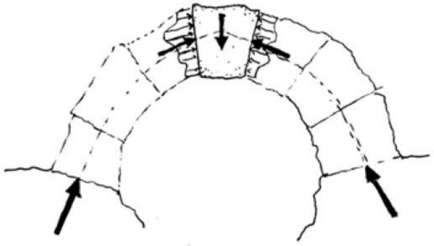
Infinite compressive, no tensile strength

Surface is height field $z(x, y)$



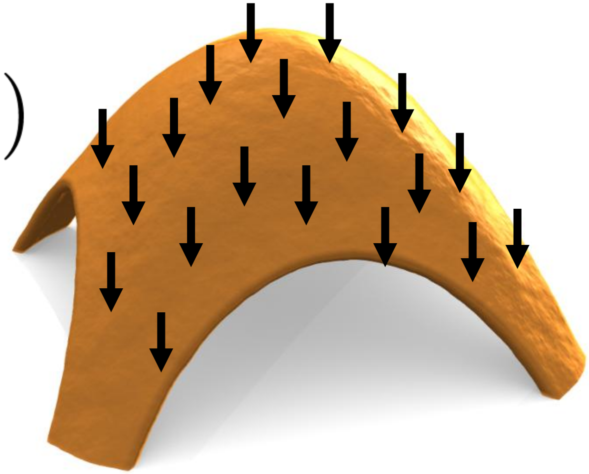
Modeling Assumptions

Safe Theorem



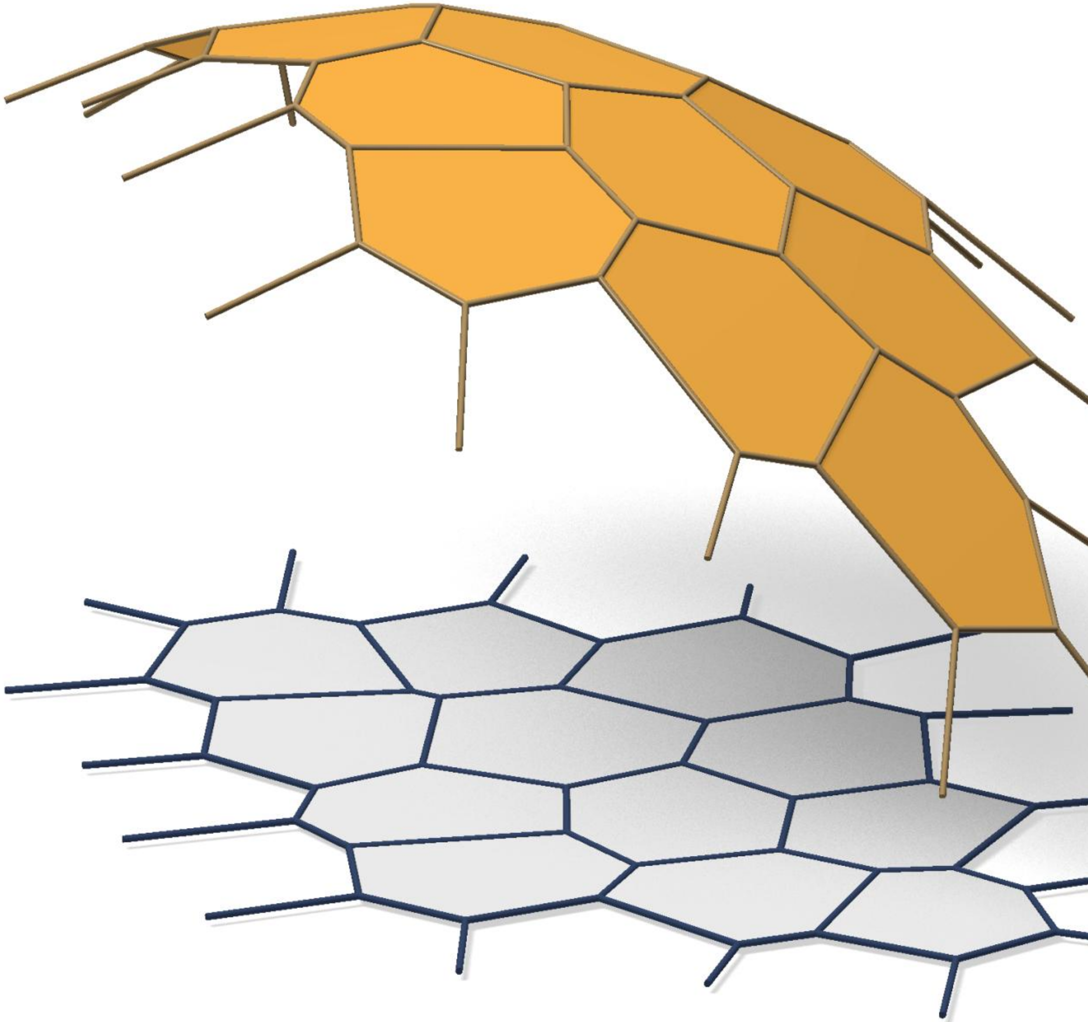
Infinite compressive, no tensile strength

Surface is height field $z(x, y)$

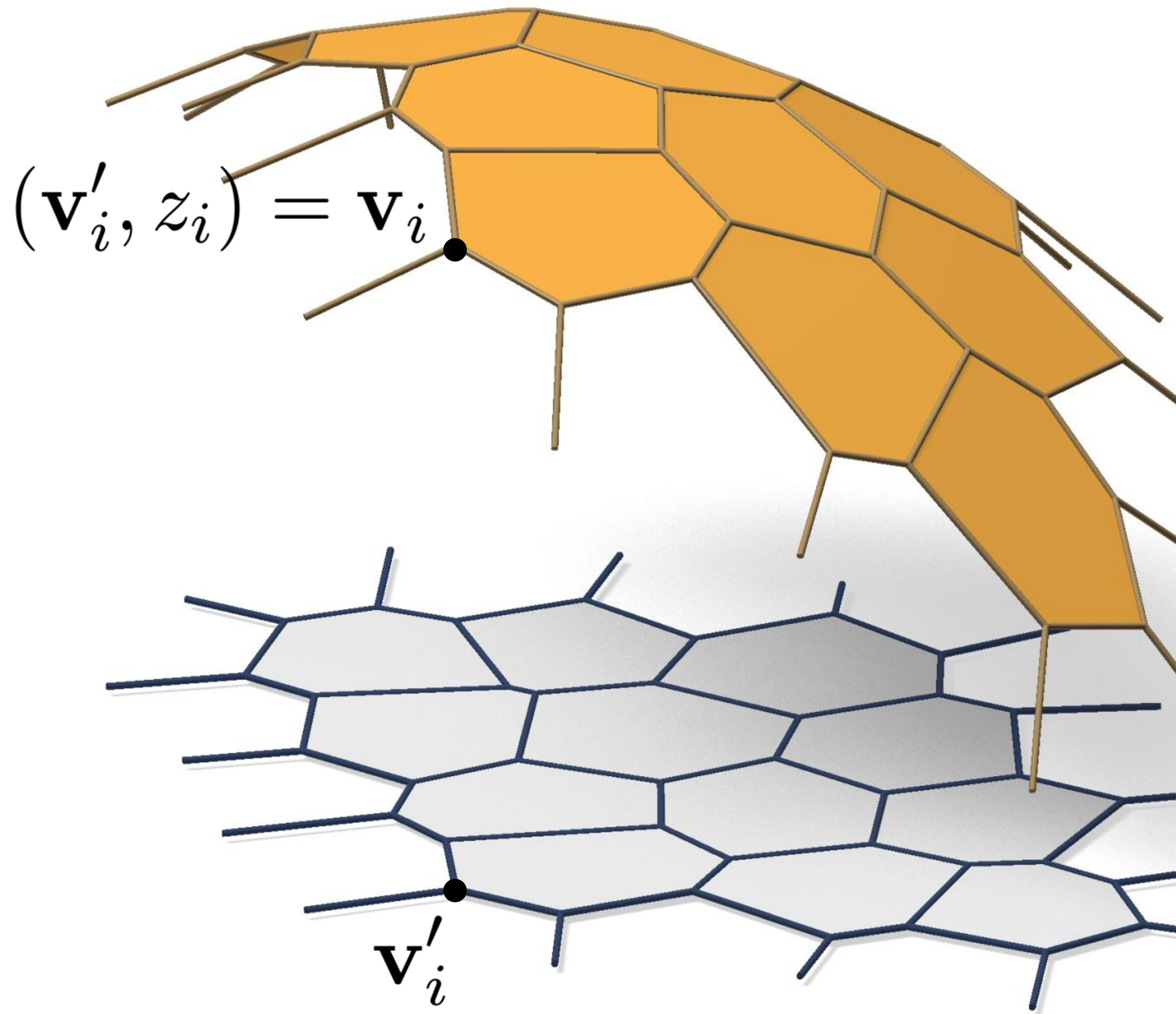


Loads are vertical

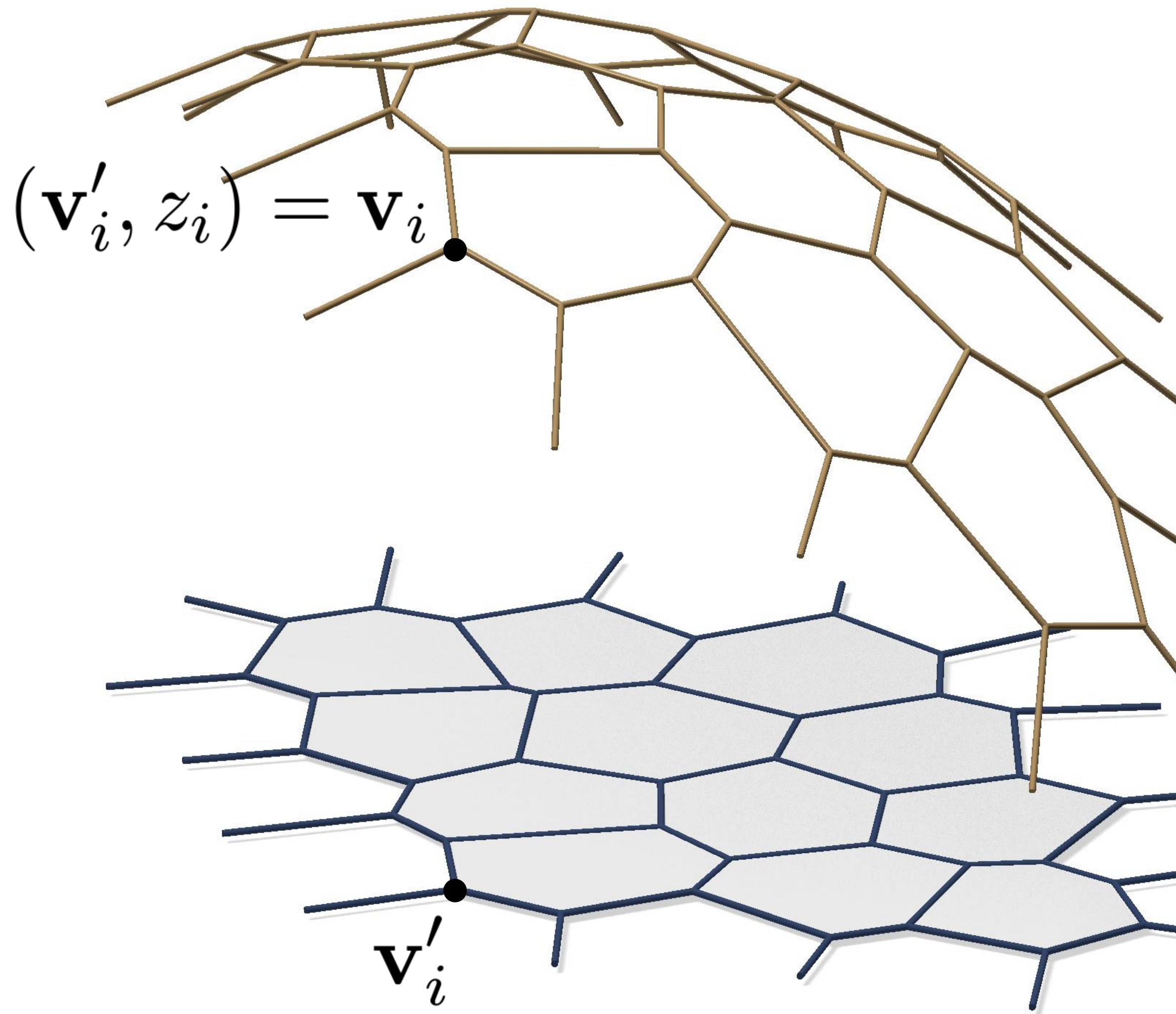
Discrete Thrust Network [Block 2009]



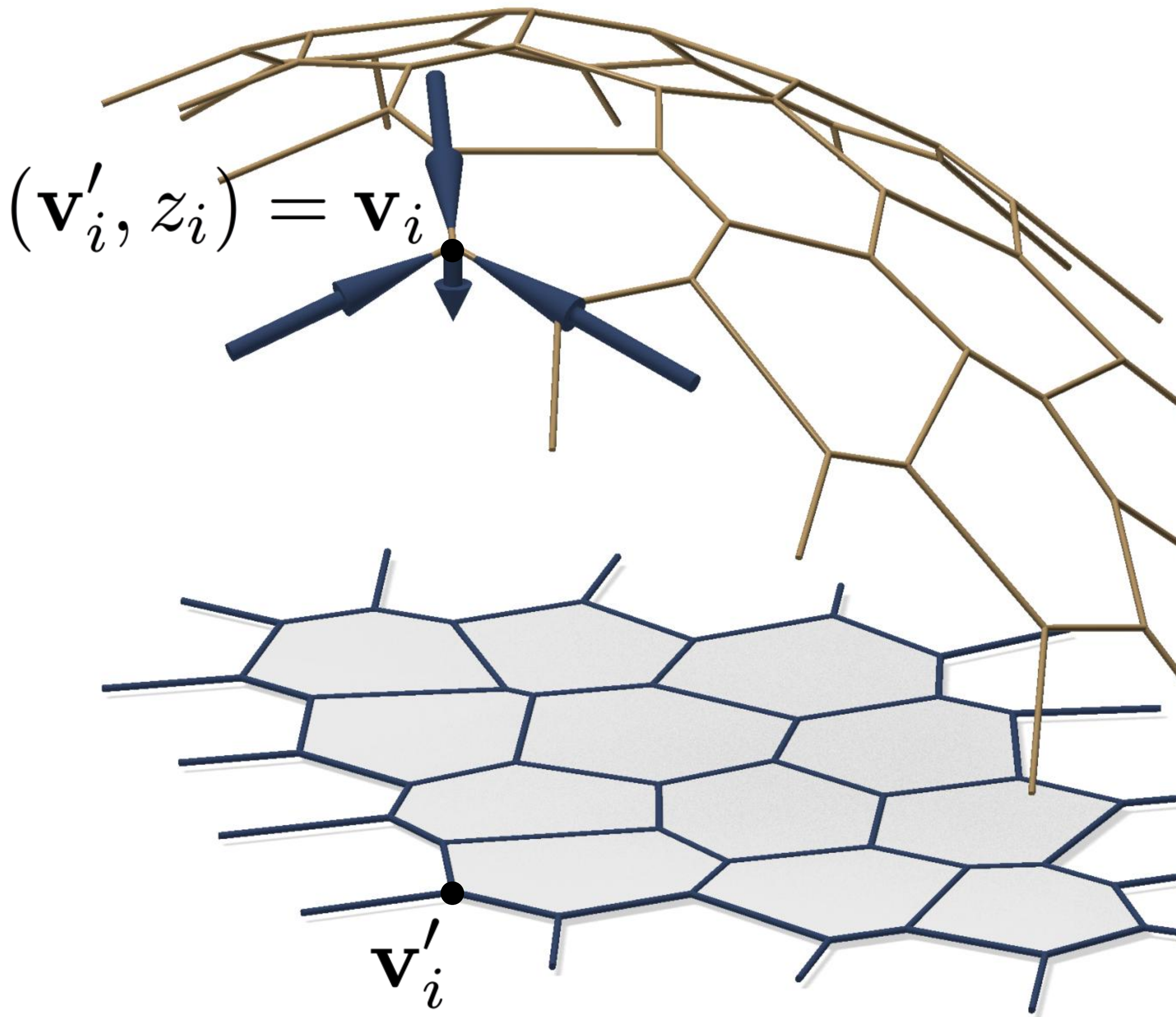
Discrete Thrust Network [Block 2009]



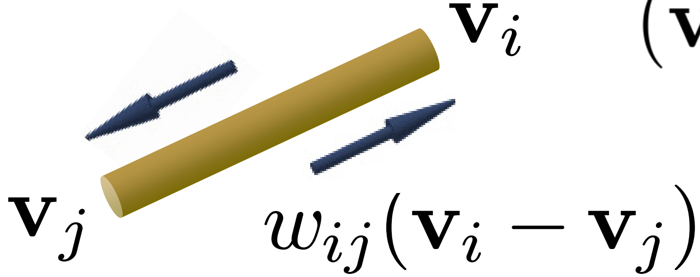
Discrete Thrust Network [Block 2009]



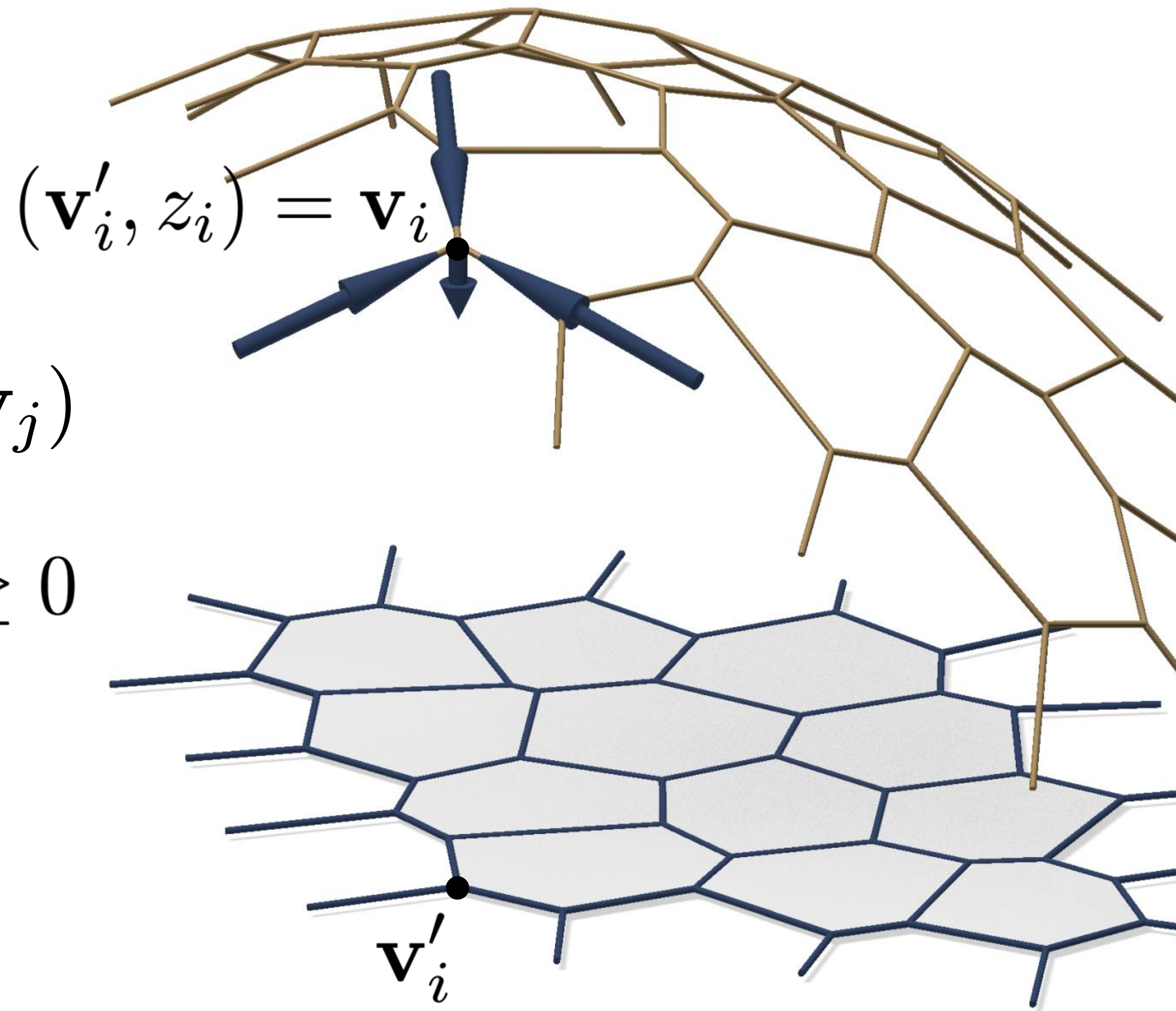
Discrete Thrust Network [Block 2009]



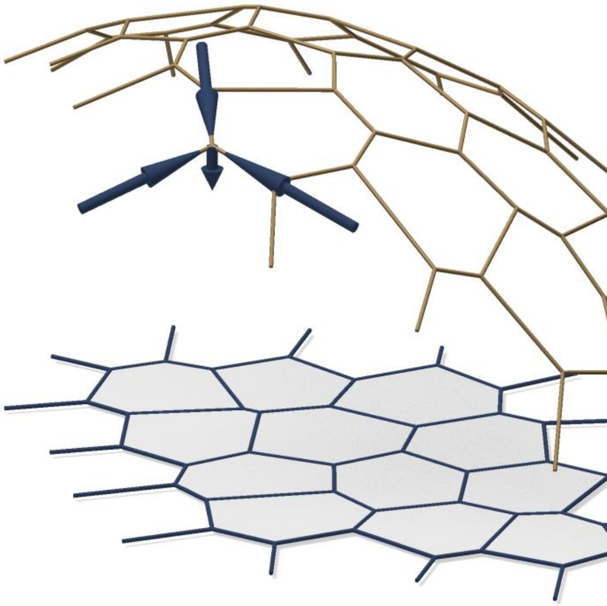
Discrete Thrust Network [Block 2009]



compressive: $w_{ij} \geq 0$



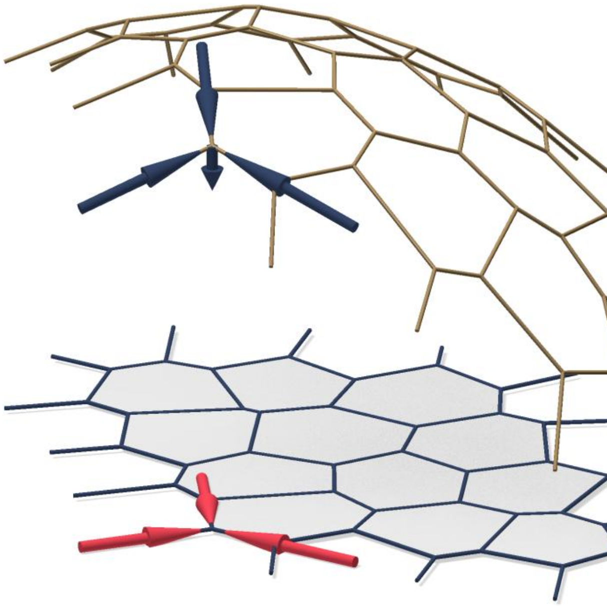
Equilibrium Equations



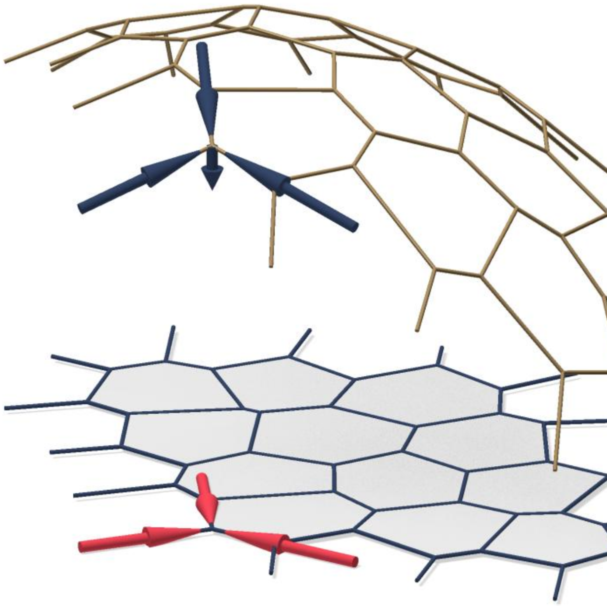
Equilibrium Equations

Horizontal equilibrium:

$$\sum_{j\sim i} w_{ij}(\mathbf{v}'_i - \mathbf{v}'_j) \ = \ 0$$



Equilibrium Equations



Horizontal equilibrium:

$$\sum_{j \sim i} w_{ij}(\mathbf{v}'_i - \mathbf{v}'_j) \;=\; 0$$

Vertical equilibrium:

gravity of lumped mass

$$\sum_{j \sim i} w_{ij}(z_i - z_j) \;=\; -A_i F_i$$

Historical Perspective

Photo: Flikr user "tillnm"

Antoni Gaudí
1852-1926

Historical Perspective

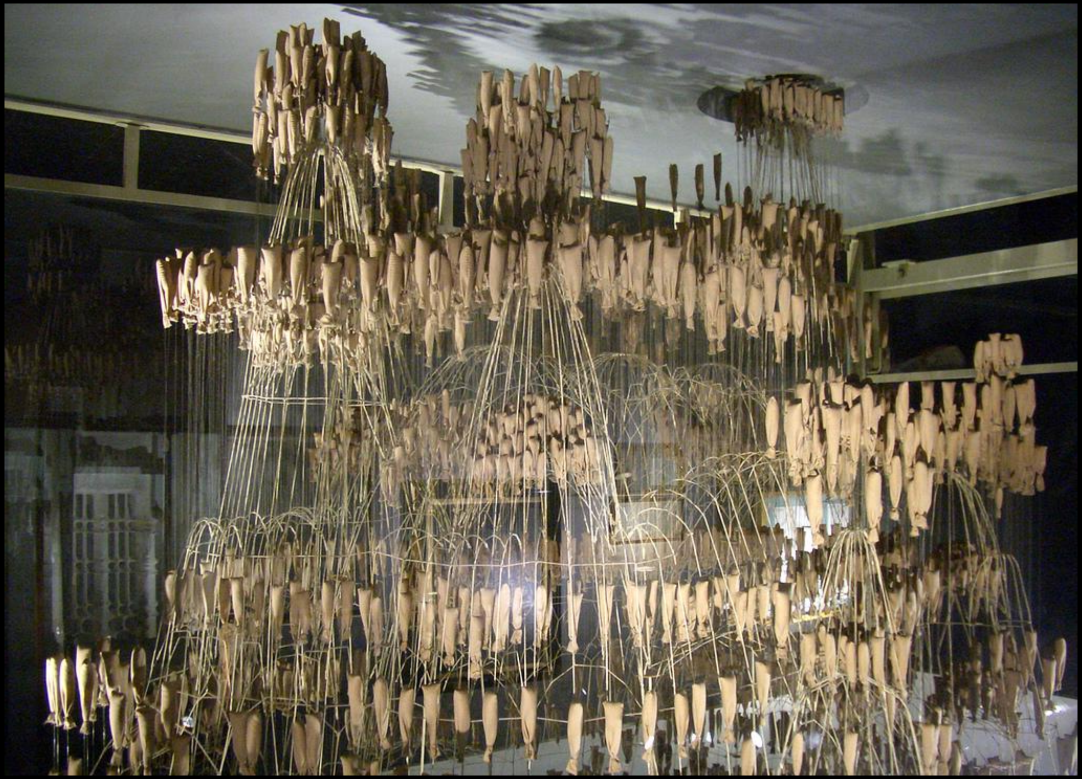
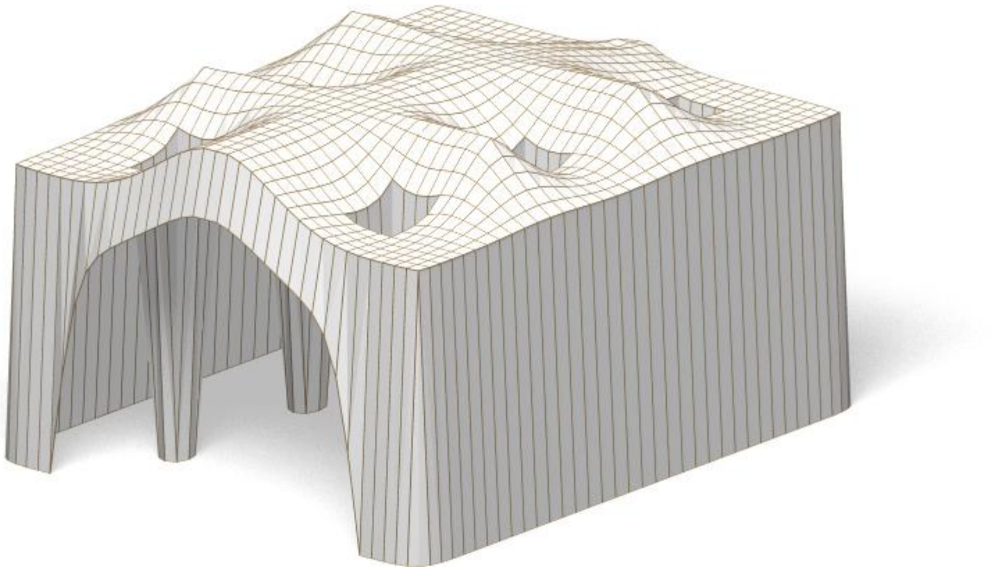


Photo: Flikr user “tillnm”

Colonia Güell, Gaudí

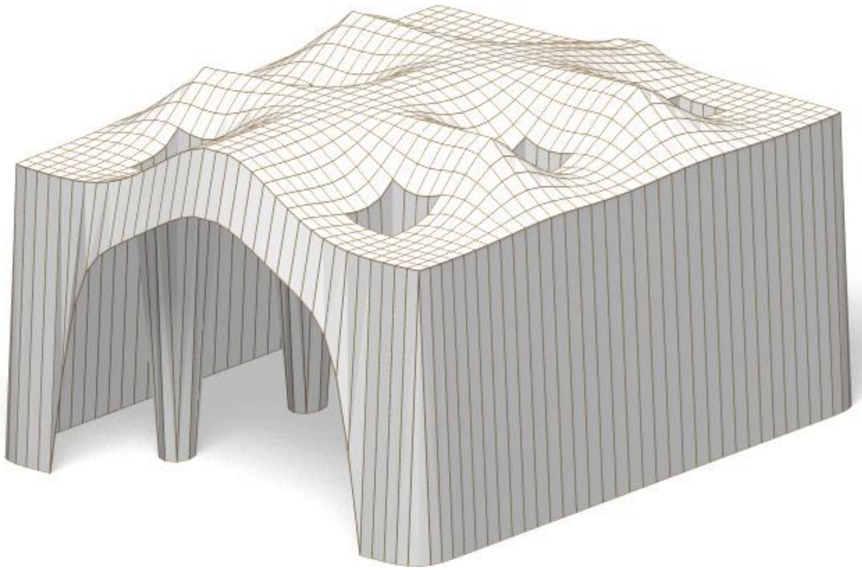
Design Paradigm

Unstable initial design



Design Paradigm

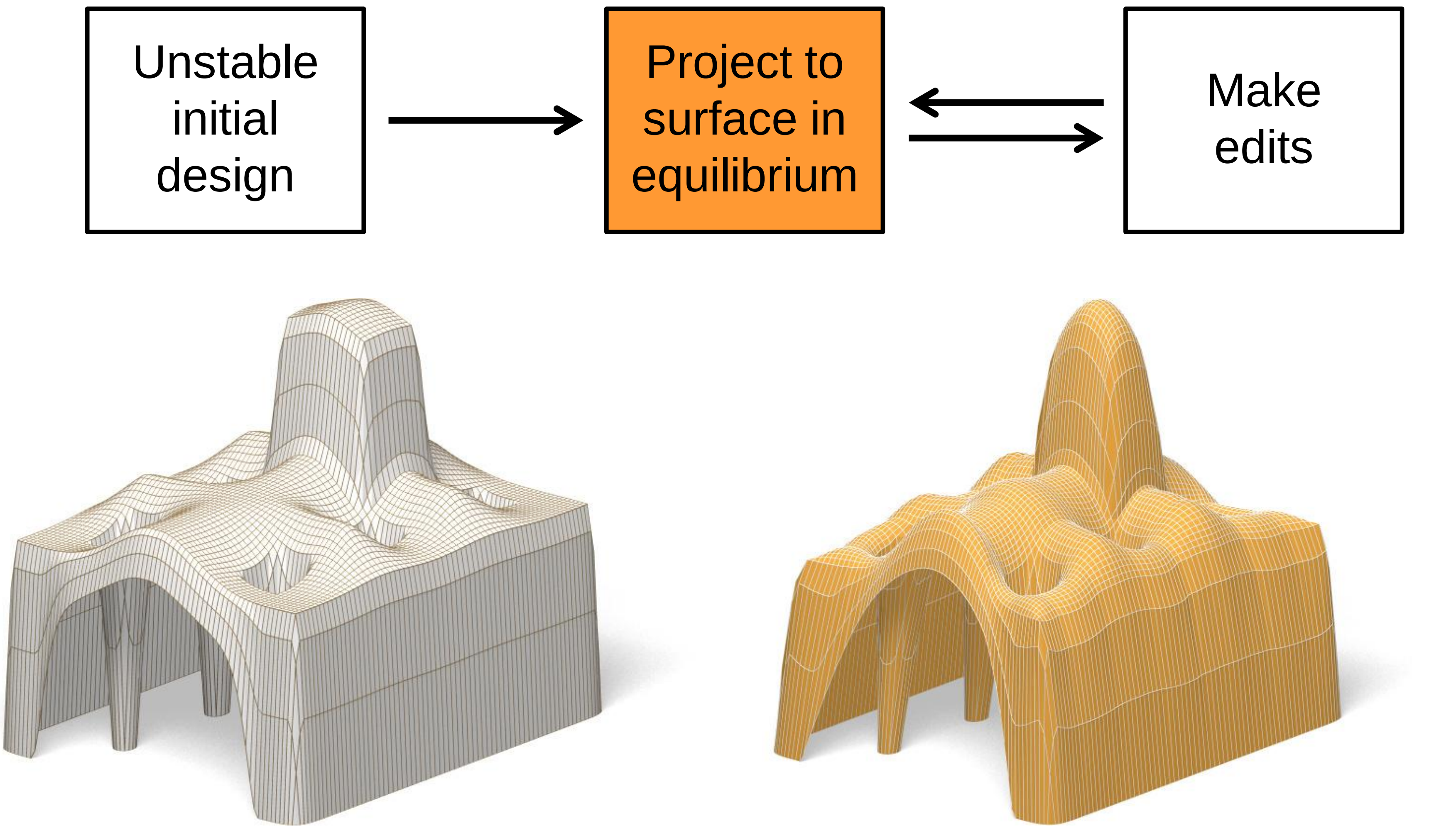
Unstable initial design → Project to surface in equilibrium



Design Paradigm

Unstable initial design → Project to surface in equilibrium → Make edits

Design Paradigm



Surface Projection

recall:

$$\sum w_{ij}(z_i - z_j) = -A_i F_i$$

$$\sum w_{ij}(\mathbf{v}'_i - \mathbf{v}'_j) = 0$$

$$w_{ij} \geq 0$$

Until convergence:

nonlinear in positions

1. Calculate vertical forces $A_i F_i$

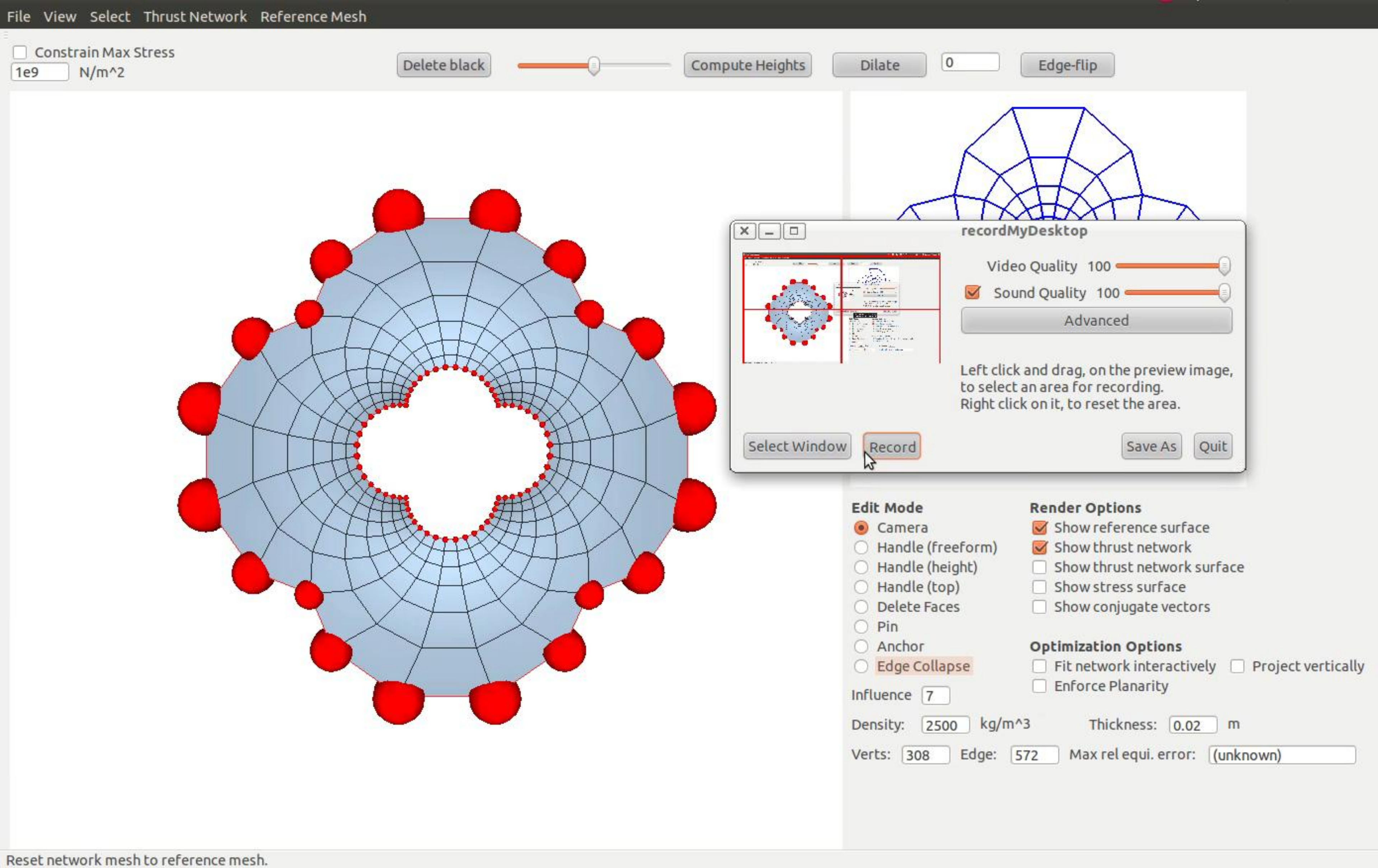
2. With positions fixed, find best weights

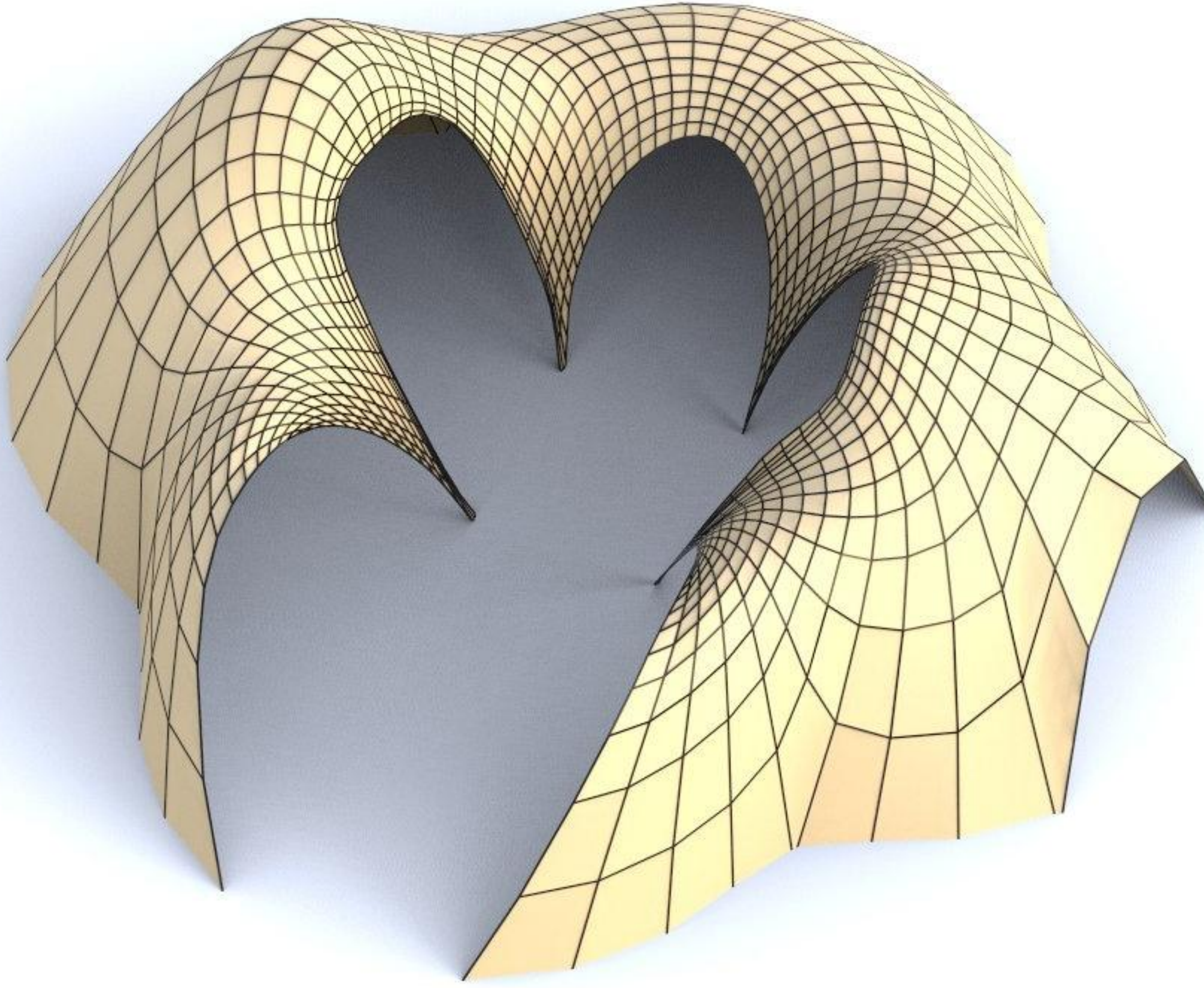
$$\min_{w_{ij}} \|\mathbf{F}_{\text{net}}\|^2 \qquad \text{s.t.} \qquad w_{ij} \geq 0$$

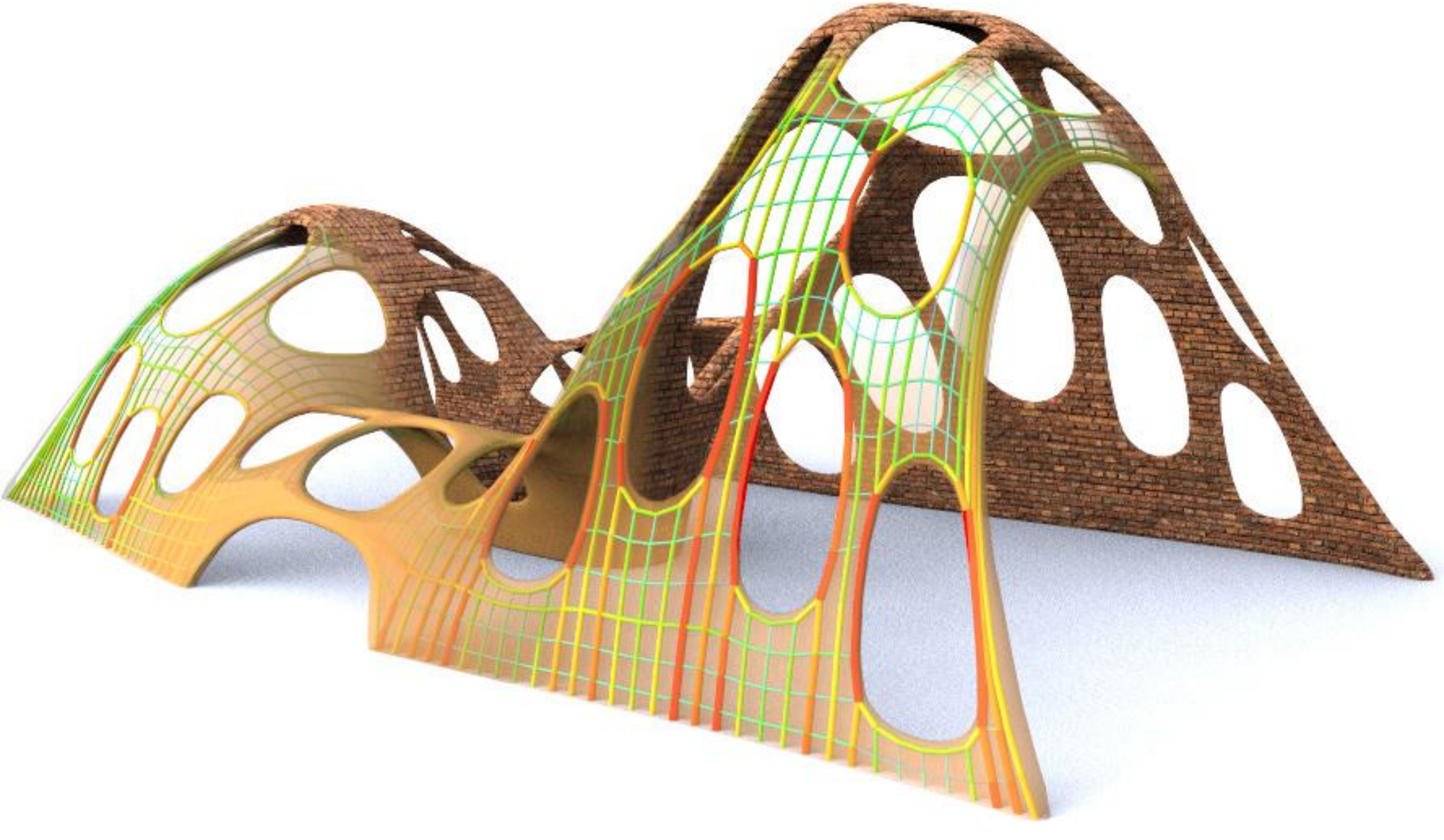
3. With weights fixed, perturb positions

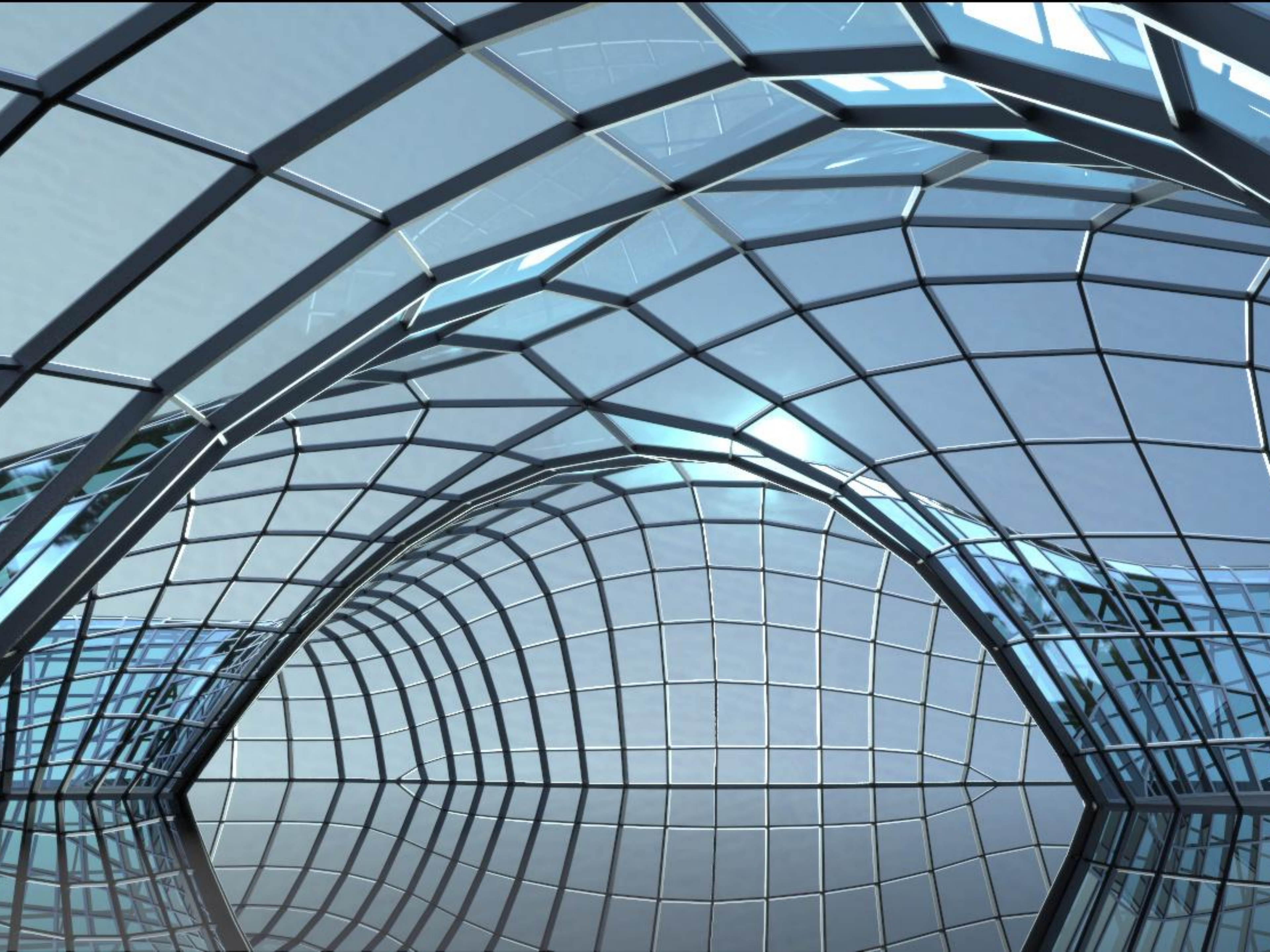
$$\min_{\mathbf{u}} \|\mathbf{F}_{\text{net}}\|^2 + \alpha\|\mathbf{u}\|^2 + \beta\|\mathbf{u}\cdot\hat{\mathbf{n}}\|^2$$

penalize total and normal motion


File View Select Thrust Network Reference Mesh
Constrain Max Stress
1e9 N/m^2
Delete black
Compute Heights
Dilate
0
Edge-flip
recordMyDesktop
Video Quality 100
Sound Quality 100
Advanced
Left click and drag, on the preview image, to select an area for recording.
Right click on it, to reset the area.
Select Window
Record
Save As
Quit
Edit Mode
Camera
Handle (freeform)
Handle (height)
Handle (top)
Delete Faces
Pin
Anchor
Edge Collapse
Render Options
Show reference surface
Show thrust network
Show thrust network surface
Show stress surface
Show conjugate vectors
Optimization Options
Fit network interactively
Project vertically
Enforce Planarity
Influence 7
Density: 2500 kg/m^3
Thickness: 0.02 m
Verts: 308 Edge: 572 Max rel equi. error: (unknown)
Reset network mesh to reference mesh.

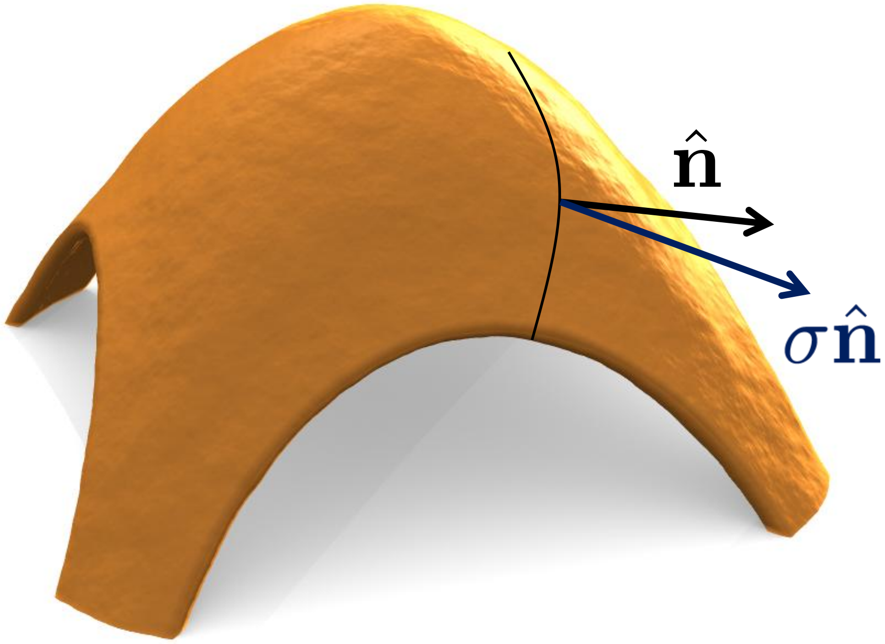






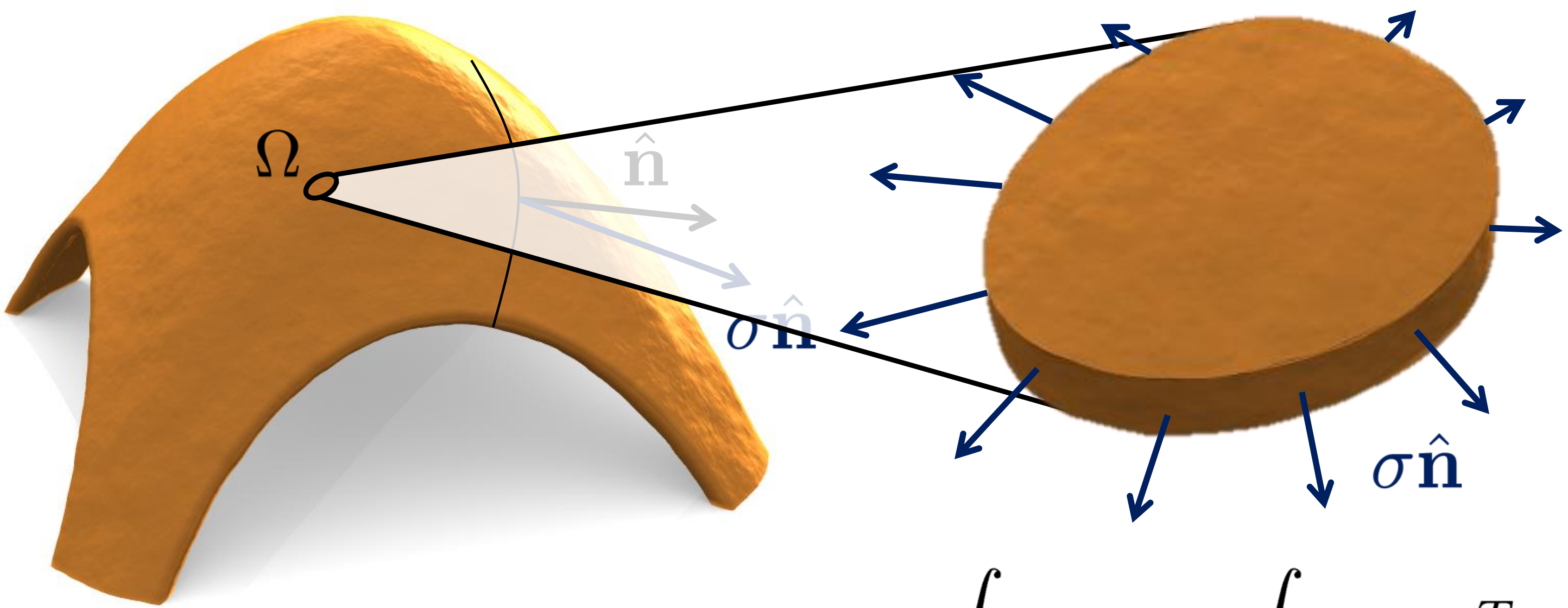
Stress Tensor

3 x 2 matrix σ measuring internal force



Membrane Equilibrium

3 x 2 matrix σ measuring internal force



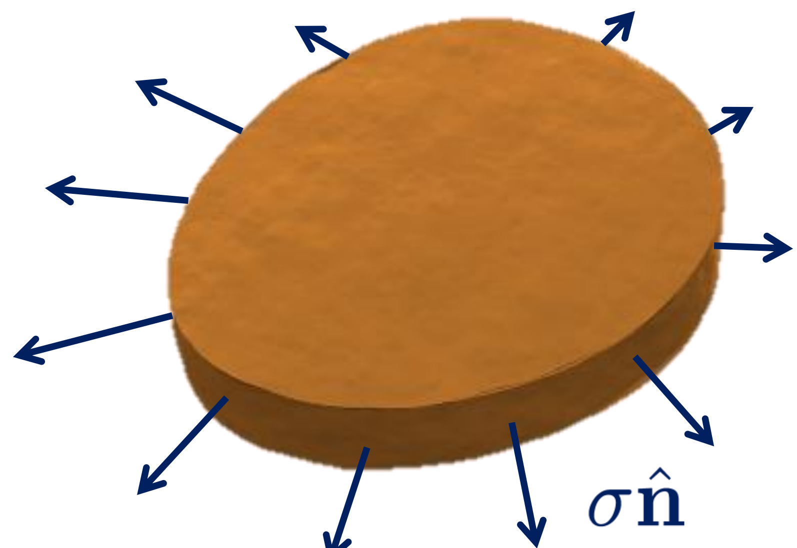
$$\mathbf{F}_{\text{int}} = \int_{\partial\Omega} \sigma\hat{\mathbf{n}} = \int_{\Omega} \operatorname{div} \sigma^T$$

Membrane Equilibrium

$$\begin{aligned} \operatorname{div}\ \sigma^T &= \mathbf{F}_{\text{ext}} \\ \operatorname{div}\ \left(M\left[I \quad \nabla z\right]\right) &= \mathbf{F}_{\text{ext}} \end{aligned}$$

$$\begin{aligned} \operatorname{div} M\nabla z &= F_z \\ \operatorname{div} M &= 0 \end{aligned}$$

M positive semidefinite



$$\mathbf{F}_{\text{int}} = \int_{\partial\Omega} \sigma\hat{\mathbf{n}} = \int_{\Omega} \operatorname{div} \sigma^T$$

Stress Laplacian

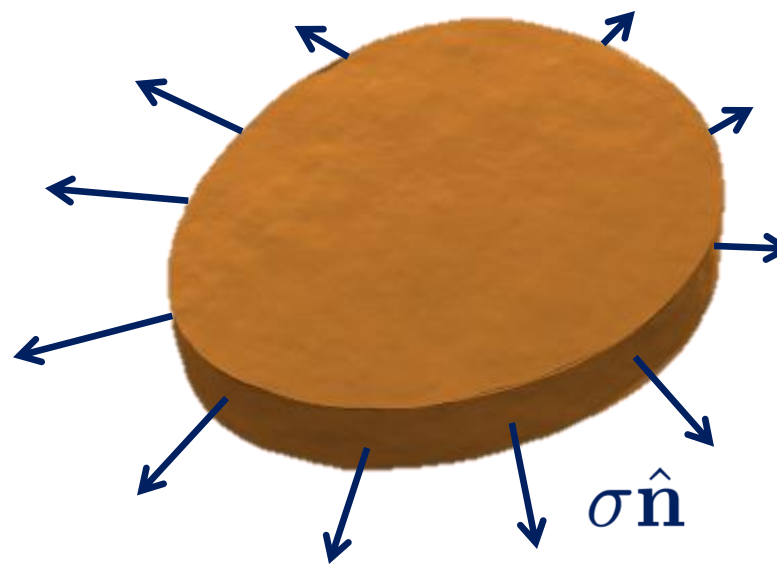
$$\operatorname{div}\,\sigma^T = \mathbf{F}_{\text{ext}}$$

$$\operatorname{div}\,\left(M\left[I \quad \nabla z\right]\right) = \mathbf{F}_{\text{ext}}$$

$\Delta_M z$ ~~$\operatorname{div} M\nabla z$~~ $= F_z$

$$\operatorname{div} M = 0$$

M positive semidefinite

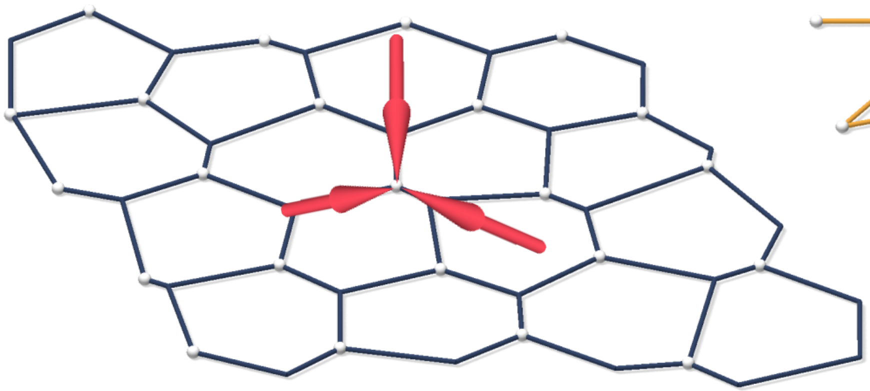


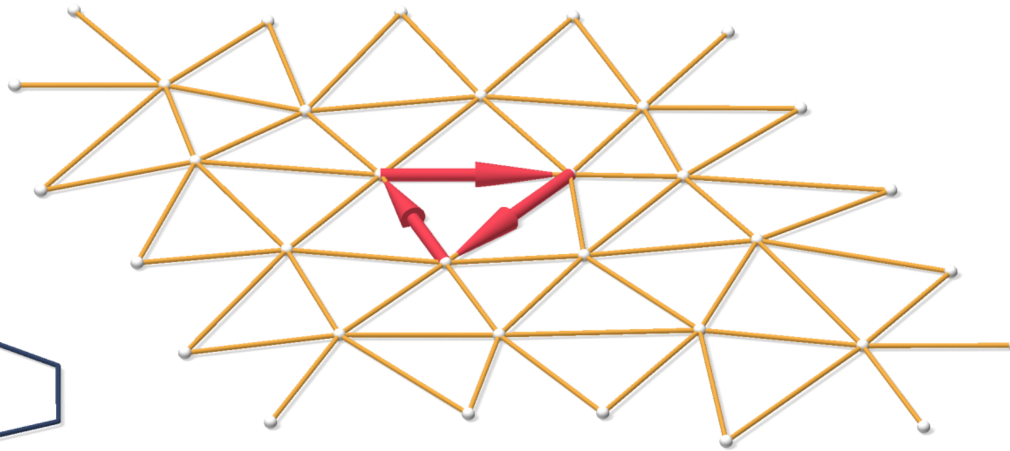
$$\mathbf{F}_{\text{int}} = \int_{\partial\Omega} \sigma\hat{\mathbf{n}} = -\int_{\Omega} \operatorname{div}\sigma^T$$

Discretization Dictionary

$z(x, y)$	z_i
$\operatorname{div} M \; = \; 0$	$\sum w_{ij}(\mathbf{v}'_i - \mathbf{v}'_j) \; = \; 0$
$\Delta_M z = \operatorname{div} M \nabla z$	$L_{\mathbf{w}} z = \sum w_{ij}(z_i - z_j)$
M positive semidefinite	$w_{ij} \geq 0$

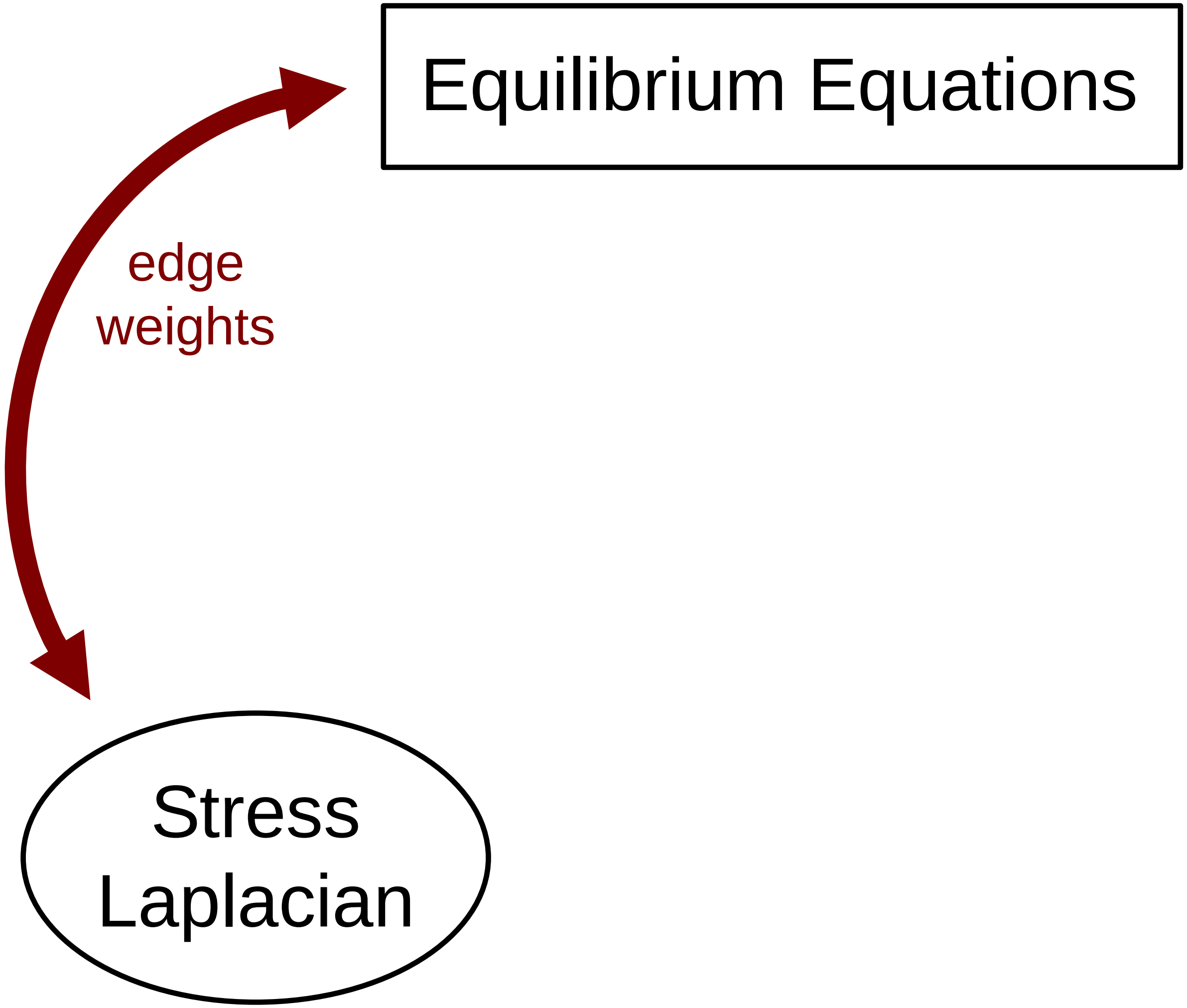
Reciprocal Diagrams [Maxwell; Ash et al 1988]

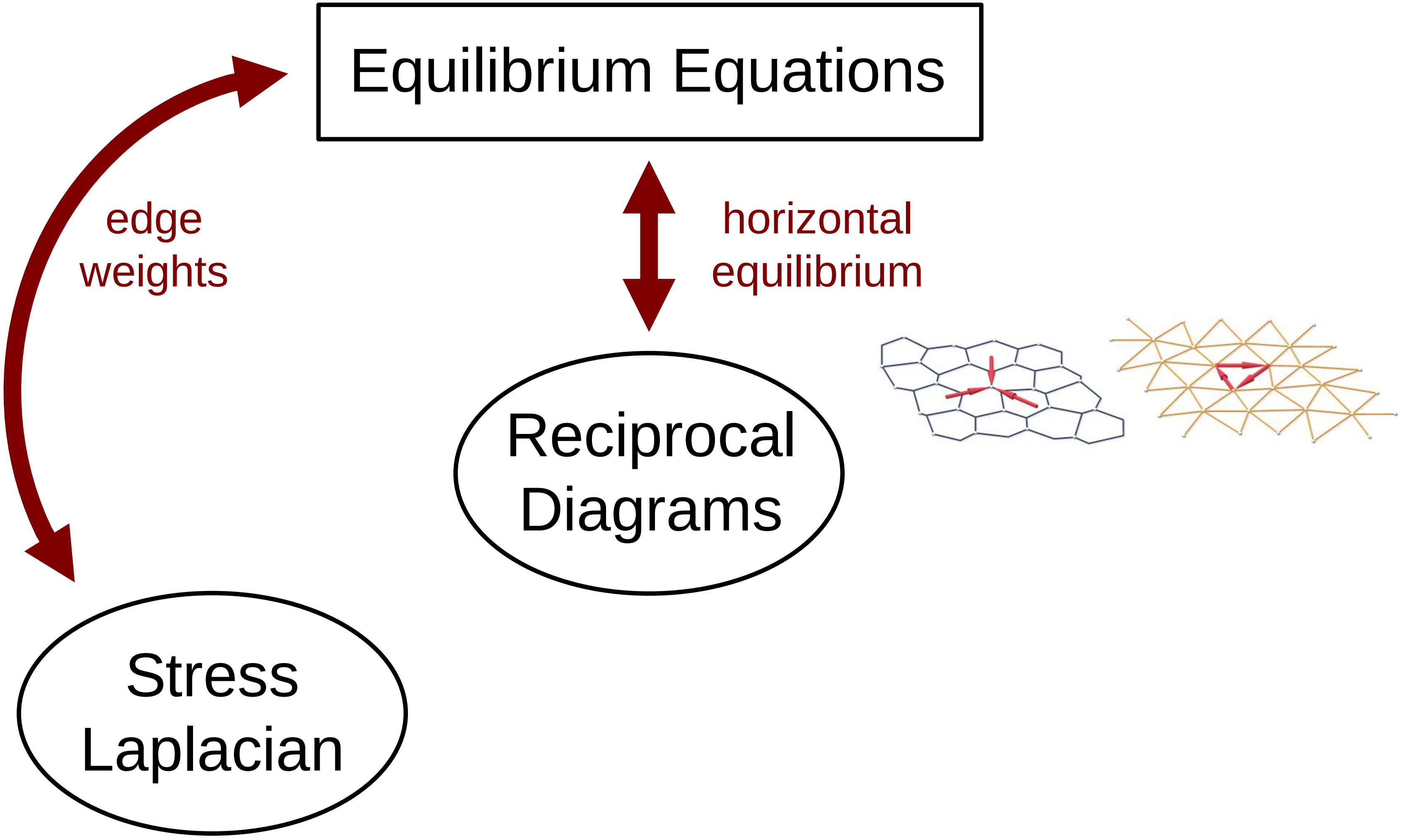


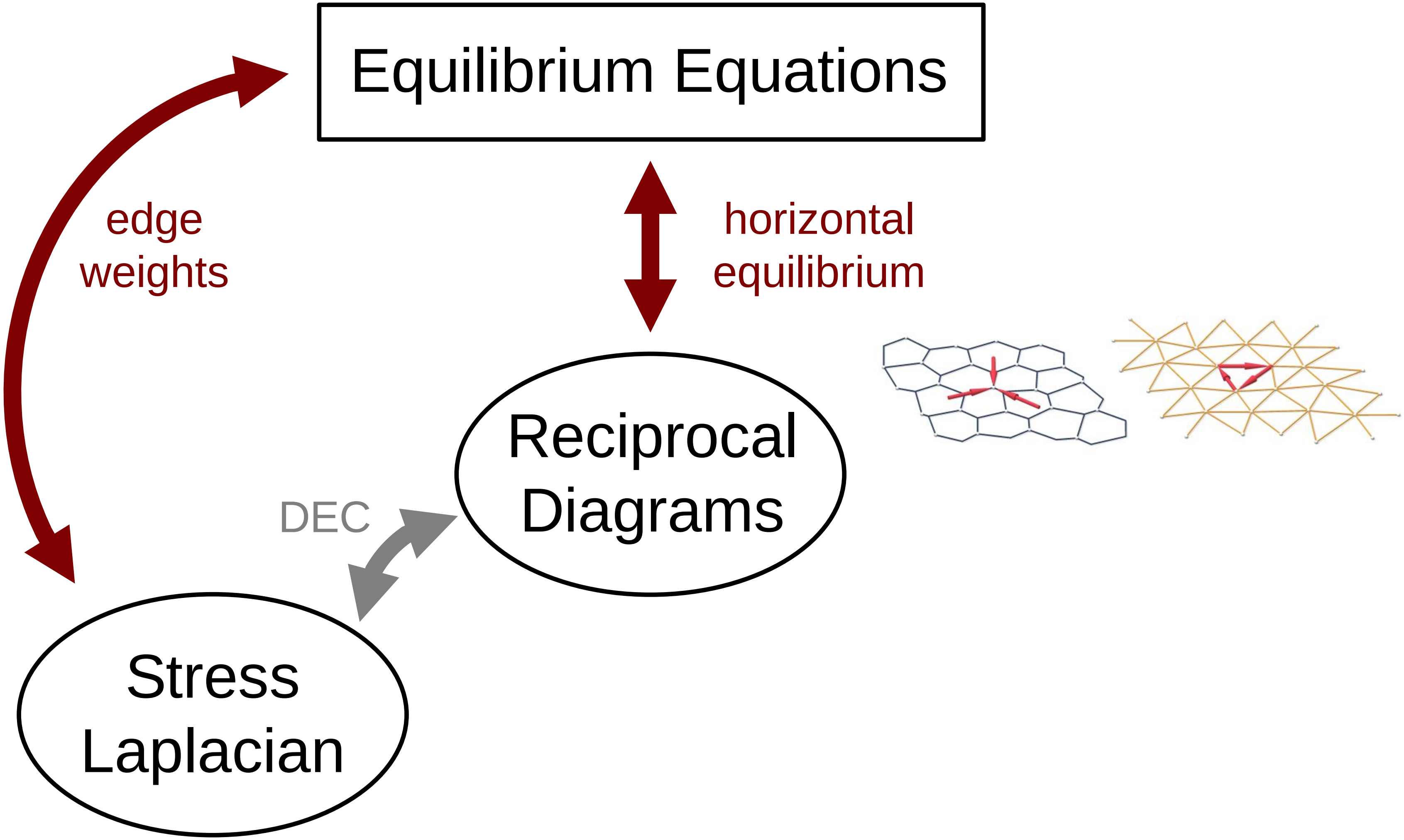


$$\sum_{j \sim i} w_{ij}(\mathbf{v}'_i - \mathbf{v}'_j) \;=\; 0$$

$$\sum_{j \sim i} w_{ij}(\mathbf{v}'_i - \mathbf{v}'_j)^{\perp} \;=\; 0$$


Equilibrium Equations
edge
weights
Stress
Laplacian


Equilibrium Equations
edge weights
horizontal equilibrium
Reciprocal Diagrams
Stress Laplacian


Equilibrium Equations
edge weights
horizontal equilibrium
Reciprocal Diagrams
DEC
Stress Laplacian

recall:

$$\operatorname{div} M \nabla z = F_z$$

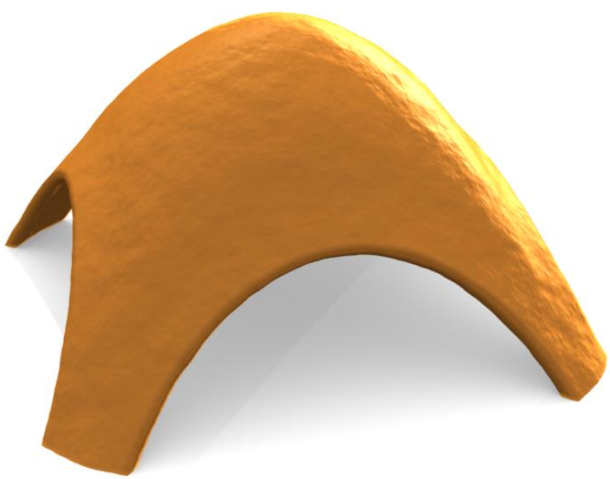
$$\operatorname{div} M = 0$$

$$M \geq 0$$

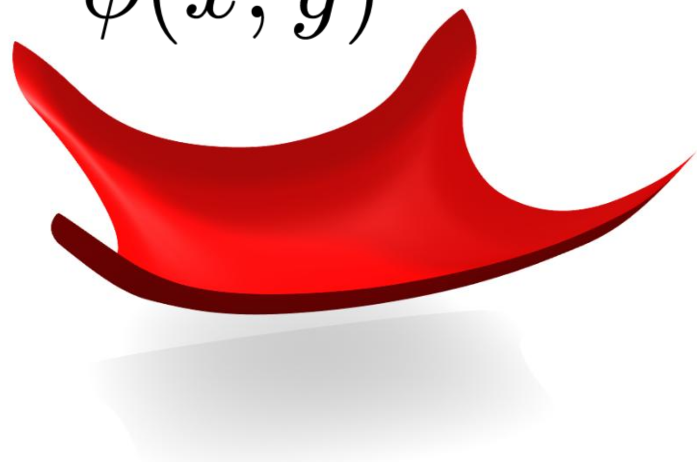
Airy Stress Potential

$$\begin{aligned} M &= \begin{bmatrix} \phi_{yy} & -\phi_{xy} \\ -\phi_{xy} & \phi_{xx} \end{bmatrix} \\ &= \left(\nabla^2 \phi\right)^{-1} \det \nabla^2 \phi \end{aligned}$$

$z(x, y)$

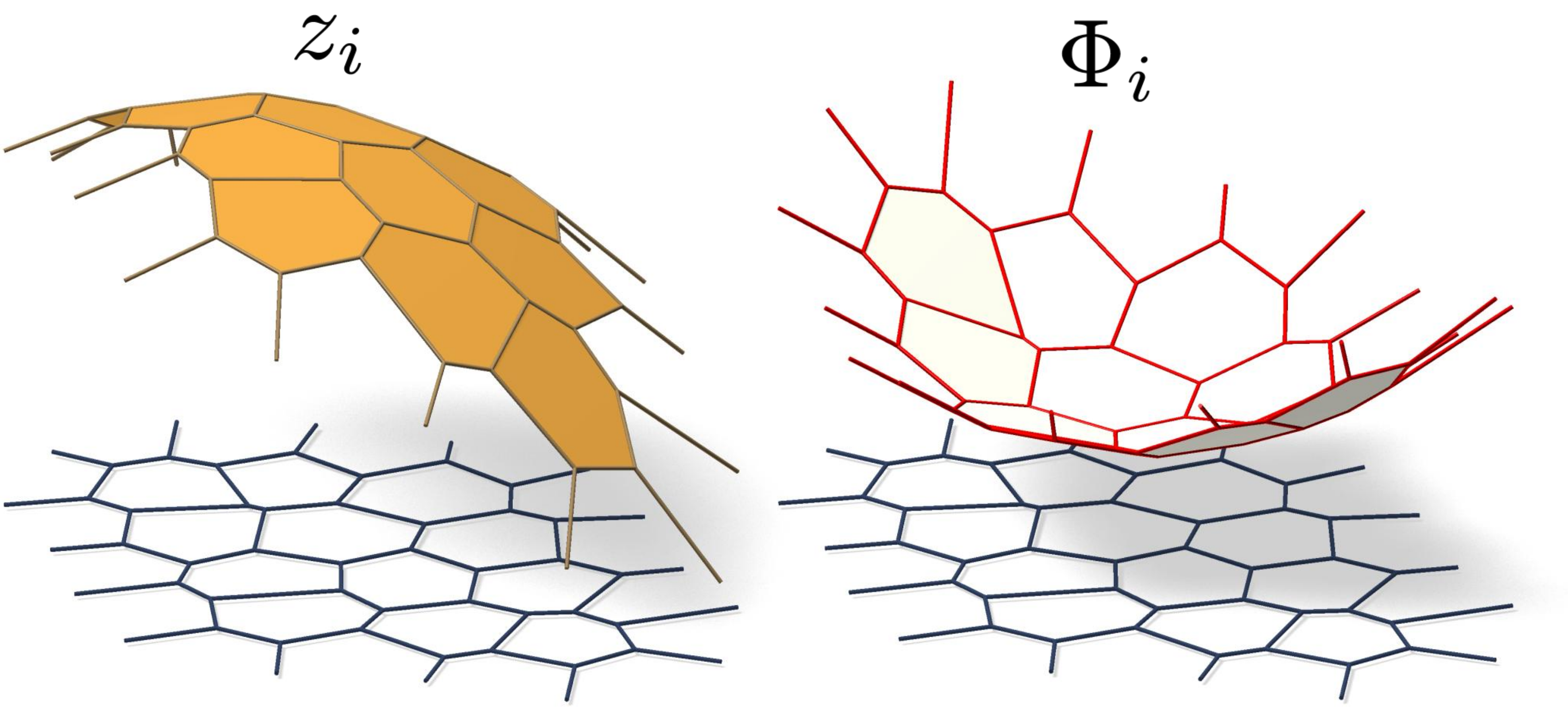


$\phi(x, y)$



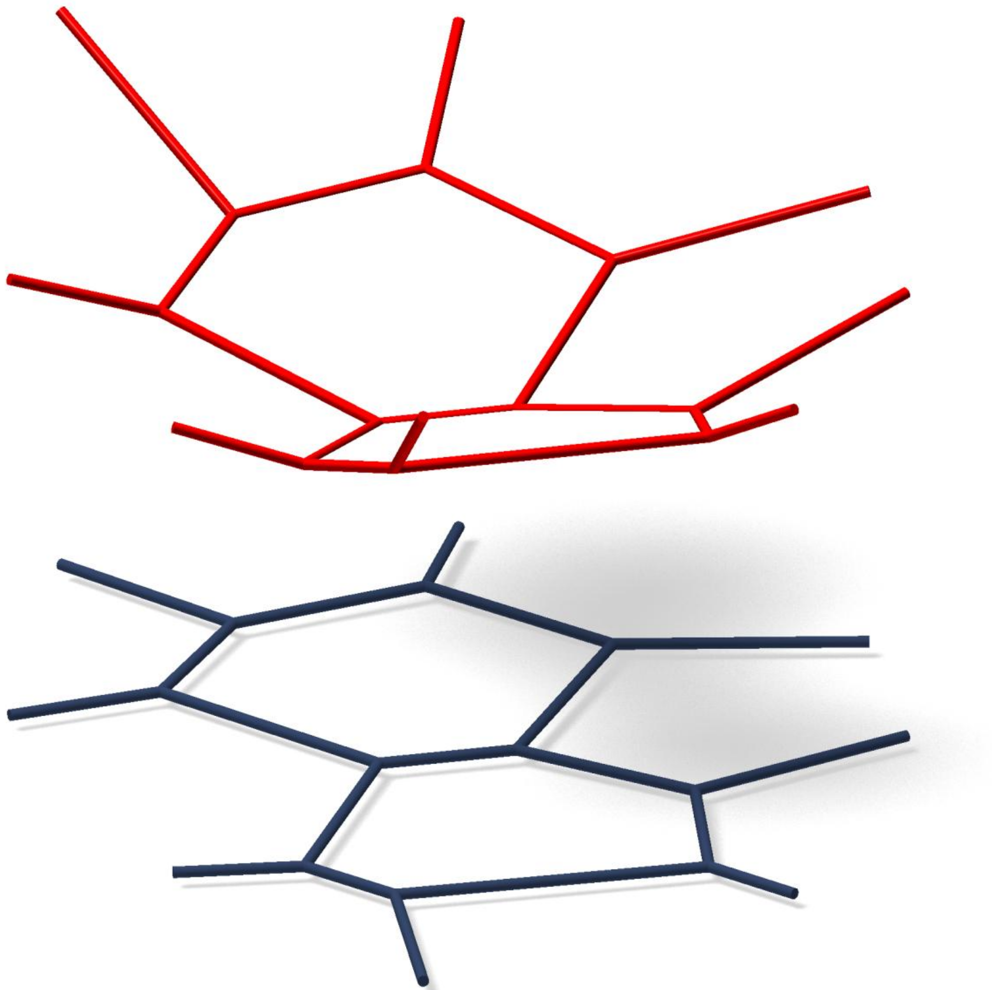
convex

Airy Stress Polytope [Fraternali 2010]

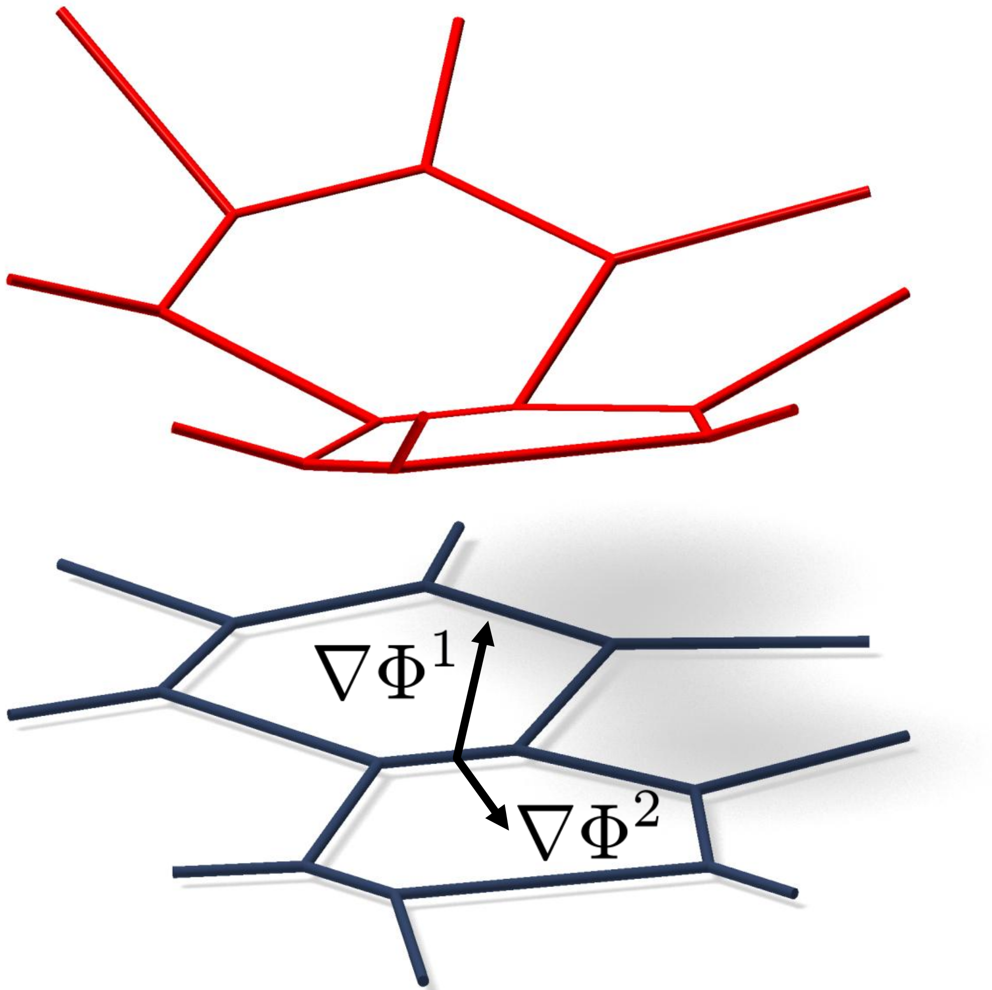


convex, planar faces

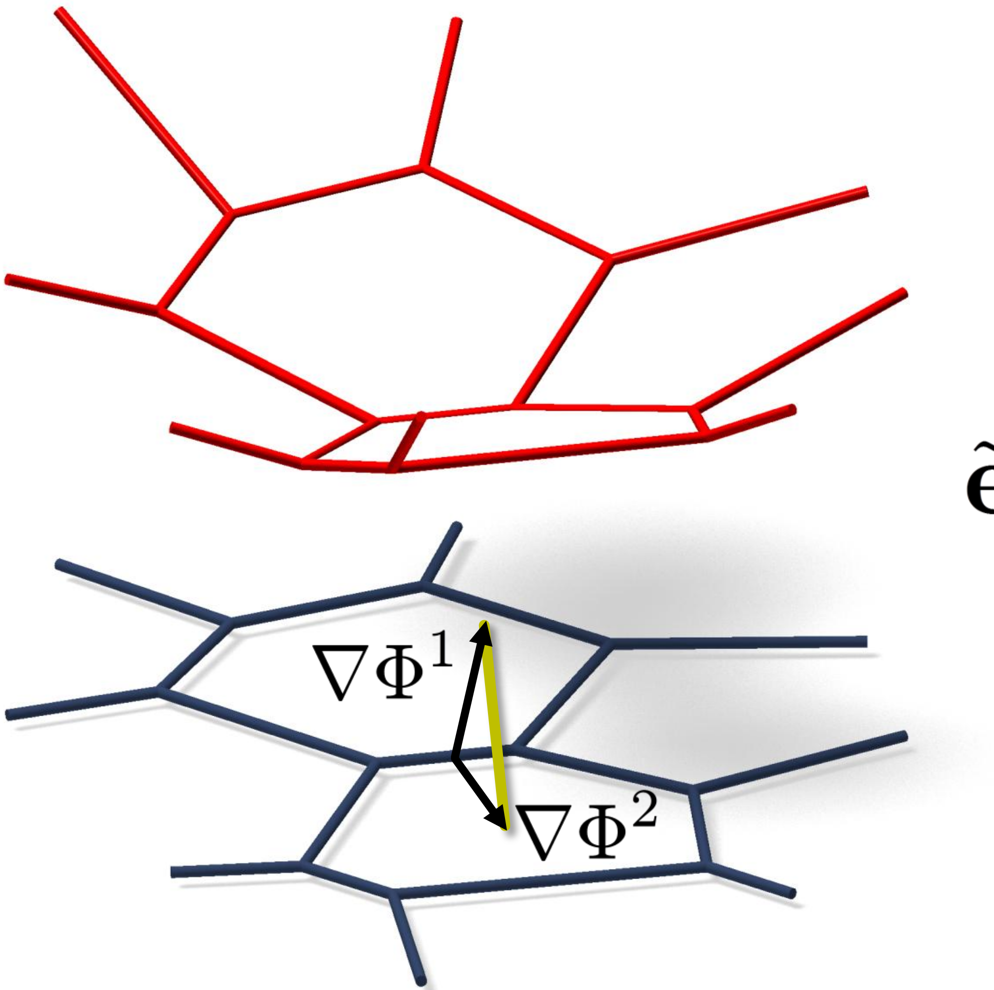
Stress Polytope: Gradient Jumps



Stress Polytope: Gradient Jumps



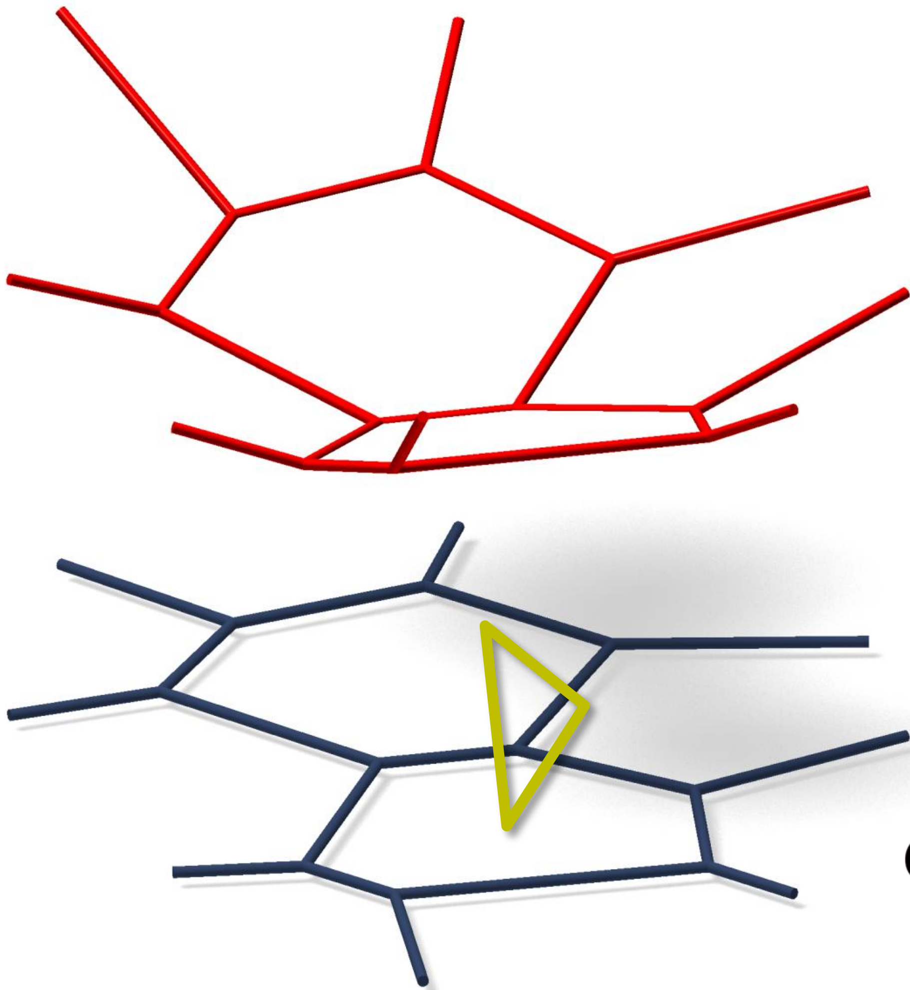
Stress Polytope: Gradient Jumps



dual edge

$$\tilde{\mathbf{e}}_{ij} = \nabla\Phi^j - \nabla\Phi^i$$

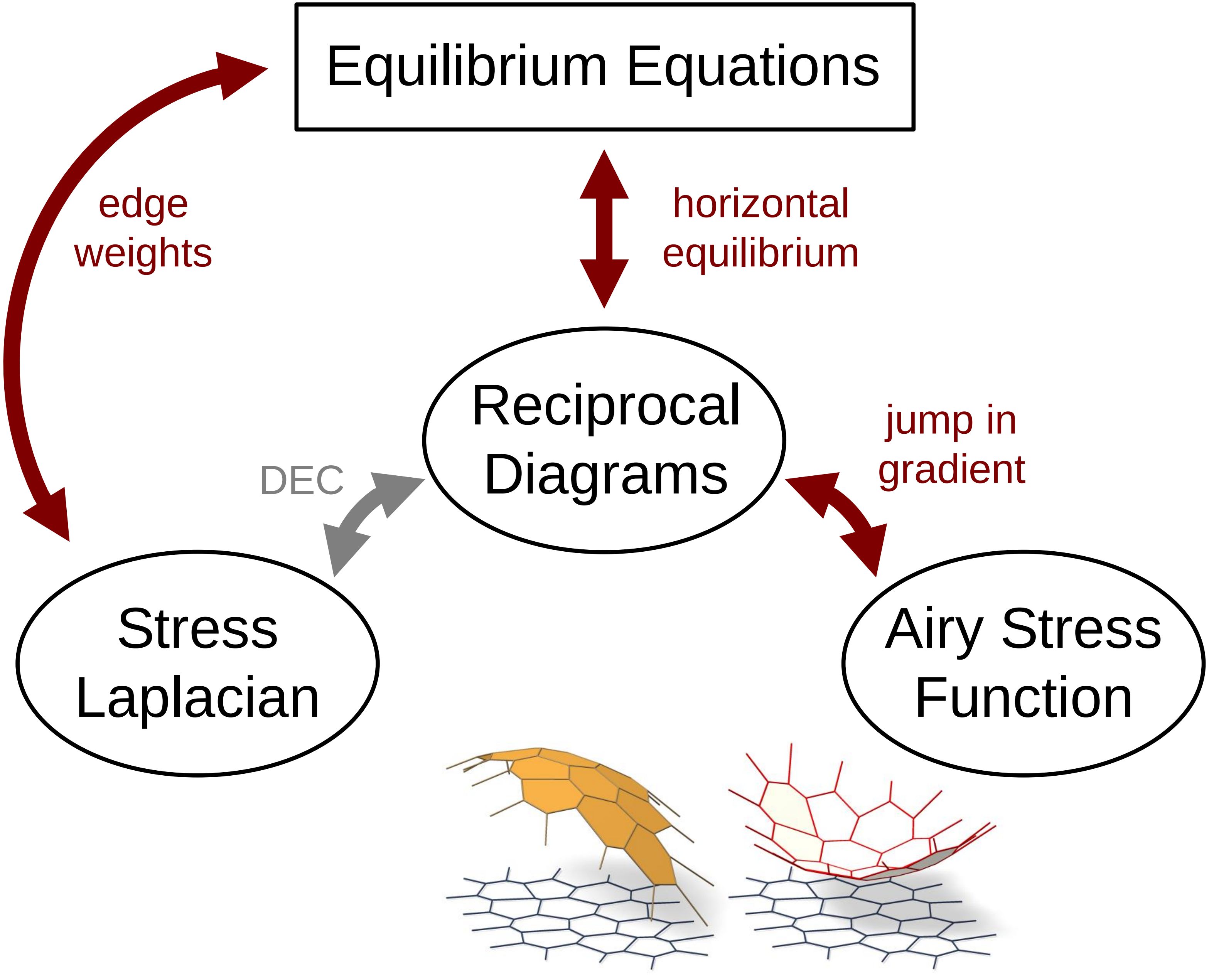
Stress Polytope: Gradient Jumps



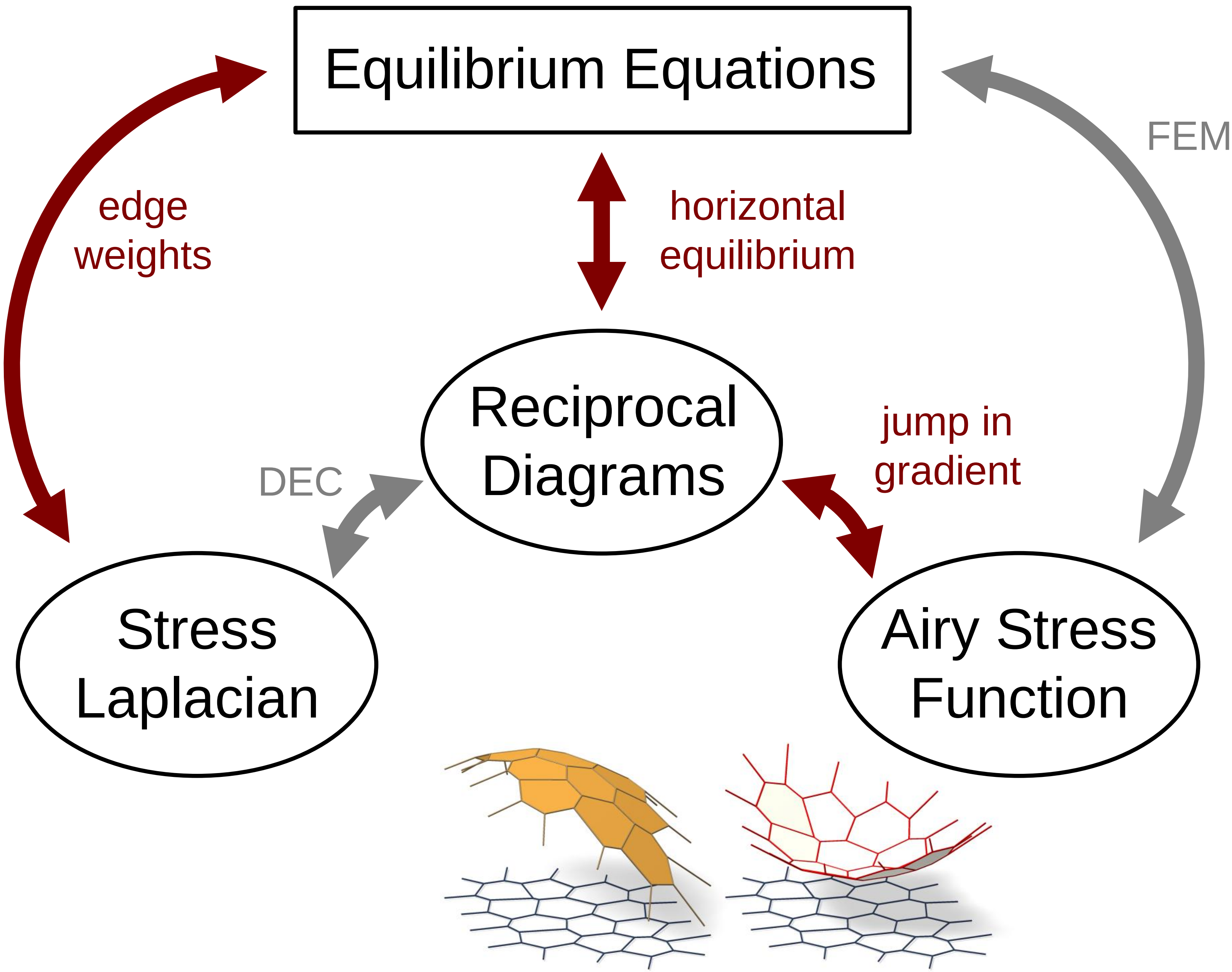
dual edge

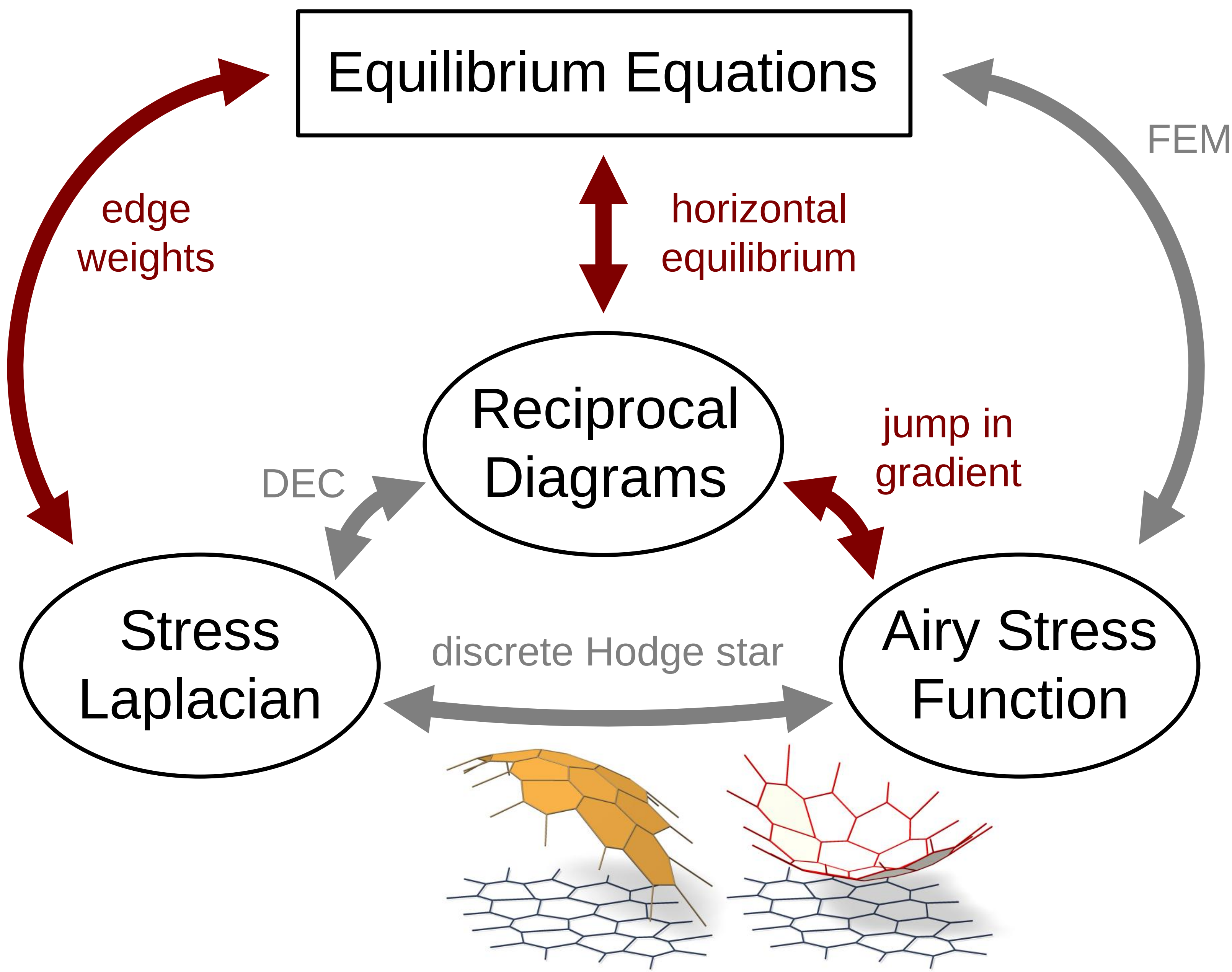
$$\tilde{\mathbf{e}}_{ij} = \nabla\Phi^j - \nabla\Phi^i$$

$$\tilde{\mathbf{e}}'_{12} + \tilde{\mathbf{e}}'_{23} + \tilde{\mathbf{e}}'_{31} = 0$$



Equilibrium Equations
edge weights
horizontal equilibrium
Reciprocal Diagrams
DEC
jump in gradient
Stress Laplacian
Airy Stress Function


Equilibrium Equations
edge weights
horizontal equilibrium
FEM
Reciprocal Diagrams
DEC
jump in gradient
Stress Laplacian
Airy Stress Function


Equilibrium Equations
edge weights
horizontal equilibrium
FEM
Reciprocal Diagrams
jump in gradient
DEC
Stress Laplacian
discrete Hodge star
Airy Stress Function

Relative Curvatures

recall:

$$\operatorname{div} M\nabla z = F_z$$

$$\operatorname{div} M = 0$$

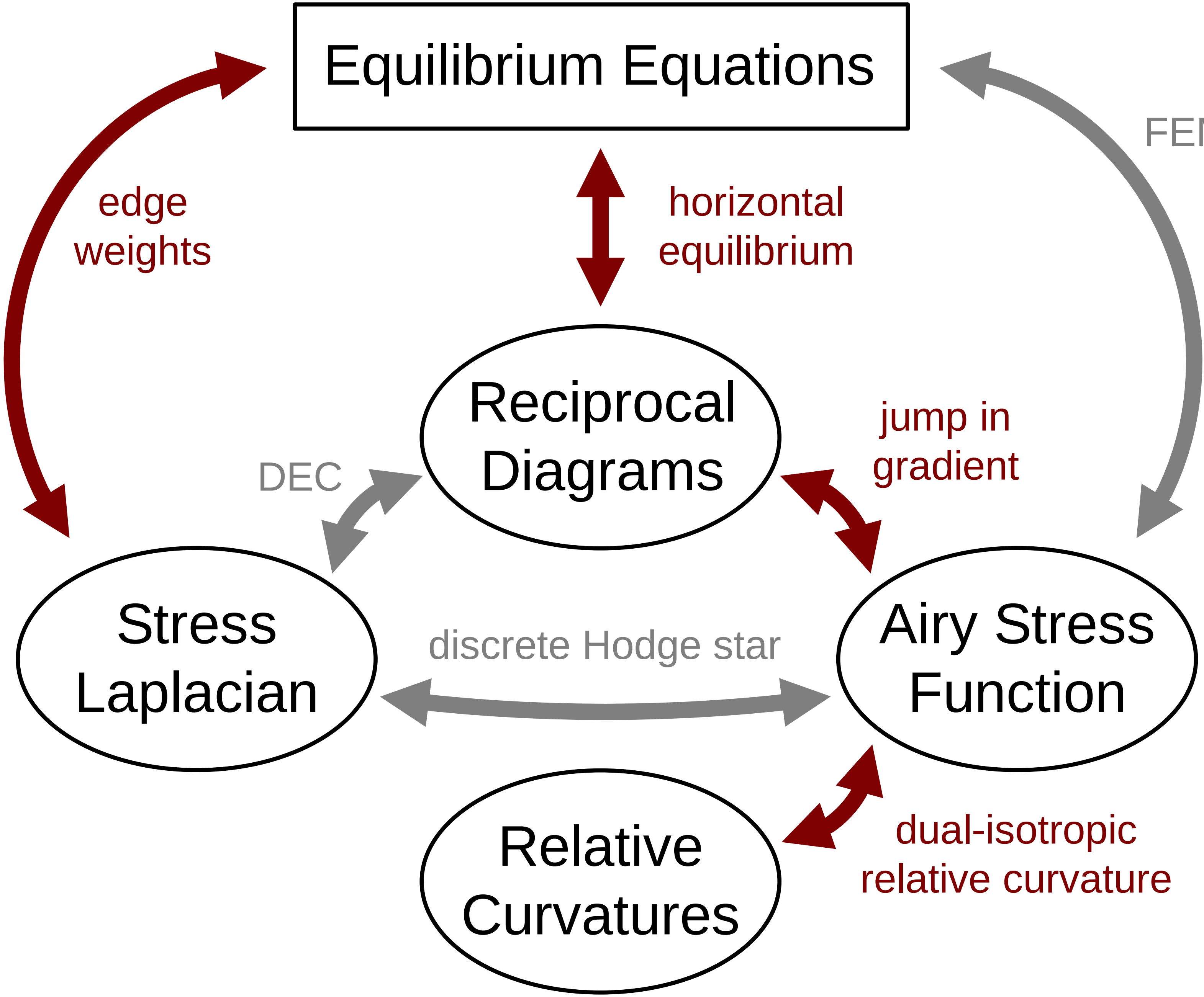
$$M \geq 0$$

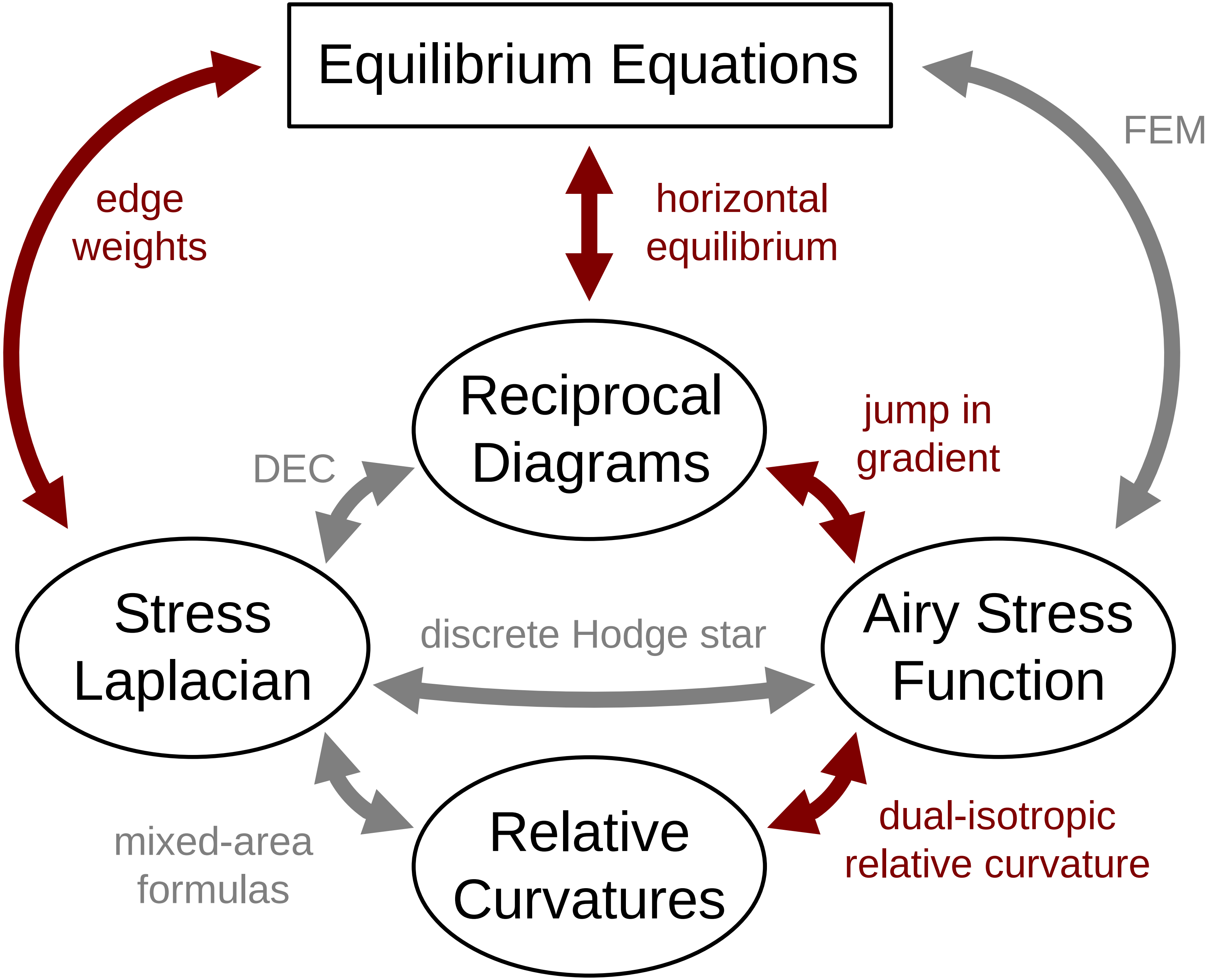
$$\operatorname{div} M\nabla z = \operatorname{tr}\left[\left(\nabla^2\phi\right)^{-1}\left(\nabla^2 z\right)\right]\det\nabla^2\phi$$

looks like curvatures!

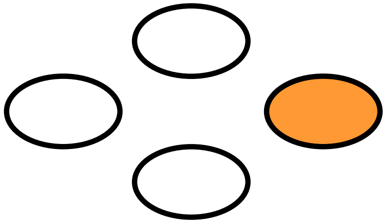
$$2H^{z,\phi}K^{\phi,(x^2+y^2)/2} = F_z$$

"dual-isotropic relative mean and Gaussian curvatures"

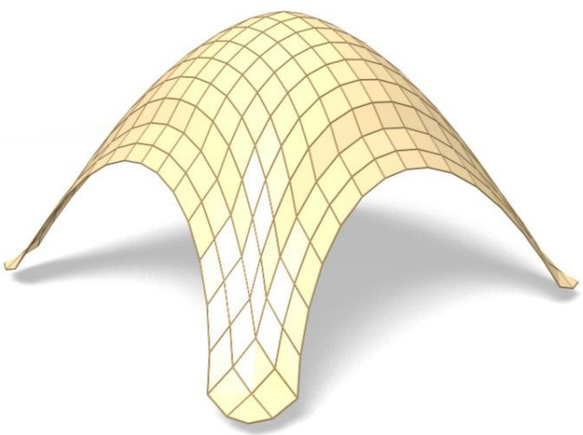

Equilibrium Equations
edge weights
horizontal equilibrium
FEM
Reciprocal Diagrams
jump in gradient
DEC
Stress Laplacian
discrete Hodge star
Airy Stress Function
Relative Curvatures
dual-isotropic relative curvature


Equilibrium Equations
edge weights
horizontal equilibrium
FEM
Reciprocal Diagrams
jump in gradient
DEC
Stress Laplacian
discrete Hodge star
Airy Stress Function
mixed-area formulas
Relative Curvatures
dual-isotropic relative curvature

PQ Remeshing

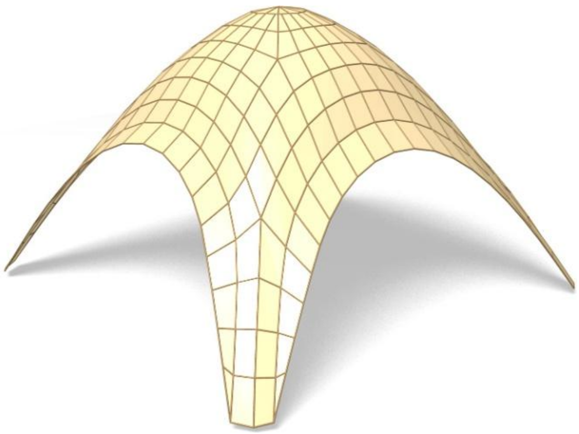


Airy Stress Function

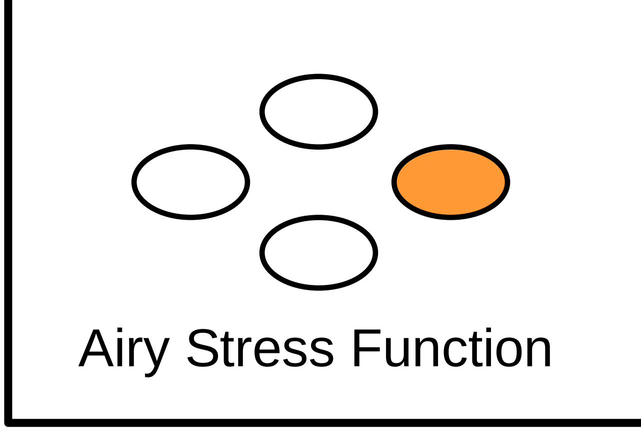


Original

Naïve Optimization



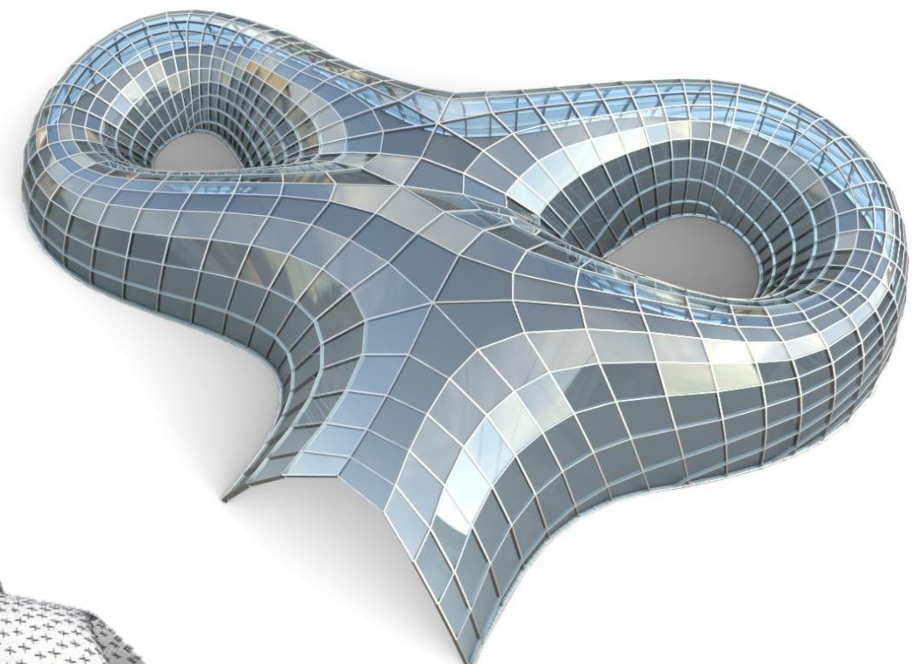
Using Airy Polytope

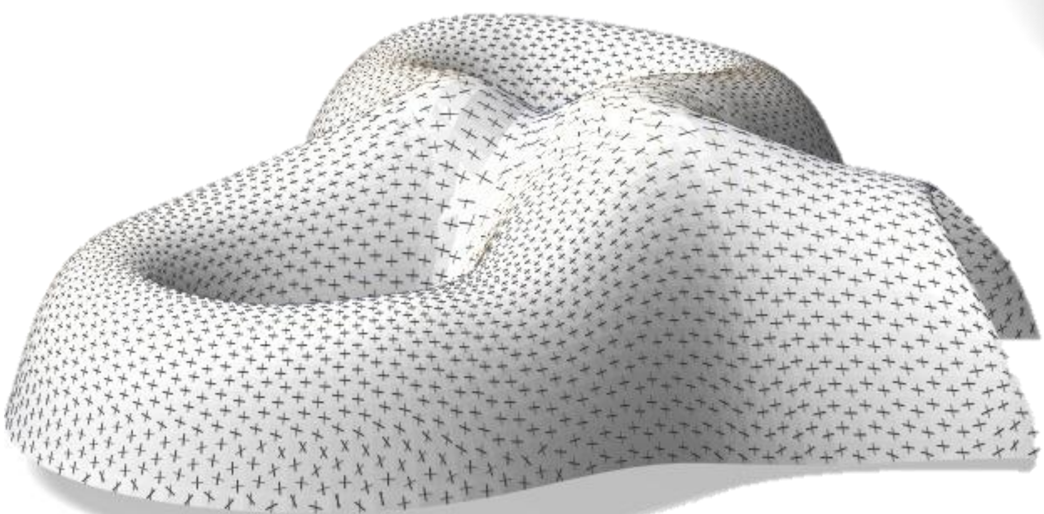


PQ Remeshing

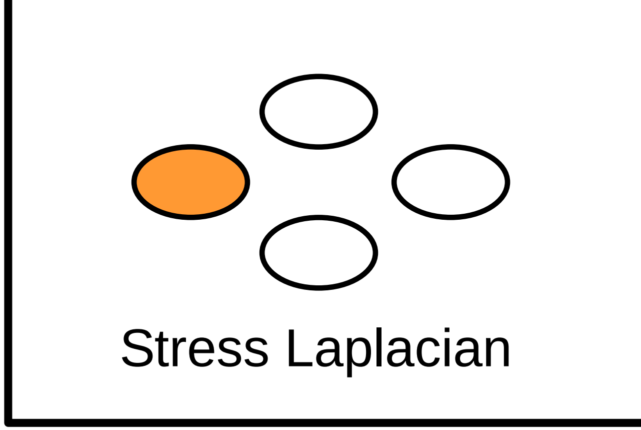
Remesh along eigenvectors of

$$(\nabla^2\phi)^{-1}(\nabla^2 z)$$

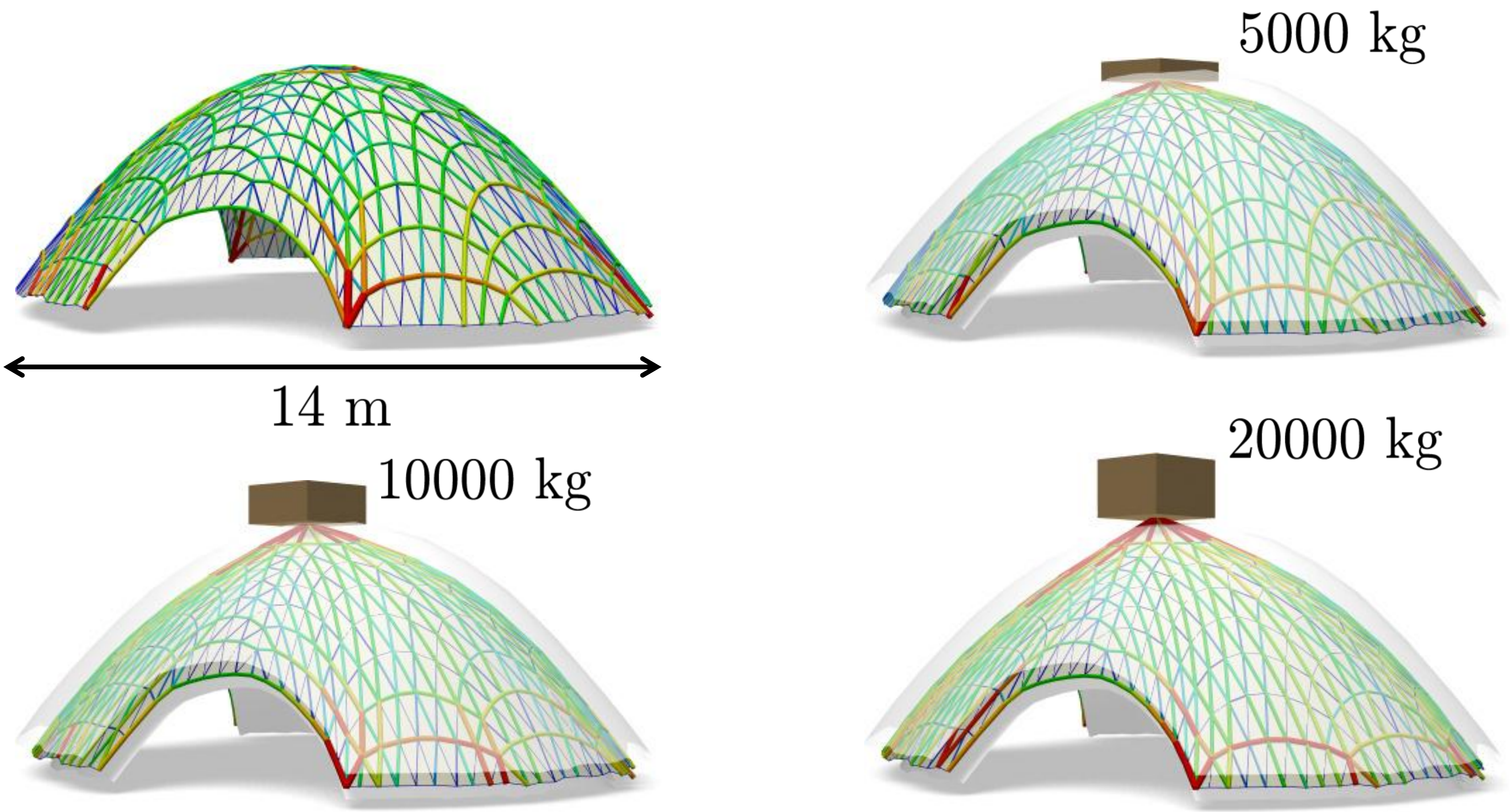


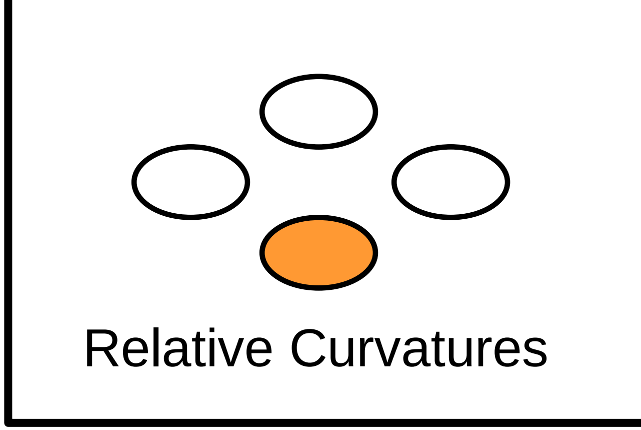


Loading of Surface



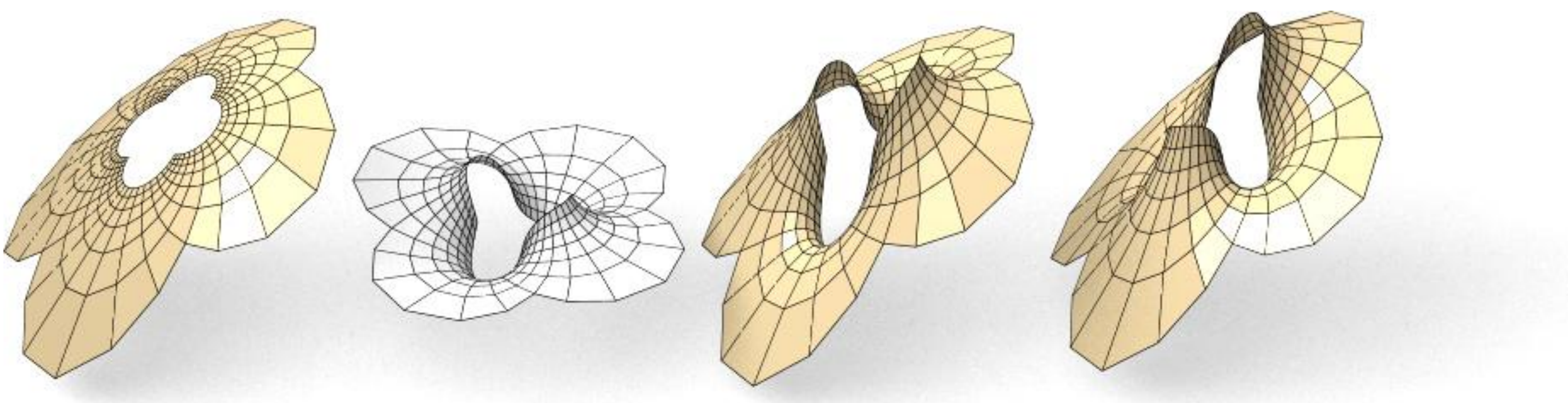
Shell thickness $0.1\ \mathrm{m}, \rho = 2500\ \mathrm{kg/m^3}$





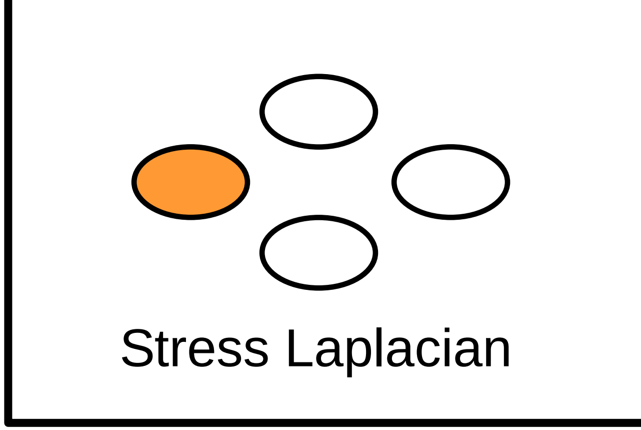
Special Families

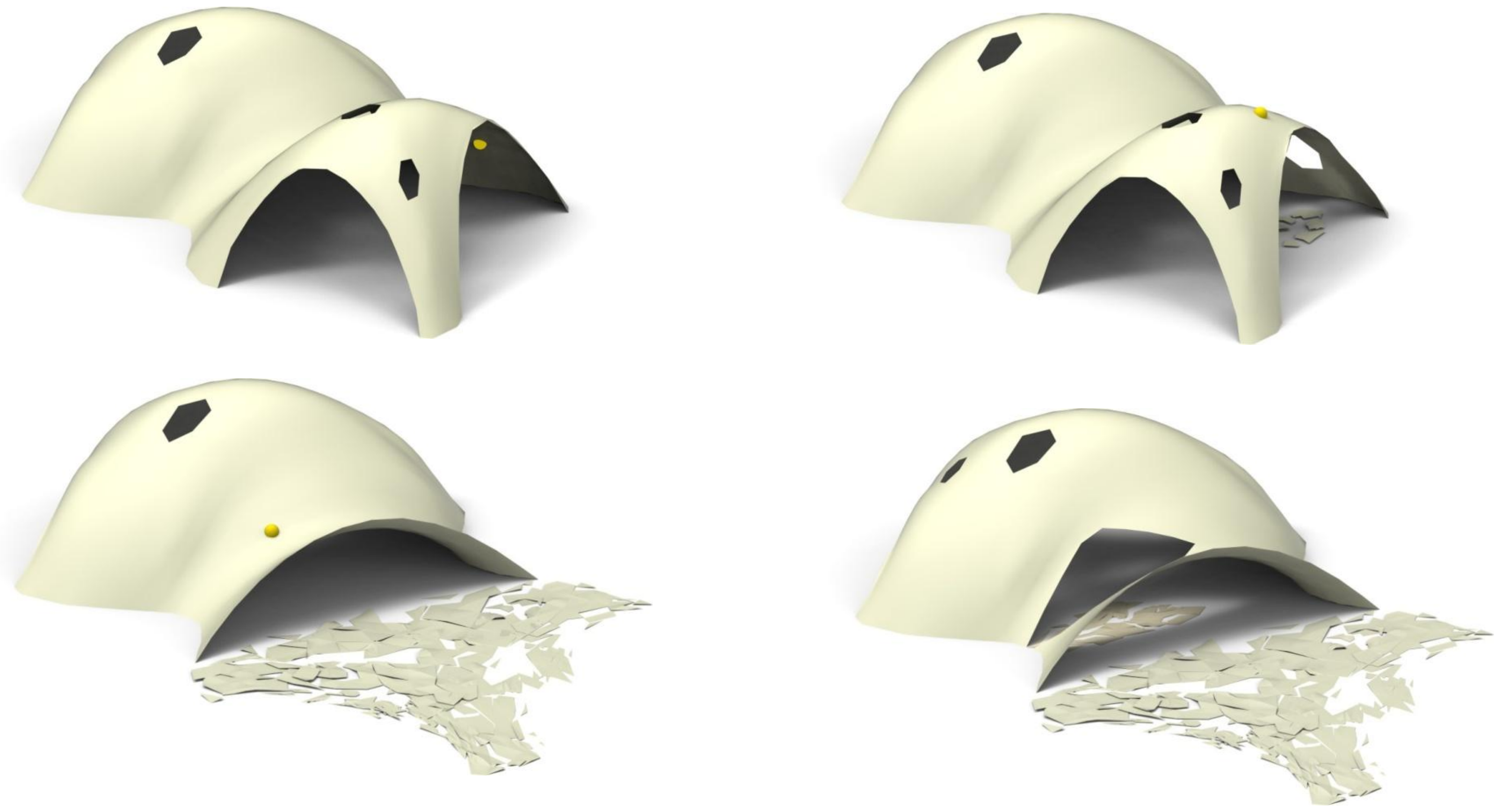
Recall: $2H^{z,\phi}K^{\phi,(x^2+y^2)/2} = F_z$



Template $H^{z,\phi} = 0$ Linear Combinations

Surface Destruction





What's Next?

Non-vertical loads

wind, earthquakes, ...

Non-manifold surfaces

T junctions, flying buttresses, ...

Can we further exploit geometrization of equilibrium?

Thank You!

Miklós Bergou
Philippe Block
Eitan Grinspun
Florin Isvoranu
Danny Kaufman
Niloy Mitra
Alexander Schiftner

Adobe
ATI
Autodesk
mental images
NVIDIA
Side Effects Software
Walt Disney Company
Weta Digital