Blending Isogeometric Analysis and local maximum-entropy approximants

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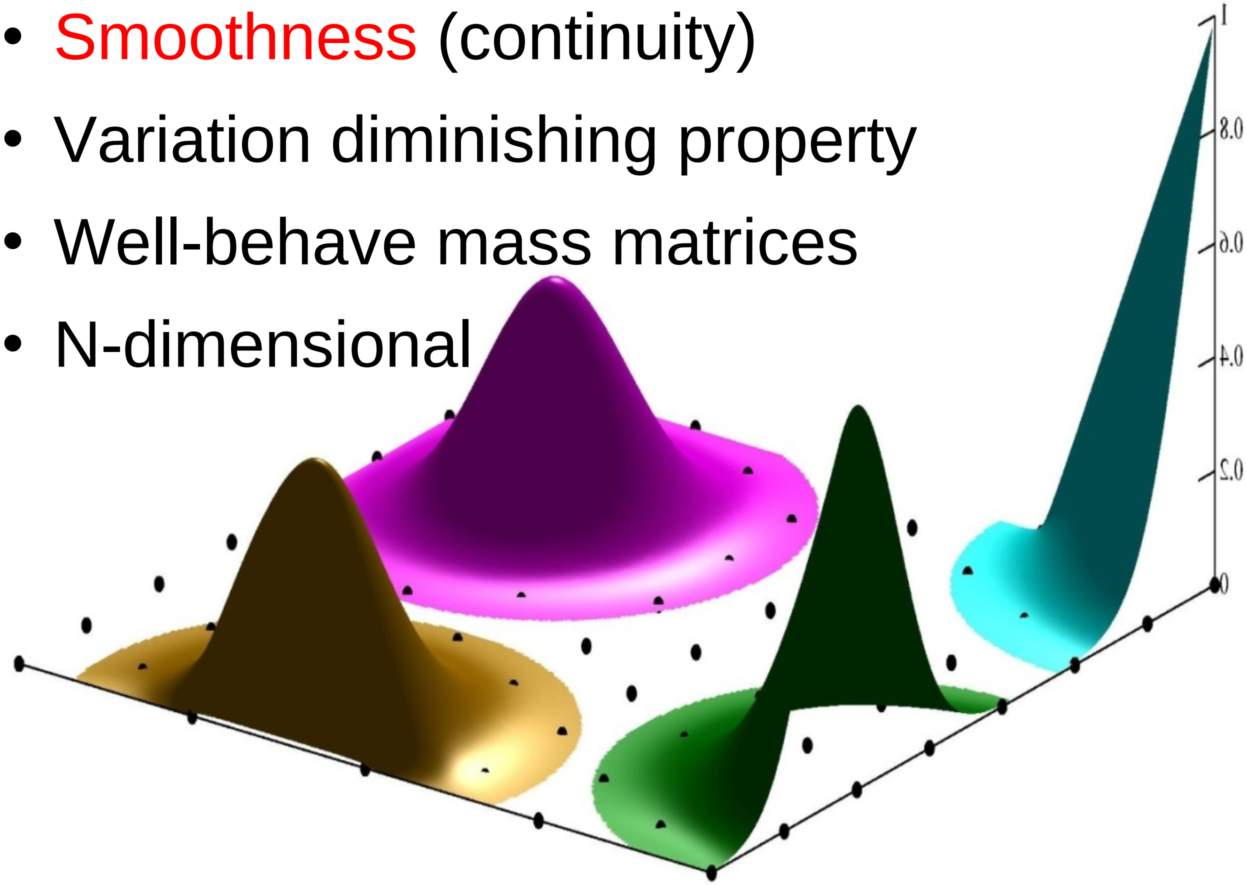
Outline

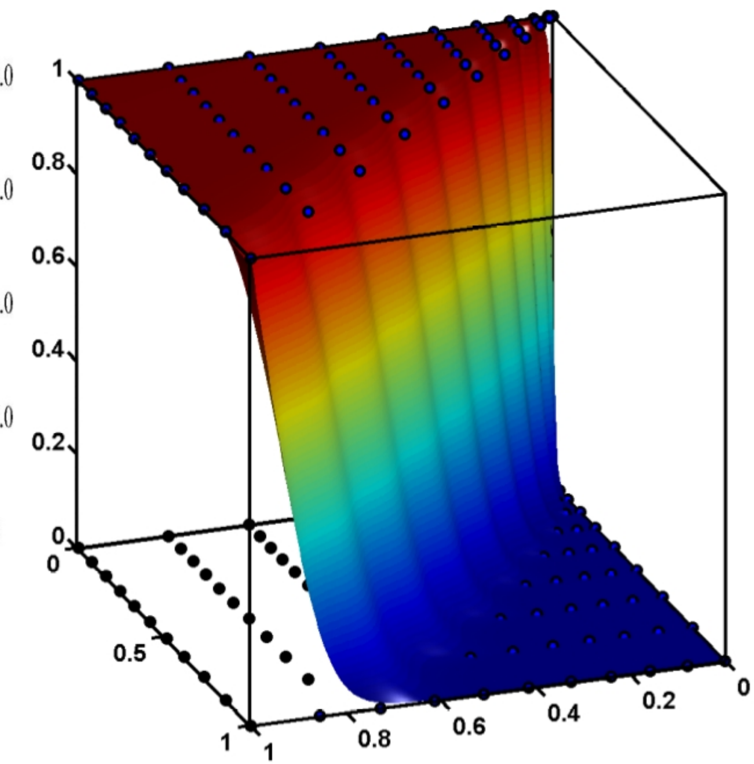
- Briefly review of approximants selected by maximum entropy
- Isogeometric analysis and local maximum-entropy approximants

Briefly review of approximants selected by maximum-entropy approximation schemes

Maximum-entropy approximants

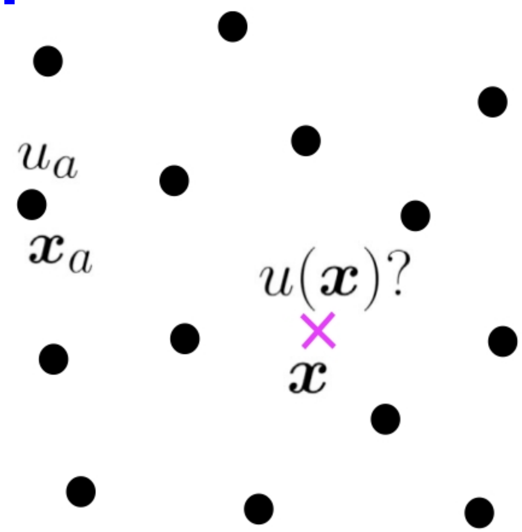
- Inspired in information theory [Sukumar, 2004] [Arroyo, 2006]
- Non-negative basis functions
- Kronecker-delta property at the boundary of the convex hull
- Smoothness (continuity)
- Variation diminishing property
- Well-behave mass matrices
- N-dimensional





Maximum-entropy approximants

- Approximation of a function



$$u(\boldsymbol{x}) \approx u^h(\boldsymbol{x}) = \sum_{a=1}^{N} p_a(\boldsymbol{x})\ u_a$$

where

- The set $\{u_a\}_{a=1,\ldots,N}$ represents the nodal data at a scattered set of points $\{\boldsymbol{x}_a\}_{a=1,\ldots,N}$
- The approximants $p_a(\boldsymbol{x})$ are regarded as unknowns

- The approximants are required to be non-negative and to satisfy zeroth and first order reproducibility conditions

$$p_a \geq 0 \qquad \sum_{a=1}^{N} p_a = 1 \qquad \sum_{a=1}^{N} p_a\ \boldsymbol{x}_a = \boldsymbol{x}$$

Maximum-entropy approximants

- Convex optimization problem

Locality (input parameter)

$$\text{For fixed } \boldsymbol{x} \text{ minimize} \quad \sum_{a=1}^{N} \beta_a p_a |\boldsymbol{x} - \boldsymbol{x}_a|^2 + \sum_{a=1}^{N} p_a \ln p_a$$

$$\text{subject to} \quad p_a \geq 0, \quad a = 1, \ldots, N$$

$$\sum_{a=1}^{N} p_a = 1, \qquad \sum_{a=1}^{N} p_a \, \boldsymbol{x}_a = \boldsymbol{x}$$

- This constrained program has a unique solution in the convex hull
- We have to solve the problem for each evaluation point (***x***)
  - Data: nodal coordinates, locality parameters, evaluation points
  - Unknowns: basis functions

Maximum-entropy approximants

- Duality methods provide an almost explicit expression of the basis functions

$$p_a(\boldsymbol{x}) = \frac{1}{Z\left(\boldsymbol{x}, \boldsymbol{\lambda}^*(\boldsymbol{x})\right)} \exp\left[-\beta_a|\boldsymbol{x}-\boldsymbol{x}_a|^2 + \boldsymbol{\lambda}^*(\boldsymbol{x})\cdot(\boldsymbol{x}-\boldsymbol{x}_a)\right]$$

where

$$Z(\boldsymbol{x}, \boldsymbol{\lambda}) = \sum_{b=1}^{N} \exp\left[-\beta_b|\boldsymbol{x}-\boldsymbol{x}_b|^2 + \boldsymbol{\lambda}\cdot(\boldsymbol{x}-\boldsymbol{x}_b)\right]$$

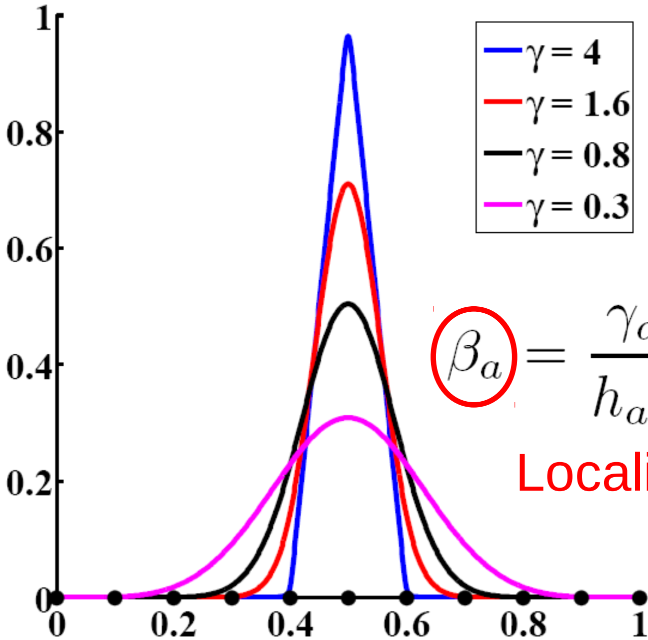
- A Lagrange multiplier must be computed for each evaluation point. It is the unique solution of the following unconstrained convex optimization problem

$$\boldsymbol{\lambda}^*(\boldsymbol{x}) = \arg\min_{\boldsymbol{\lambda}\in\mathbb{R}^d} \ln Z(\boldsymbol{x}, \boldsymbol{\lambda})$$

d: Spatial dimension

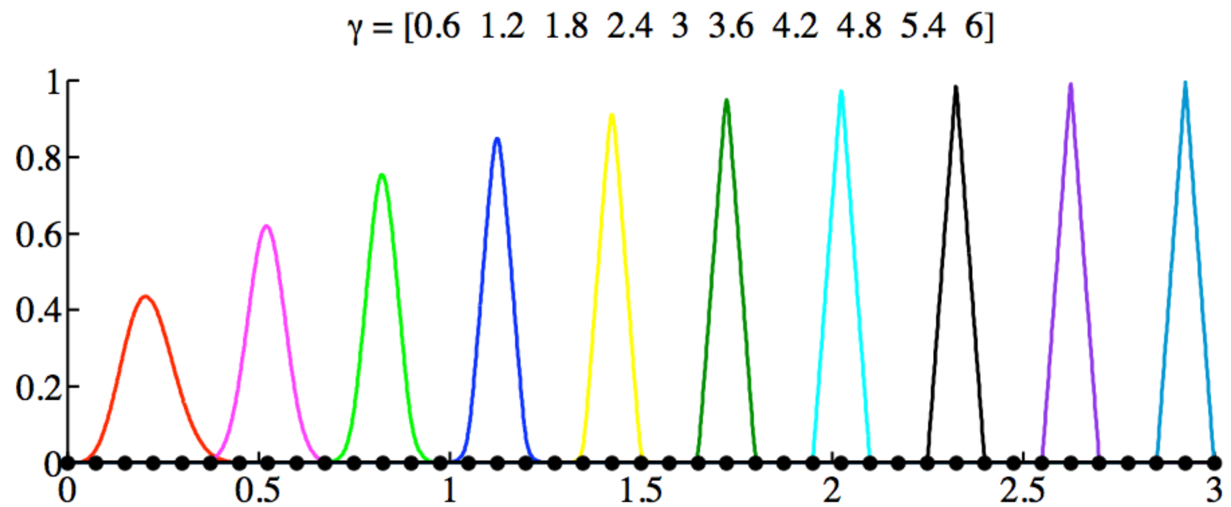
Maximum entropy approximants

- Control of locality (support size)



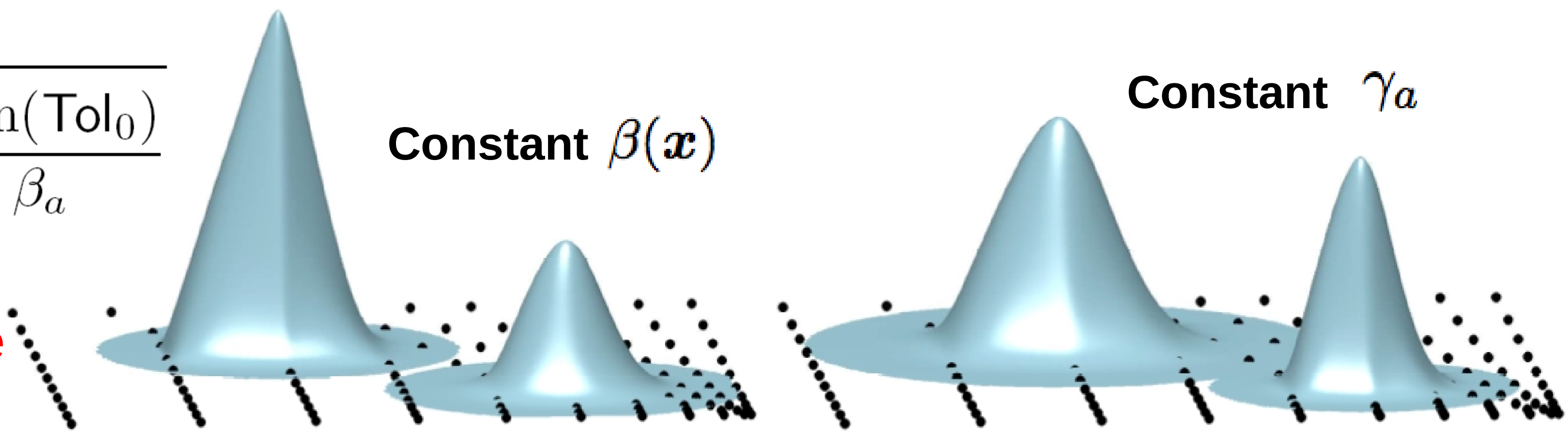
$$\beta_a = \frac{\gamma_a}{{h_a}^2}$$

Locality



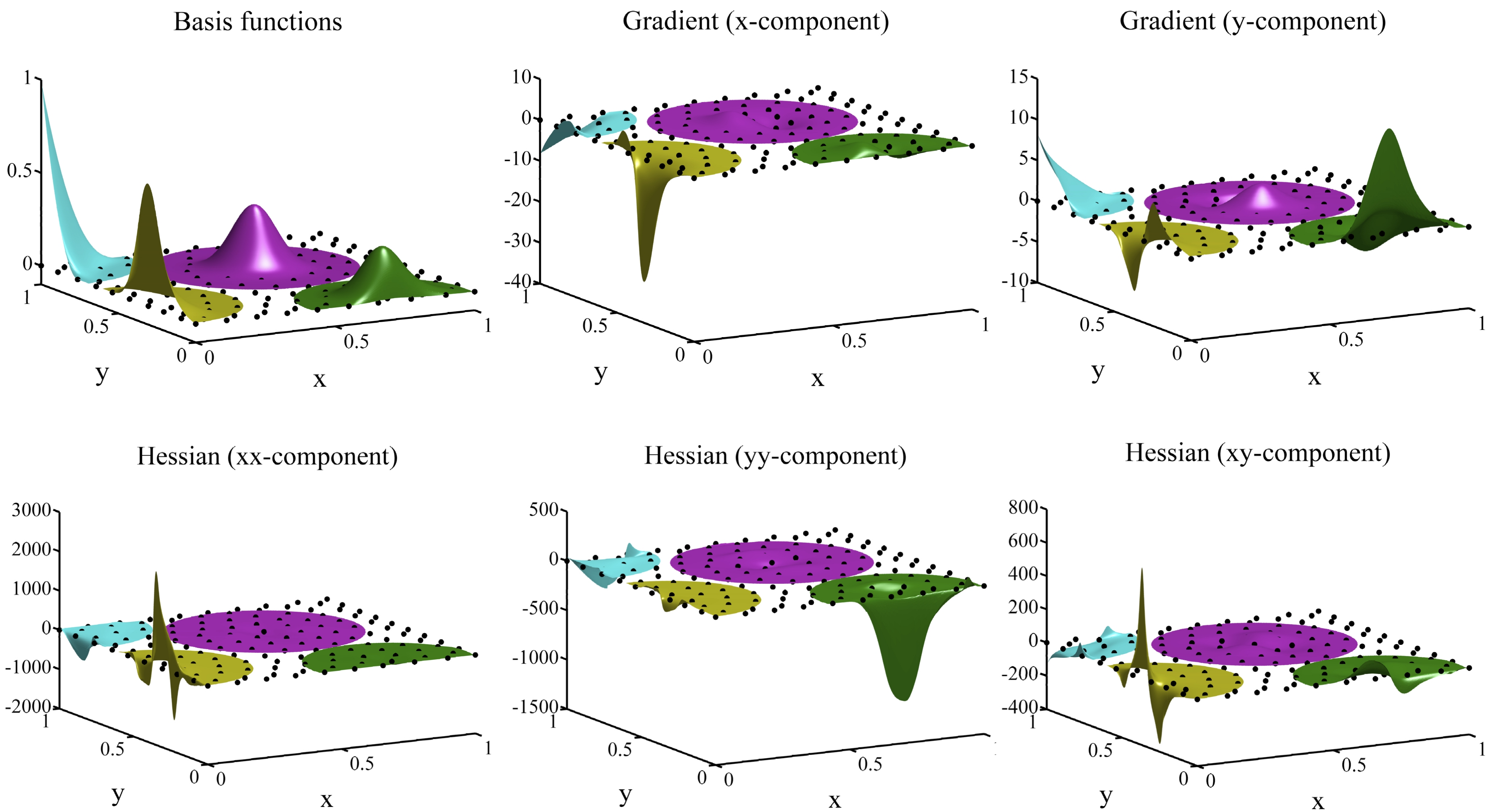
$$R_a = \sqrt{\frac{-\ln(\text{Tol}_0)}{\beta_a}}$$

Numerical support size



Maximum entropy approximants

- Derivatives can be computed analytically



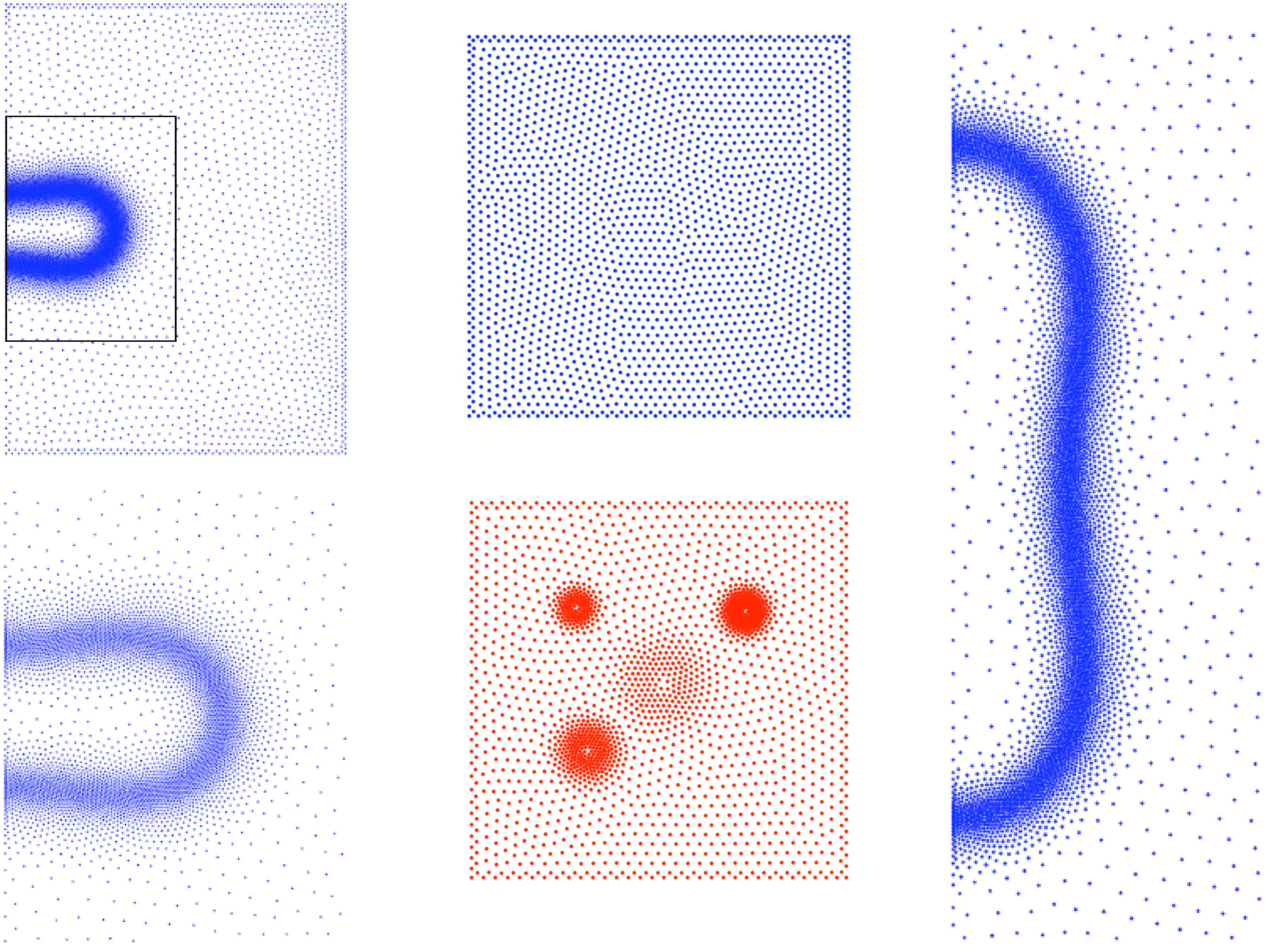
Maximum entropy approximants

- Ability to handle non-uniform and unstructured set of points

- LME are very accurate in the calculation of problems involving high order derivatives

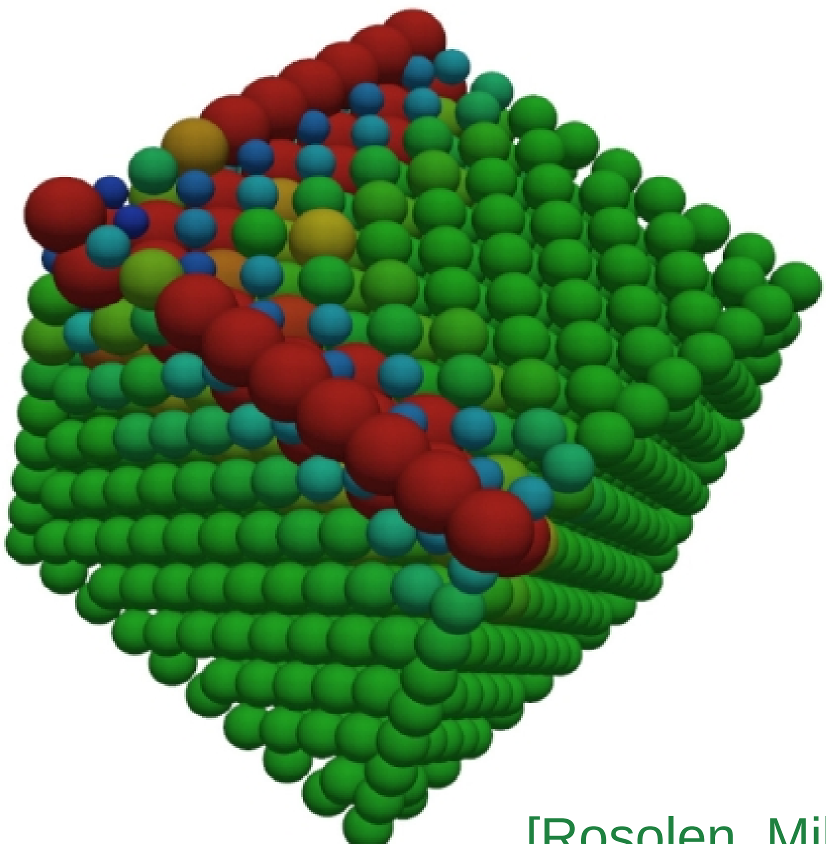
- Applications
  - Finite deformations
  - Thin-shell analysis
    - High-order PDEs
    - Linear and nonlinear
  - Variational fracture (phase-field approach)
  - Phase-field modeling of biomembranes
    - High-order PDEs
    - Statics and dynamics

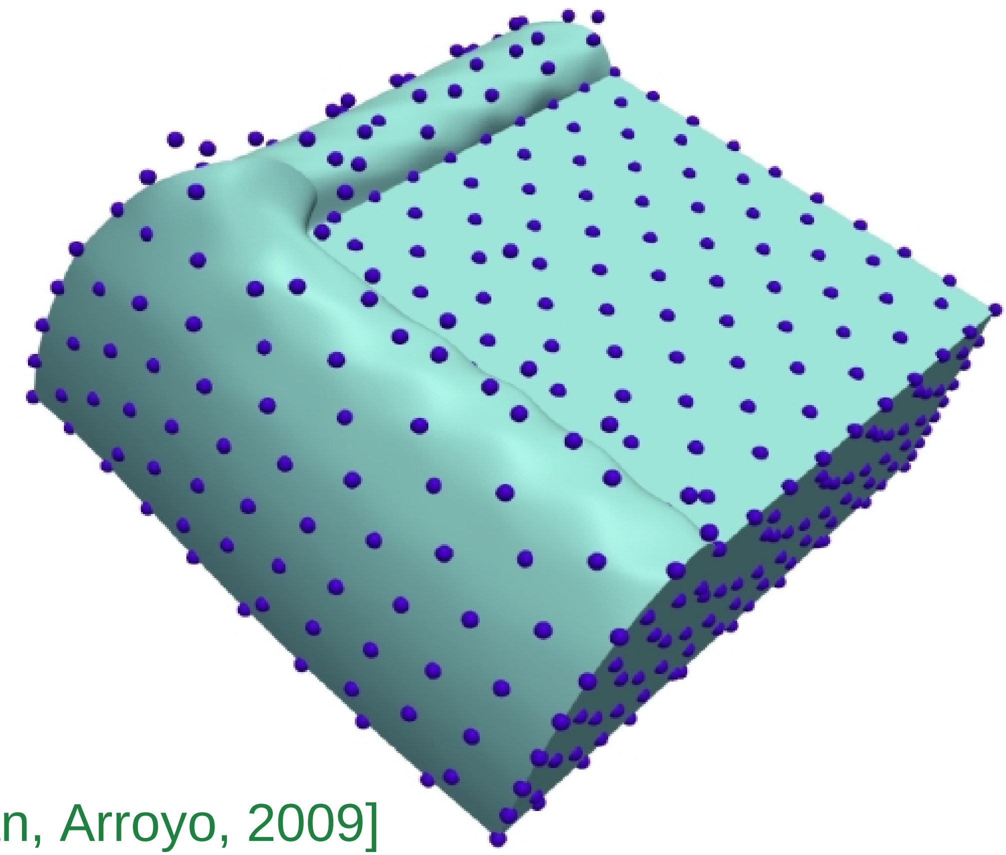
Non-uniform and unstructured set of points



Nonlinear elasticity

- Compression of a neo-Hookean hyperelastic block (nominal stretch ratio = 0.5)

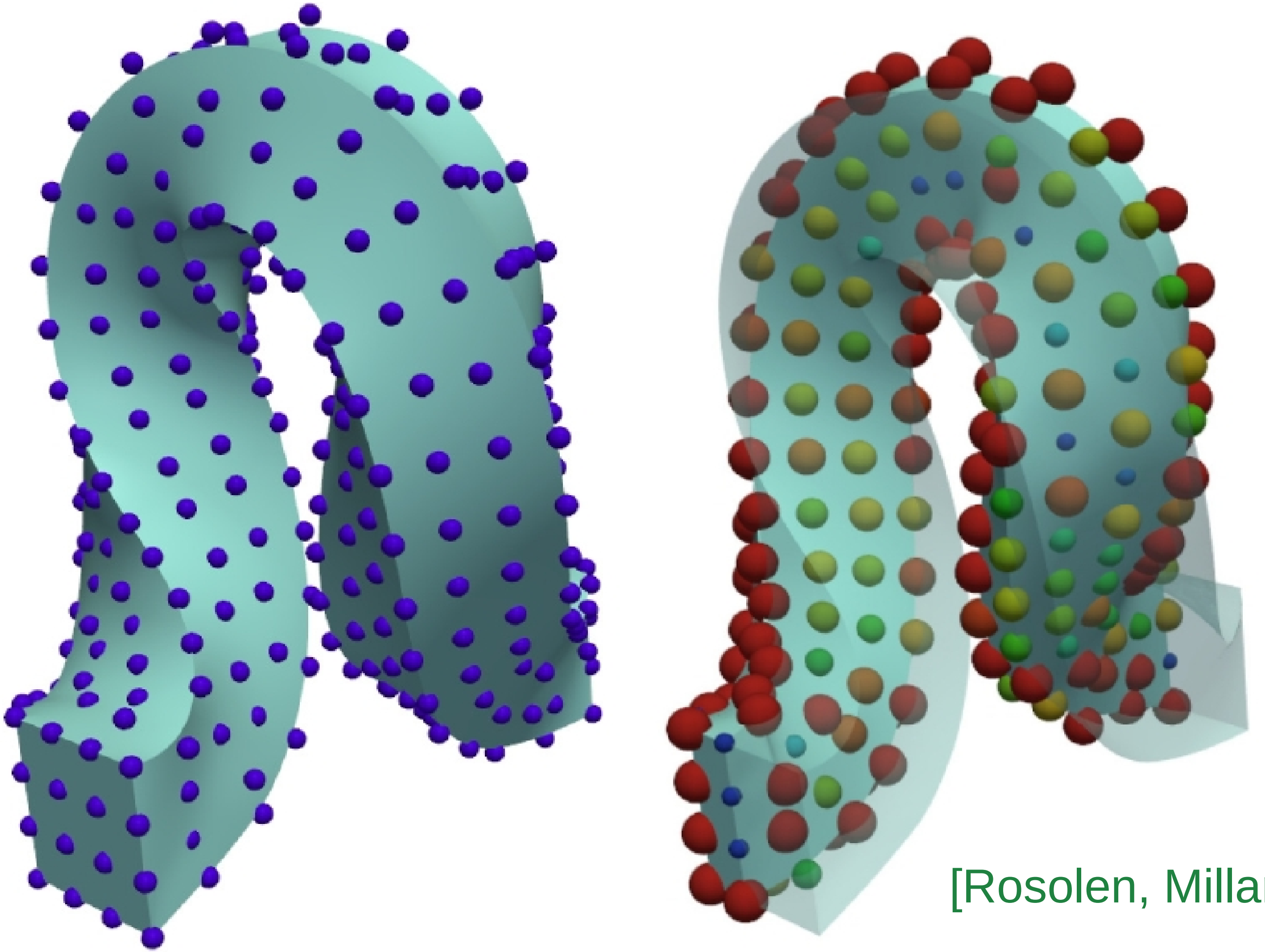




[Rosolen, Millan, Arroyo, 2009]

Nonlinear elasticity

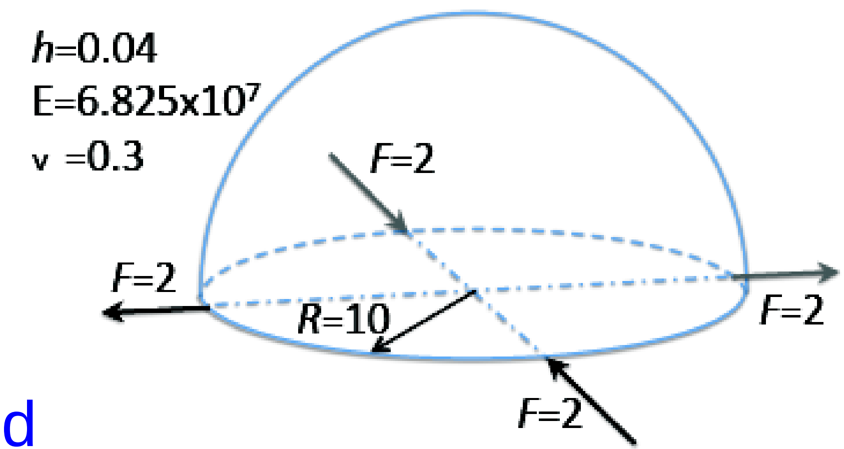
- Buckling of a neo-Hookean hyperelastic beam (nominal stretch ratio = 0.35)

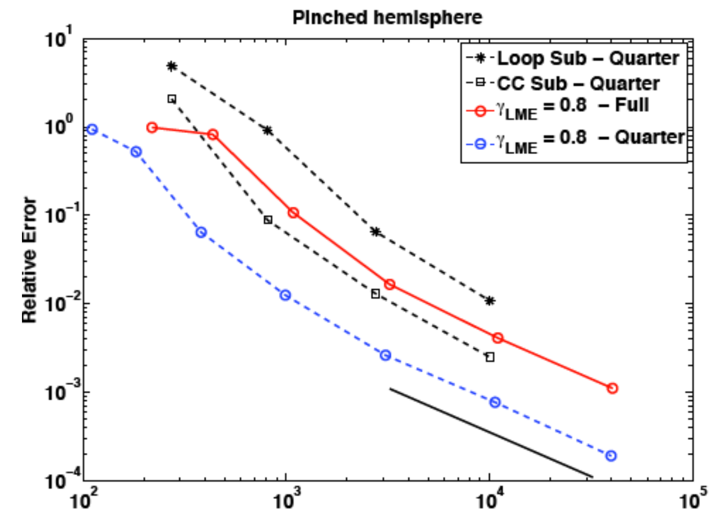


[Rosolen, Millan, Arroyo, 2009]

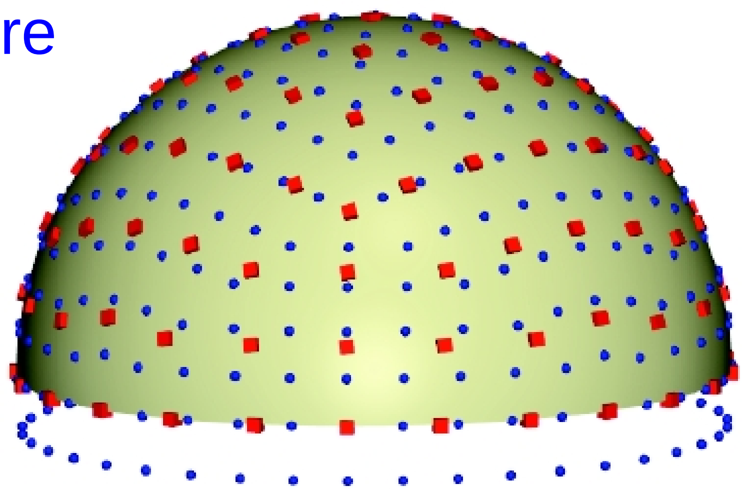
Linear thin-shell problems

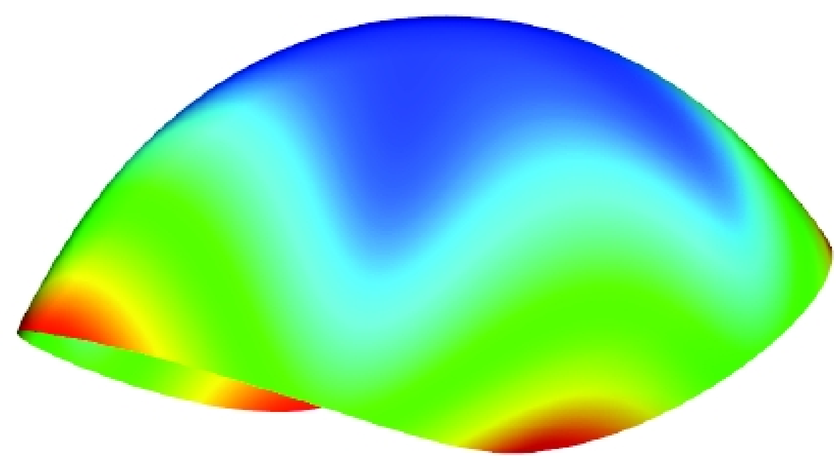
- Benchmark tests





Pinched hemisphere



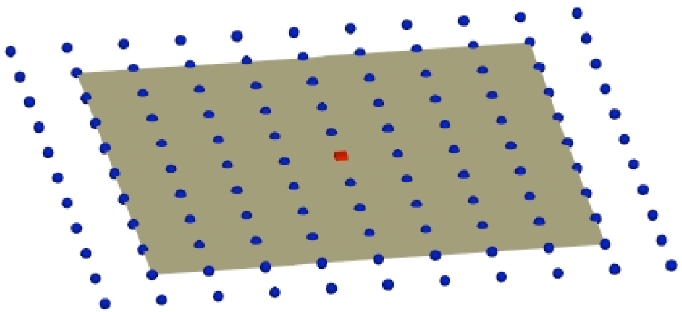


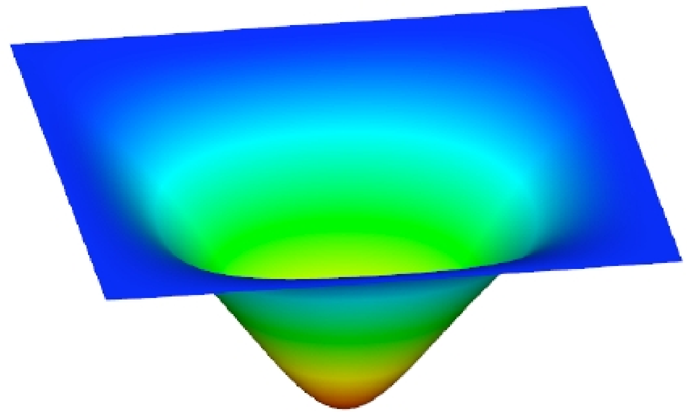
[Millan, Rosolen, Arroyo, 2011]

Linear thin-shell problems

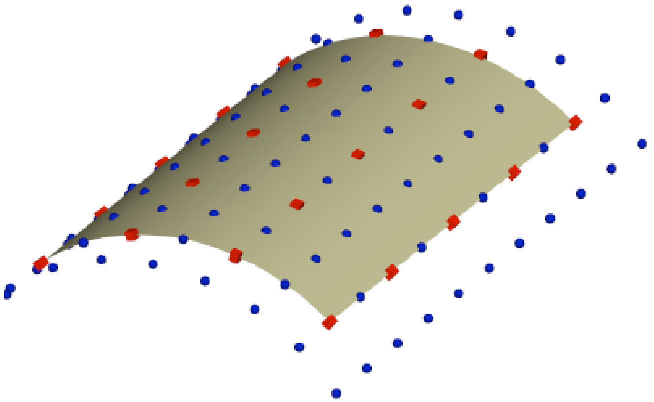
- Benchmark tests

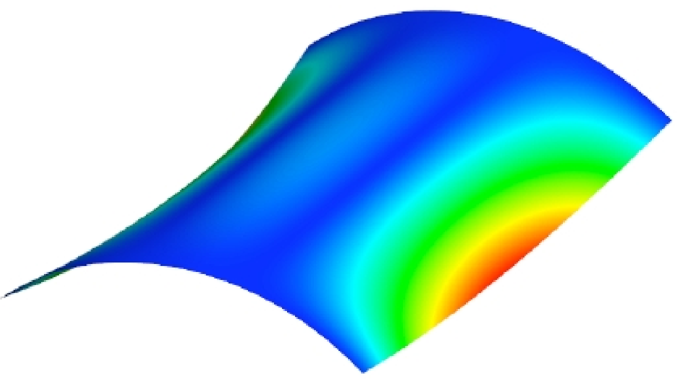
Clamped plate

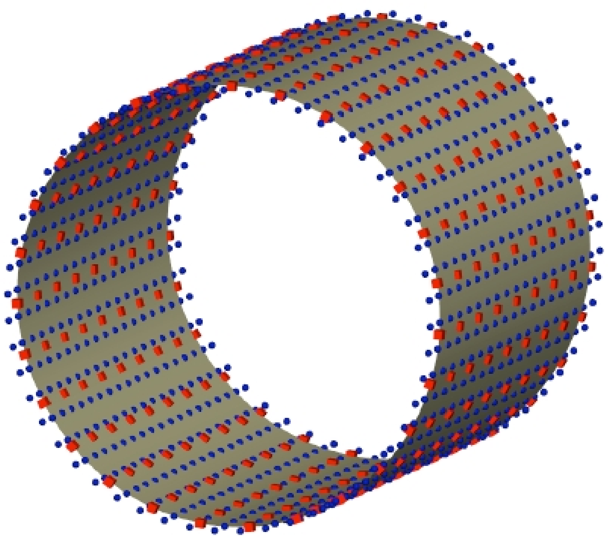


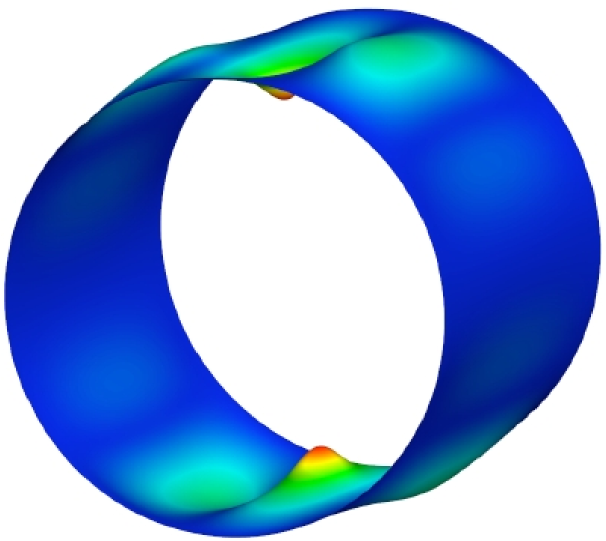


Scordelis roof







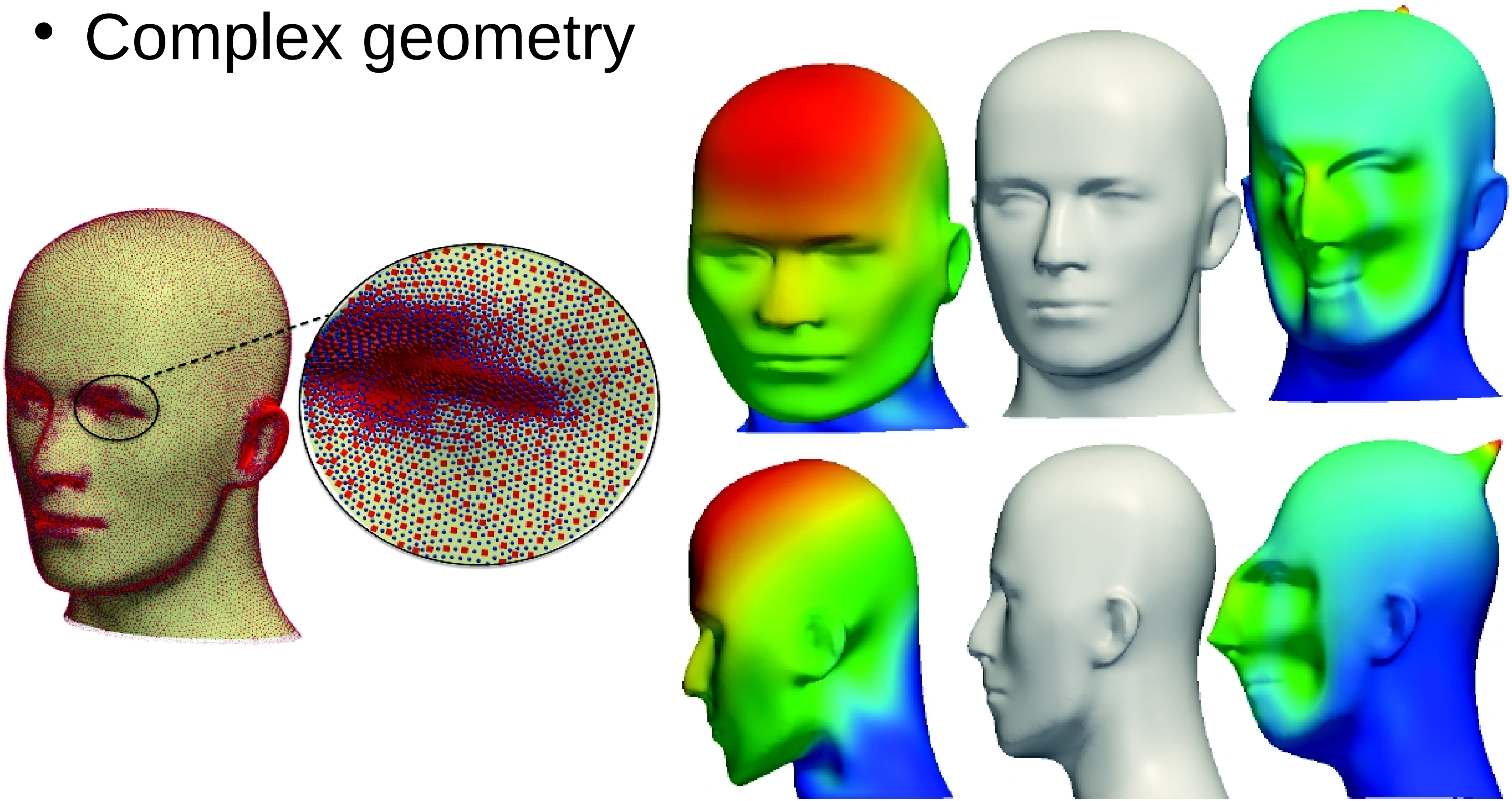


Pinched cylinder

[Millan, Rosolen, Arroyo, 2011]

Linear thin-shell problems

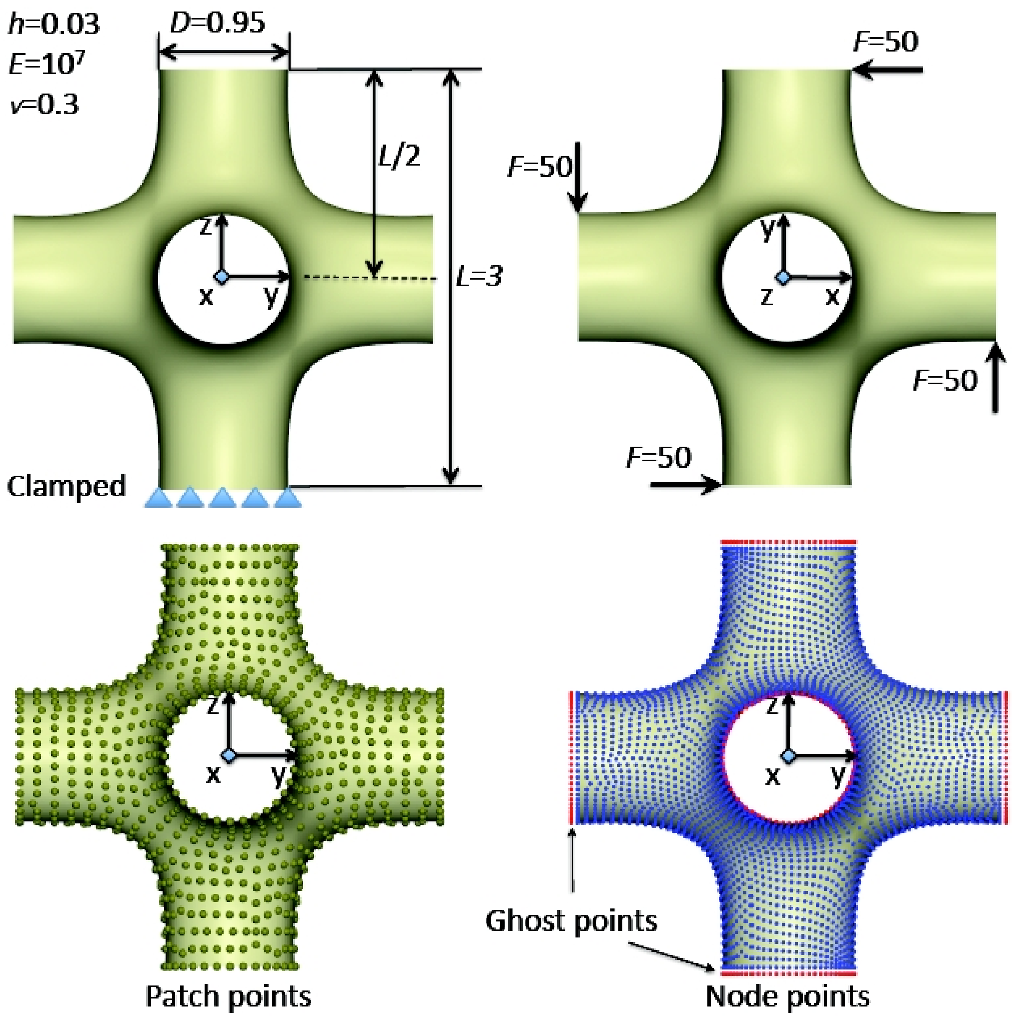
- Complex geometry

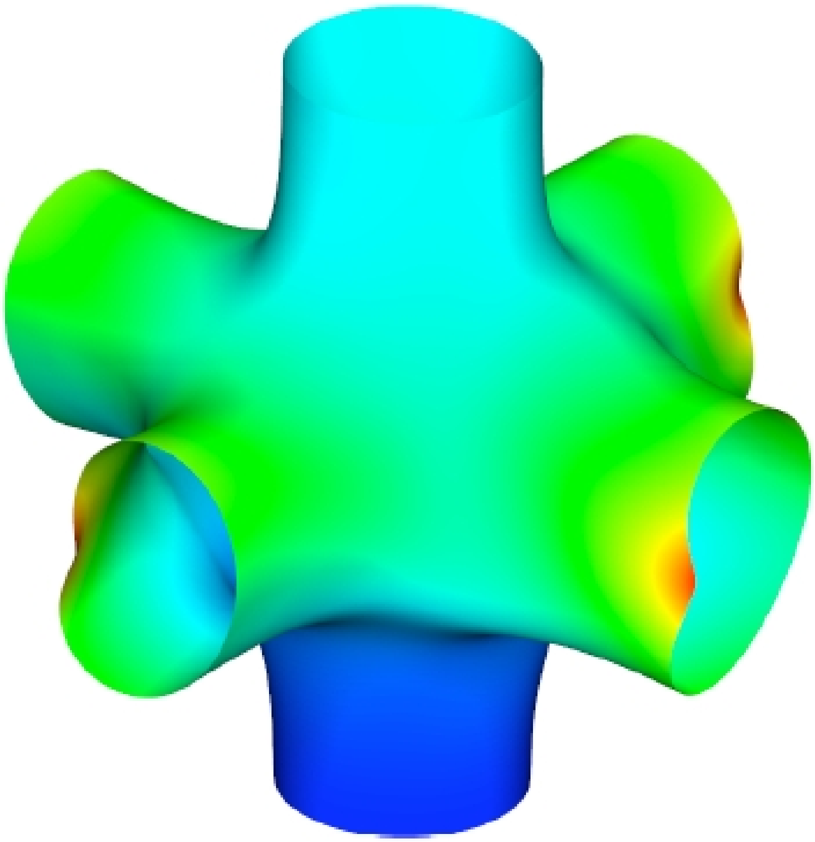


[Millan, Rosolen, Arroyo, 2011]

Linear thin-shell problems

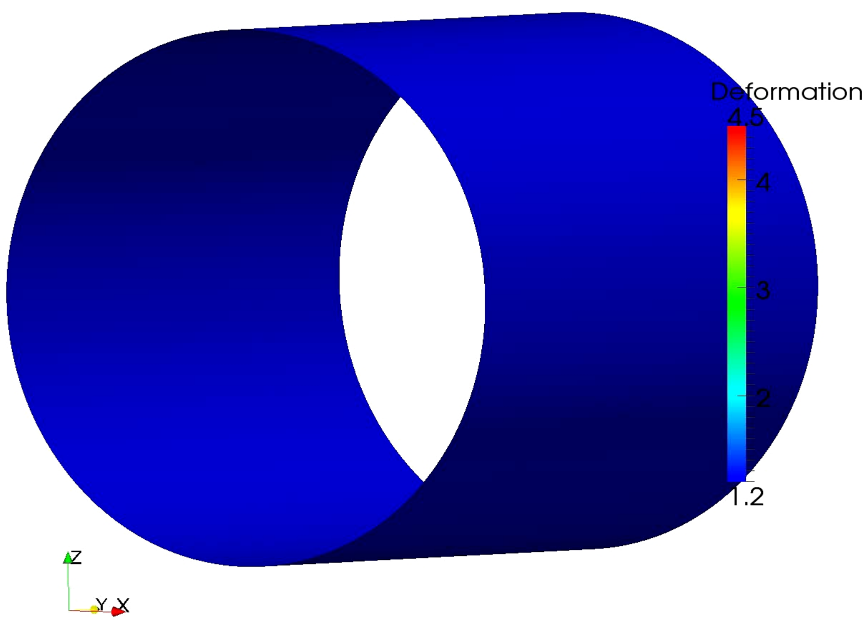
- Complex topology

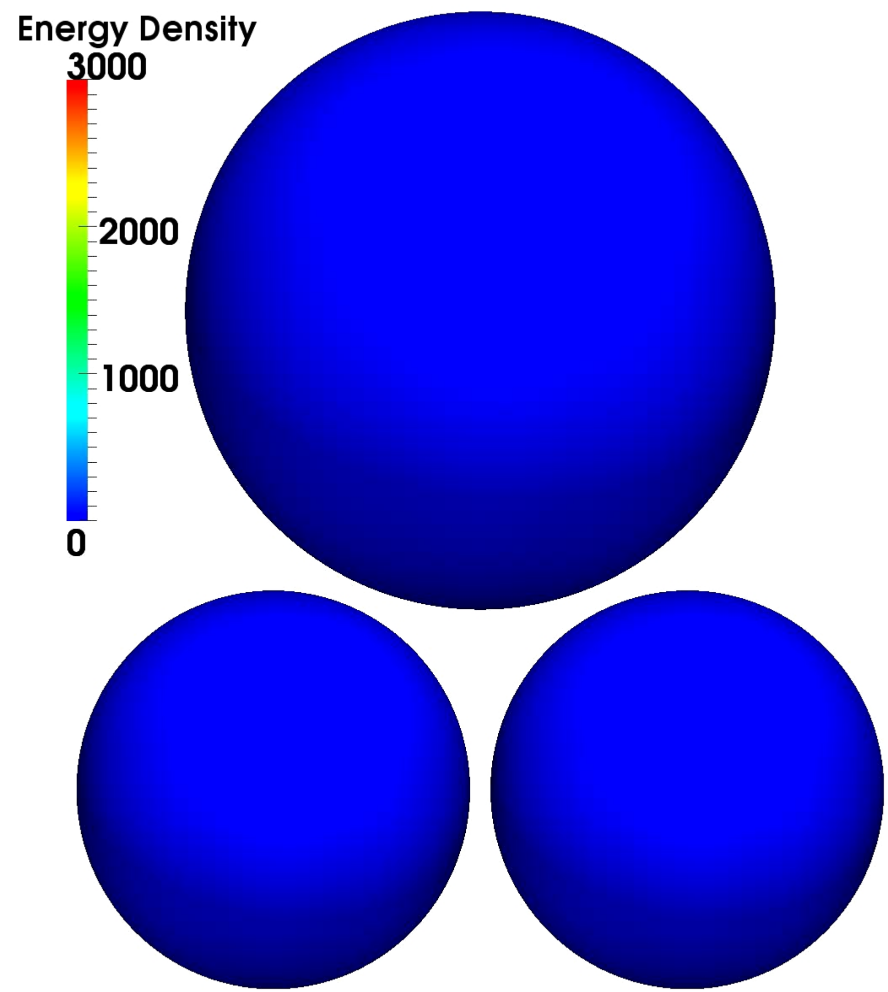




[Millan, Rosolen, Arroyo, 2011]

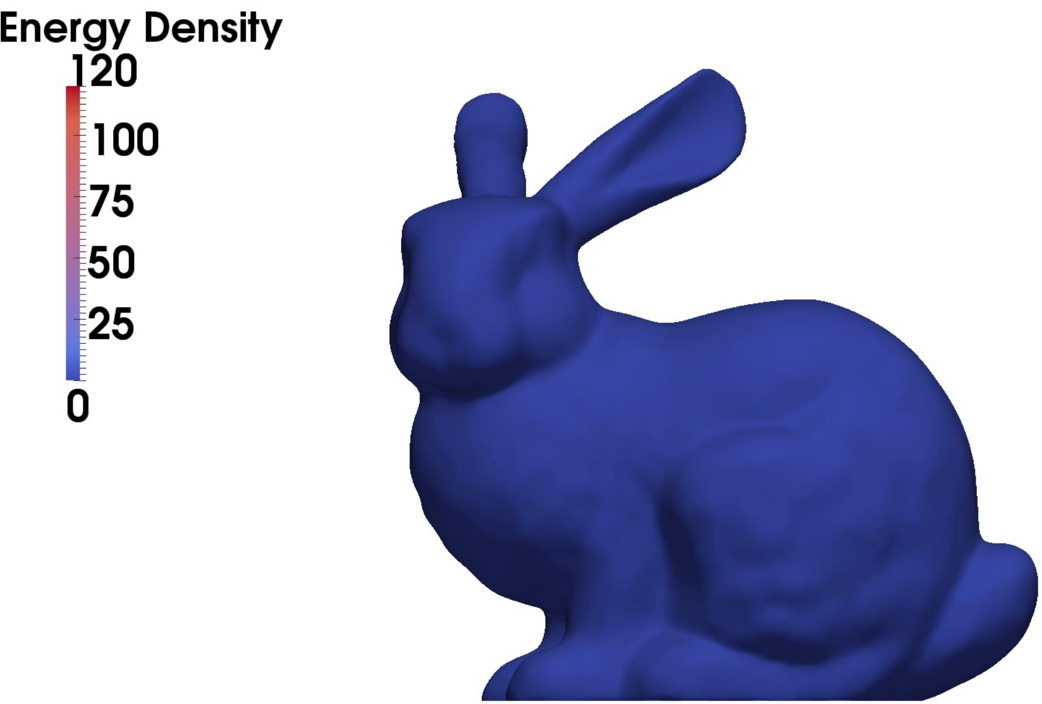
Nonlinear thin-shell problems

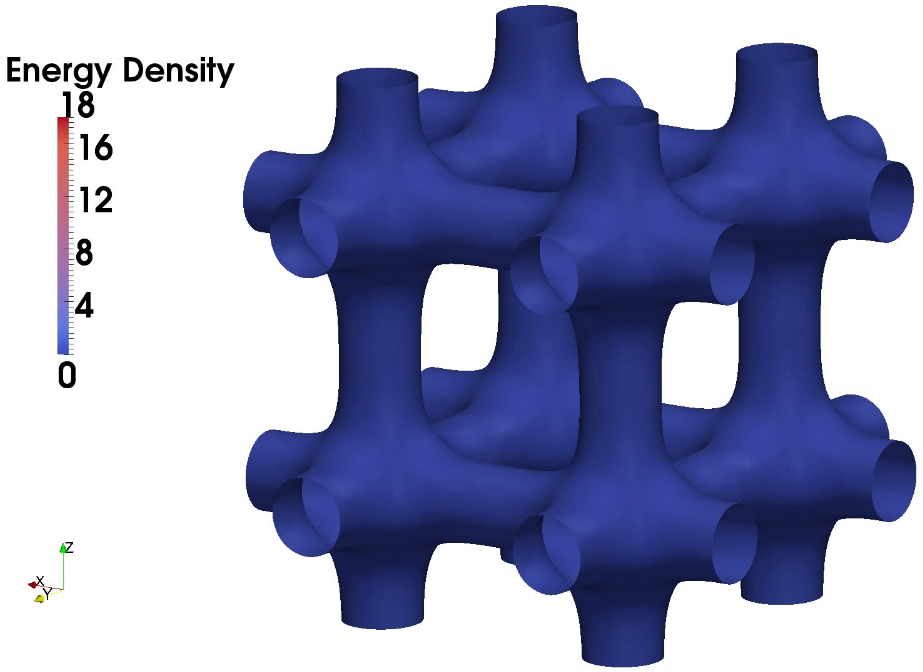




[Millan, Rosolen, Arroyo, 2012]

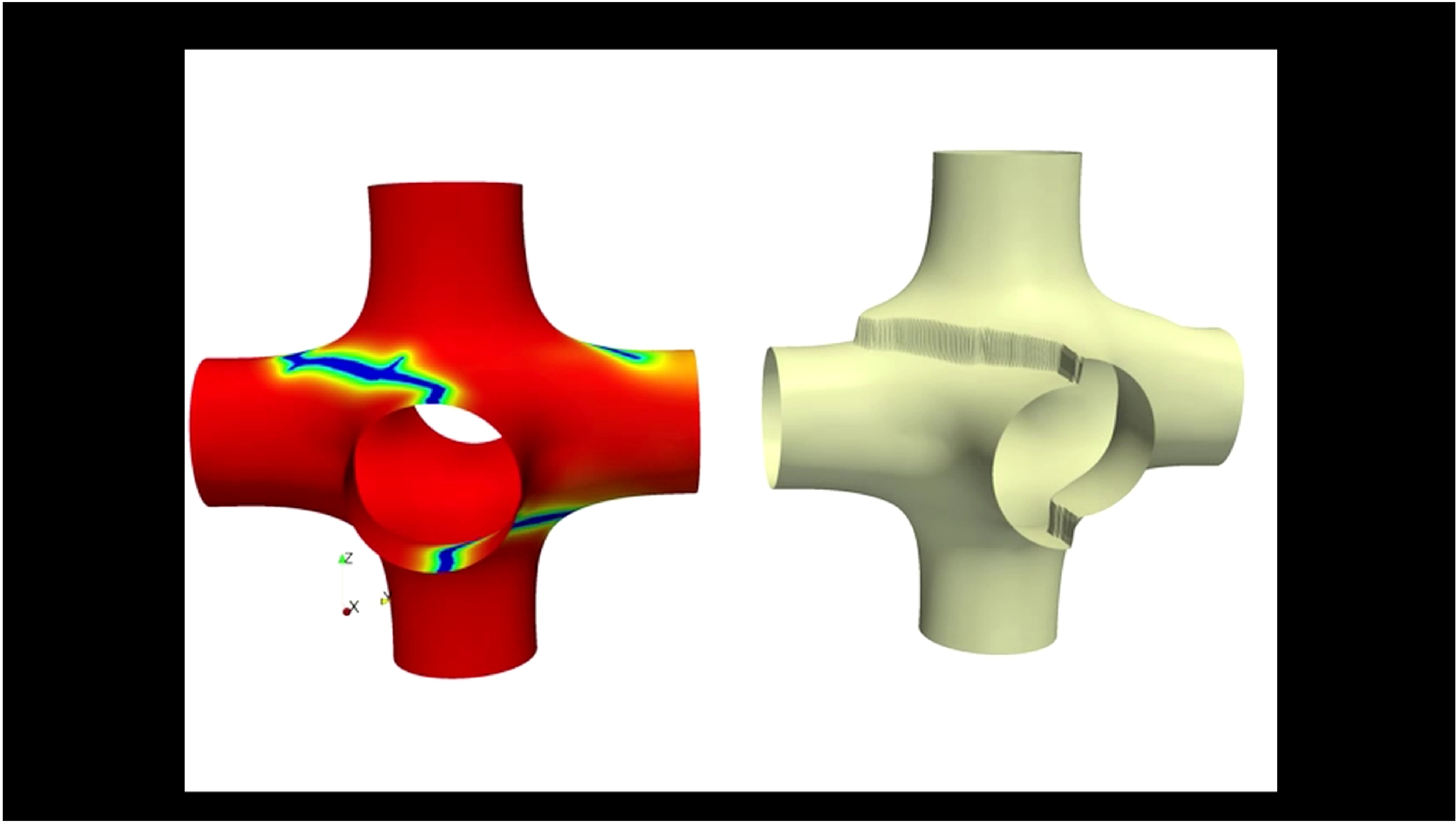
Nonlinear thin-shell problems





[Millan, Rosolen, Arroyo, 2012]

Variational Fracture (phase-field approach)

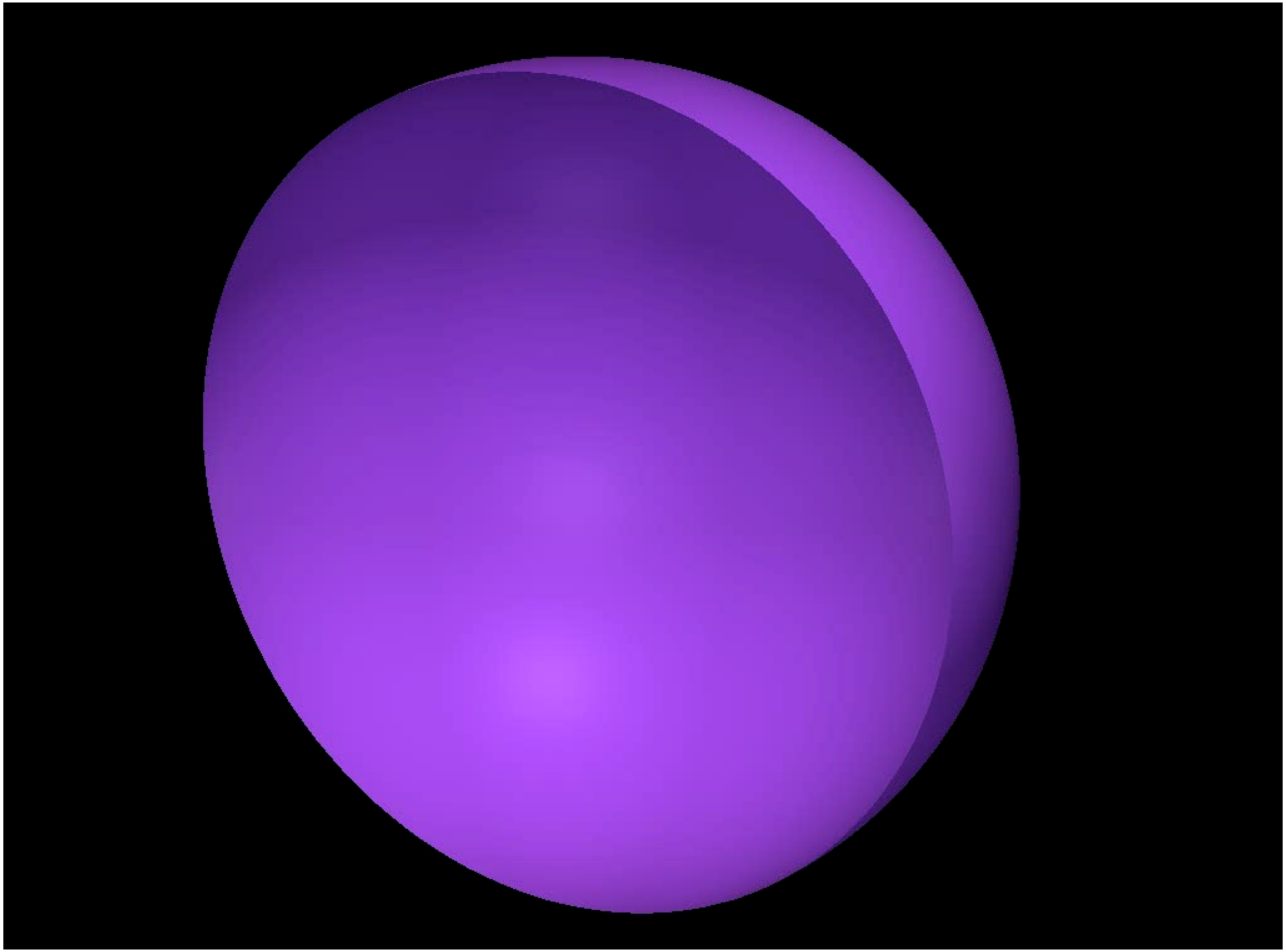


[Millan, Arroyo, et al., 2012]

Phase-field model of biomembranes

Statics

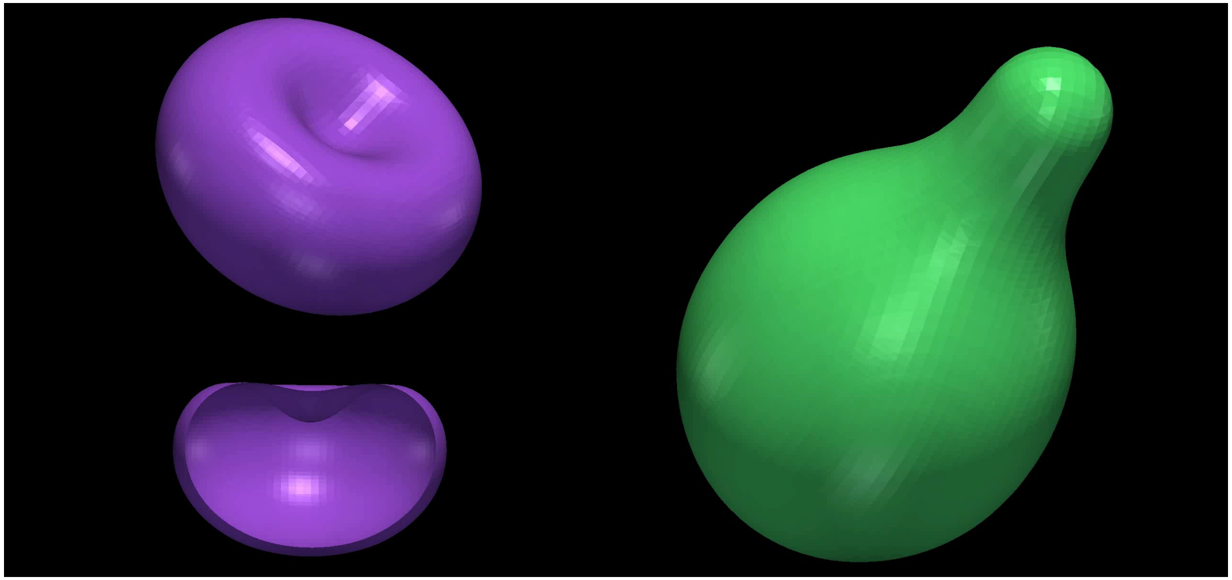
Oblate branch



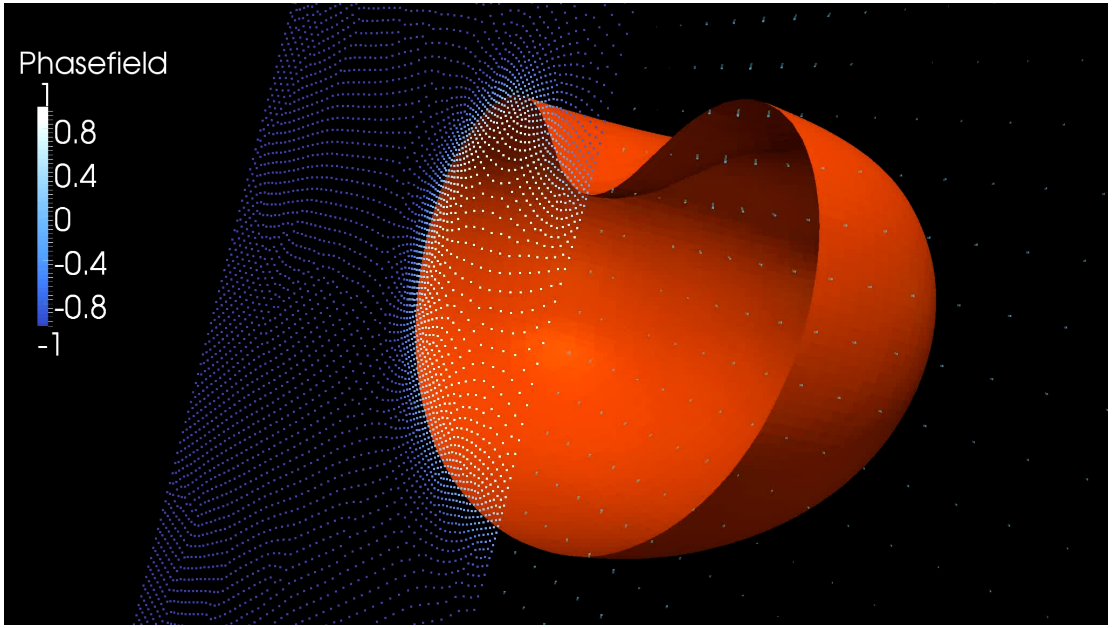
[Rosolen, Peco, Arroyo, 2012]

Phase-field model of biomembranes

Dynamics



Pear-shaped

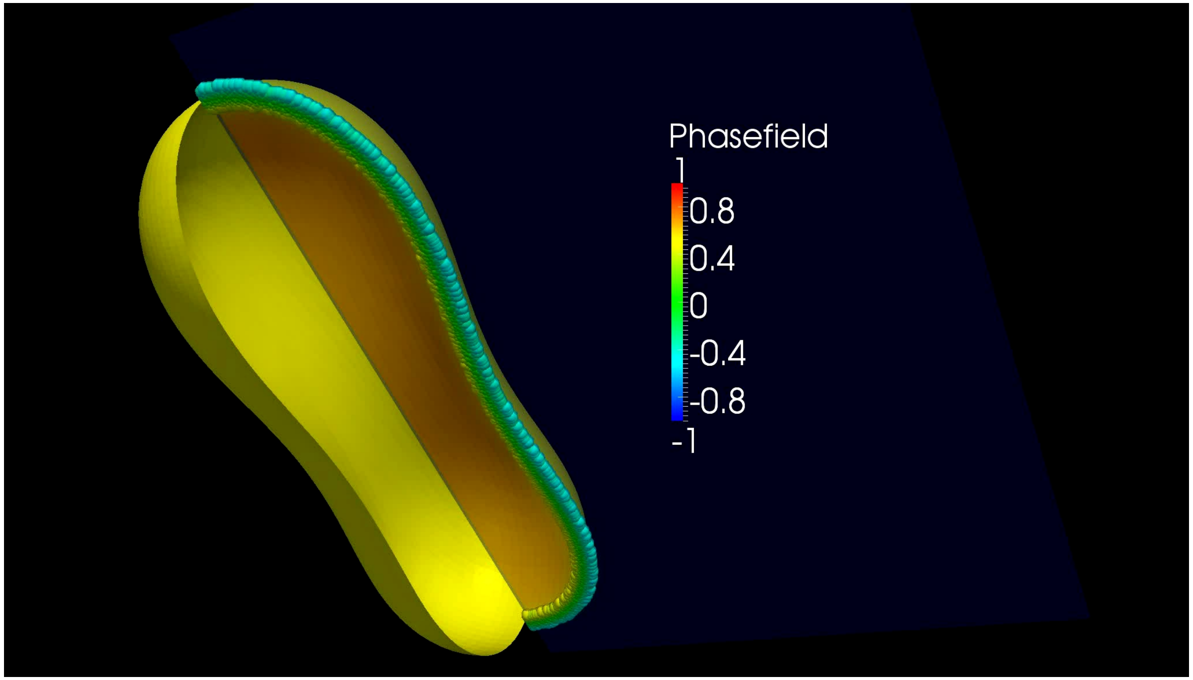


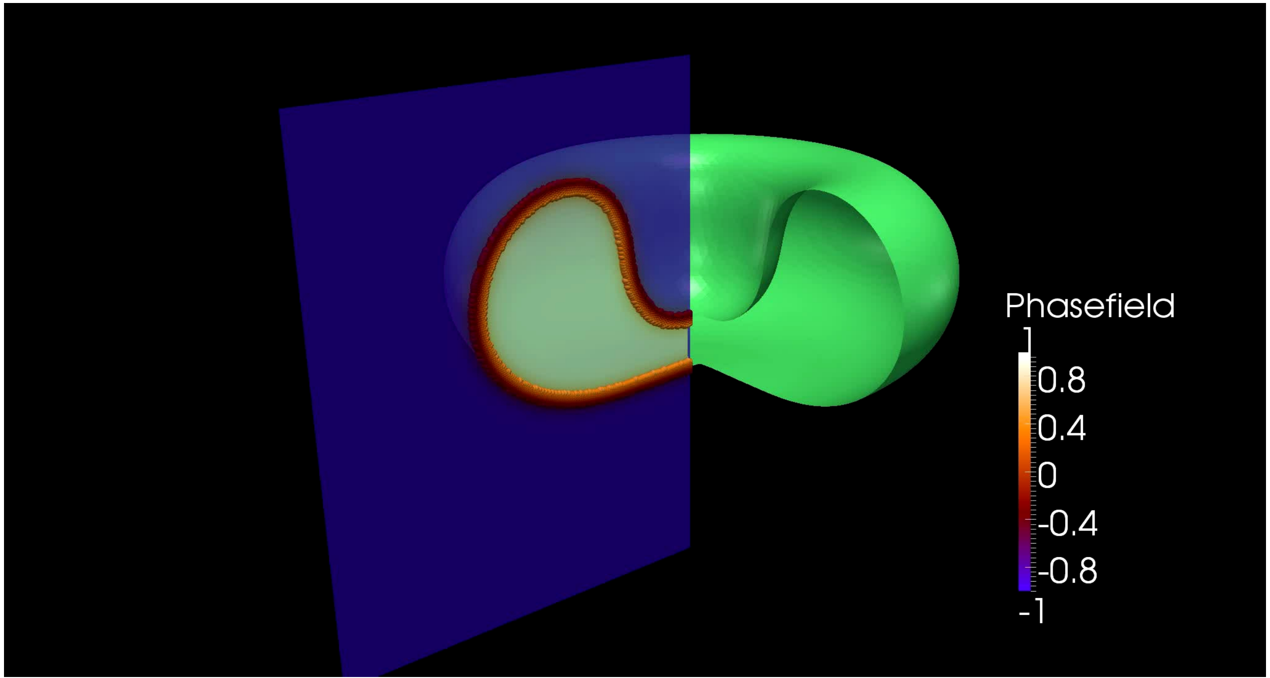
Discocyte

[Peco, Rosolen, Arroyo, 2012]

Phase-field model of biomembranes

Dynamics





Dumbell;
Pear-shaped

Stomatocyte

[Peco, Rosolen, Arroyo, 2012]

Isogeometric Analysis and maximum-entropy approximants

- In the last years, the excellent properties of smooth non-negative basis functions have motivated their use in the numerical solution of PDE

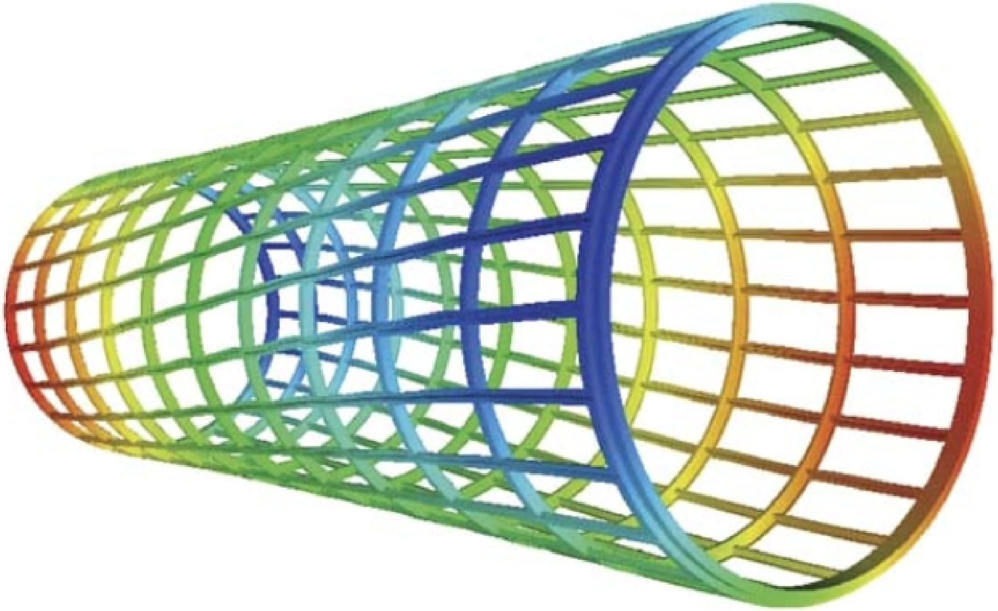
- Subdivision finite elements [Cirak et al., 2000]
  - Two dimensional approximants on unstructured grids

- Isogeometric Analysis [Hughes et al., 2005]

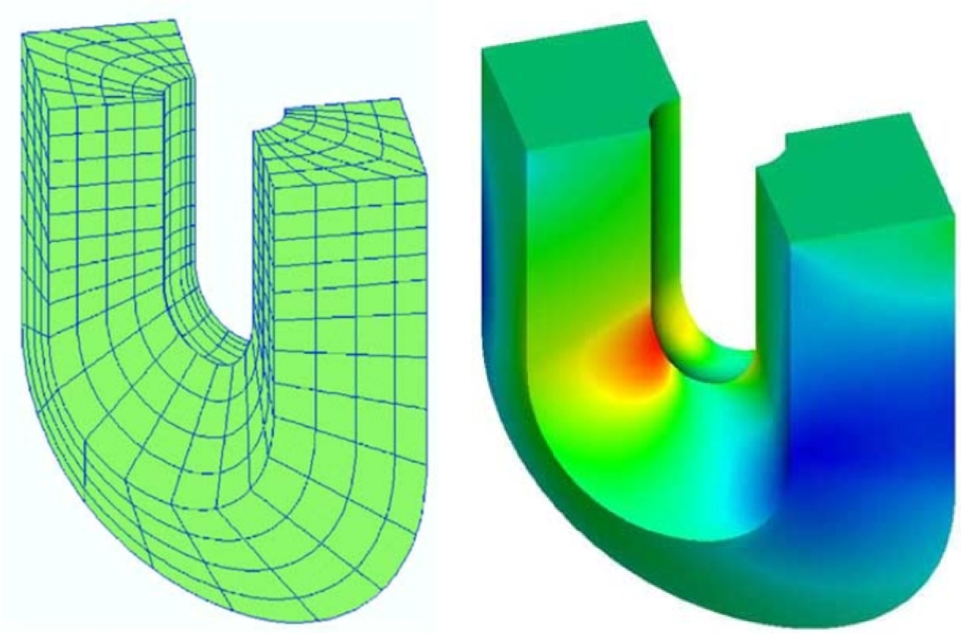
- Maximum entropy approximation schemes

Isogeometric analysis

- The same basis functions are used to describe the geometry and to interpolate the physical fields



[Cottrell, CMAME, 2006]



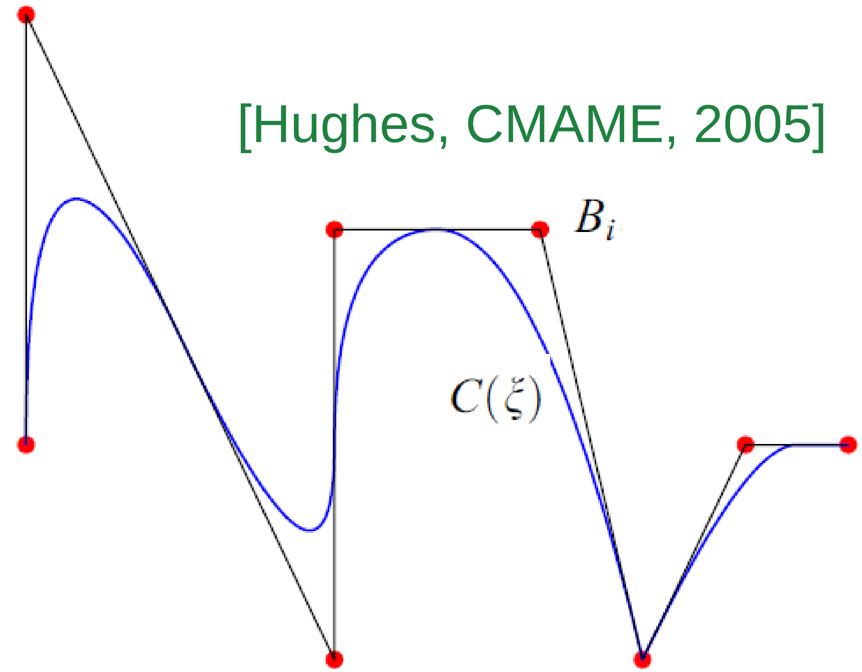
[Hughes, CMAME, 2005]

Isogeometric analysis

- Based on NURBS approximation schemes
- Description of the geometry with CAD fidelity
- (semi-)Structured grids
  - Current research in T-Splines
- Handling multiple boundaries requires significant preprocessing and techniques to have globally smooth approximation

Isogeometric analysis

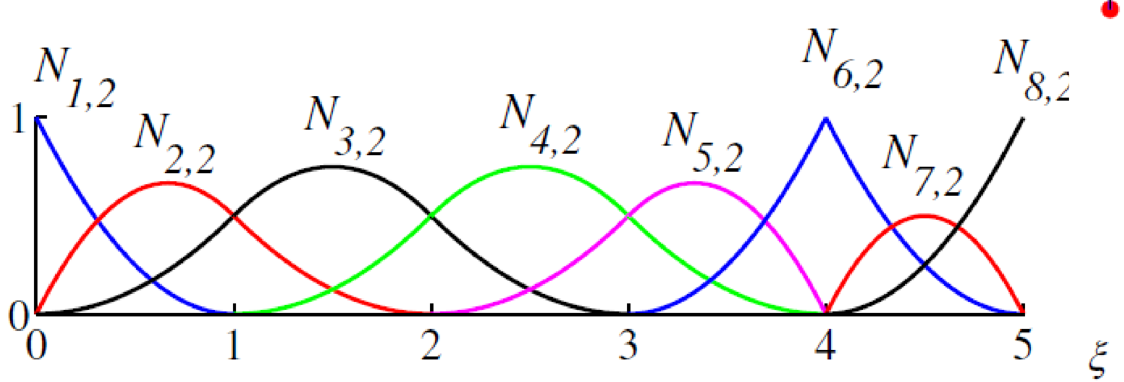
[Hughes, CMAME, 2005]



- Representation of a curve

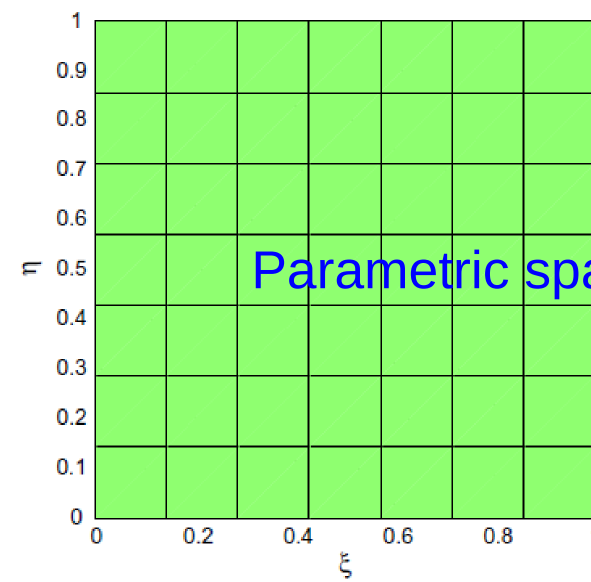
$$C(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi) B_i$$

- Parametric space: the knot vector



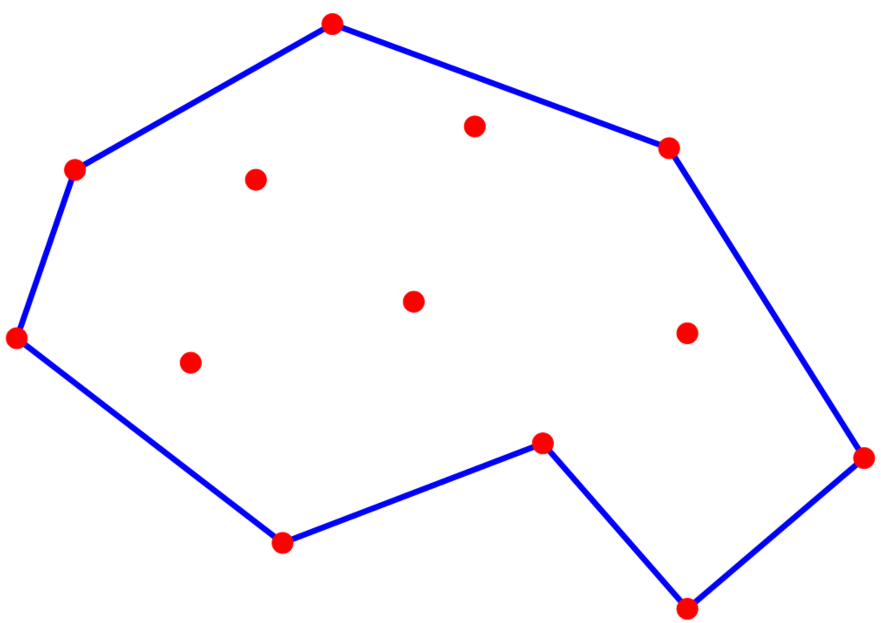
$$\Xi = \{0,0,0,1,2,3,4,4,5,5,5\}$$

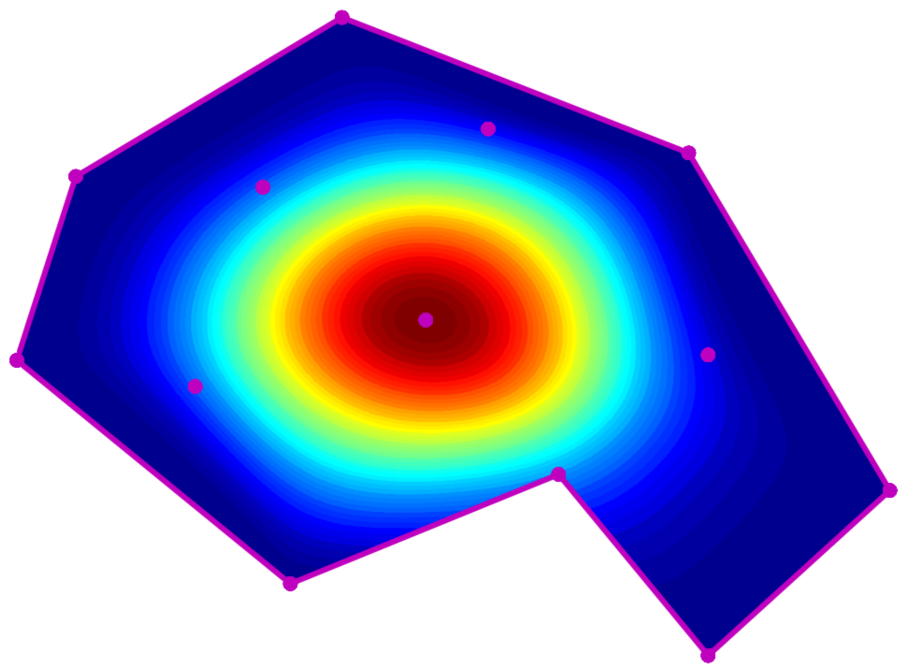
- Control points
- Higher dimensions:
  - Parametric space is local to “patches”



Intrinsic limitation of meshfree methods

- Solely with points, only polytopes can be represented
- Non high fidelity representation of the geometry



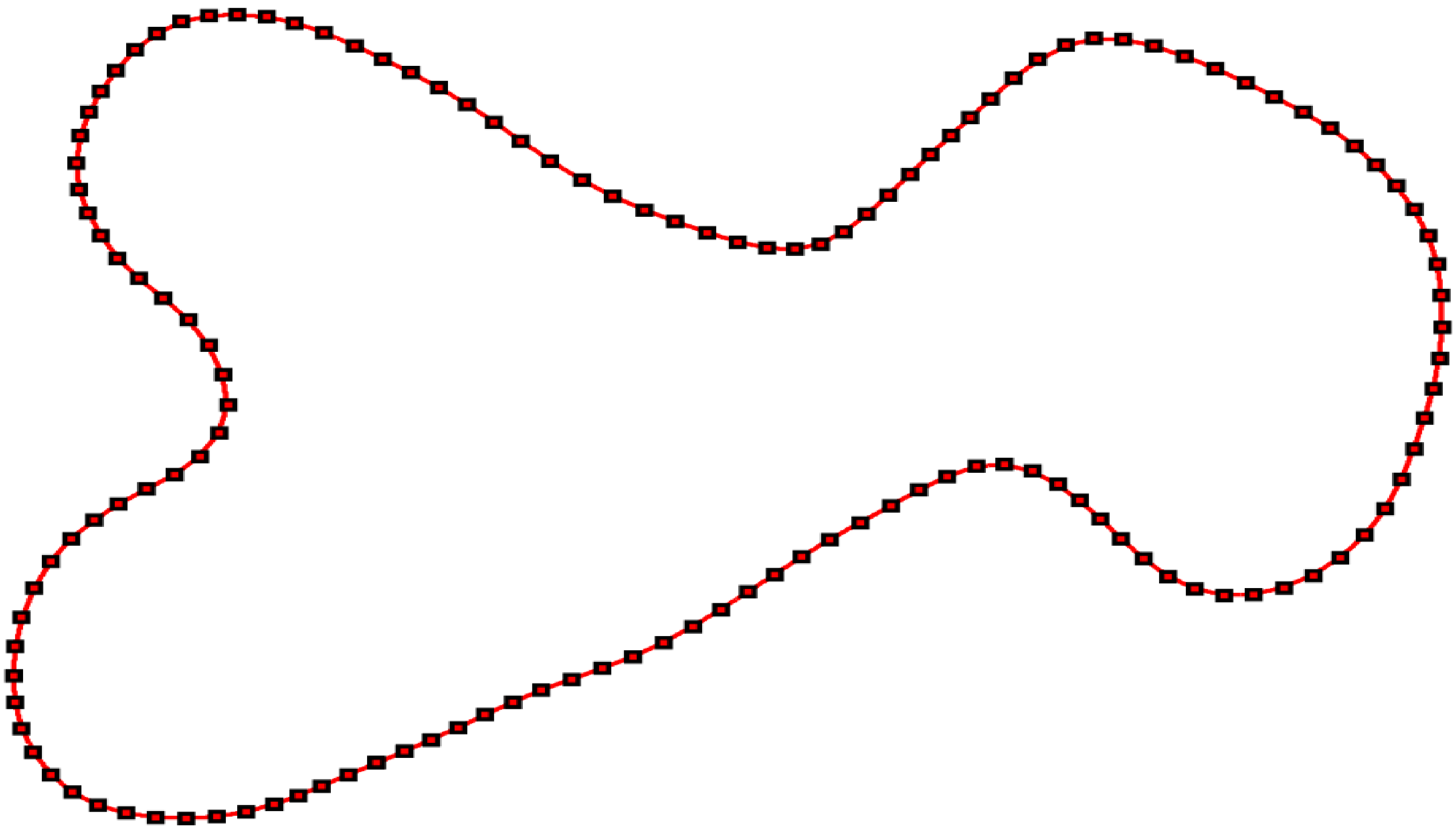


Blending B-Splines and max-ent

- Limitations of max-ent and B-Splines are complementary
  - Handling of unstructured meshes
  - Description of the geometry with high fidelity
- The convex structure is shared
  - Suggests blending through a convex optimization problem
- Related approch: NEFEM [Sevilla et al., 2010]
  - High fidelity representation of the geometry
  - Coupling between NURBS and high-order FEM-DG
  - Smoothness and positivity of the basis functions is not preserved

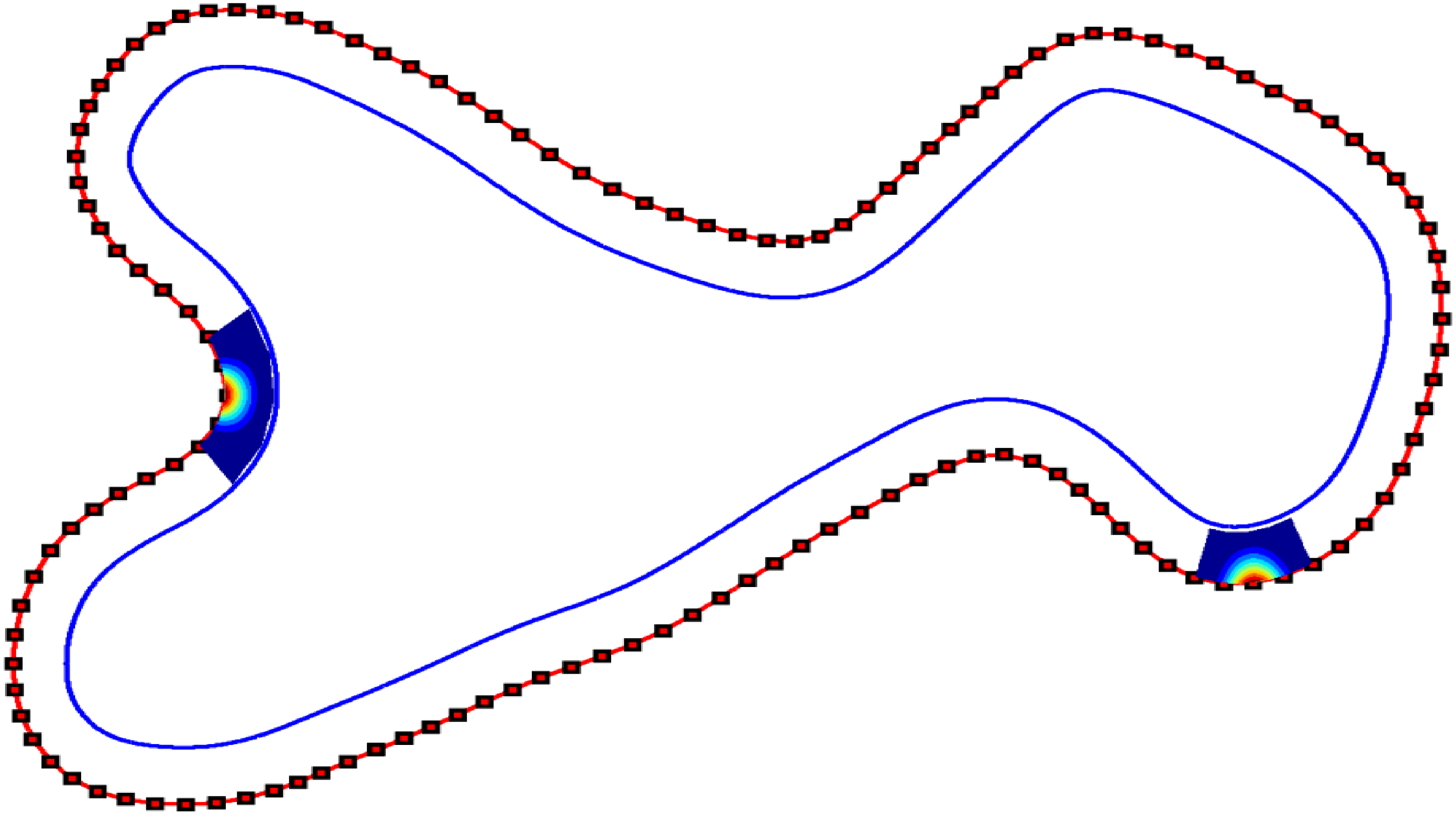
Blending B-Splines and max-ent

- CAD description of the boundary (B-Spline curve)



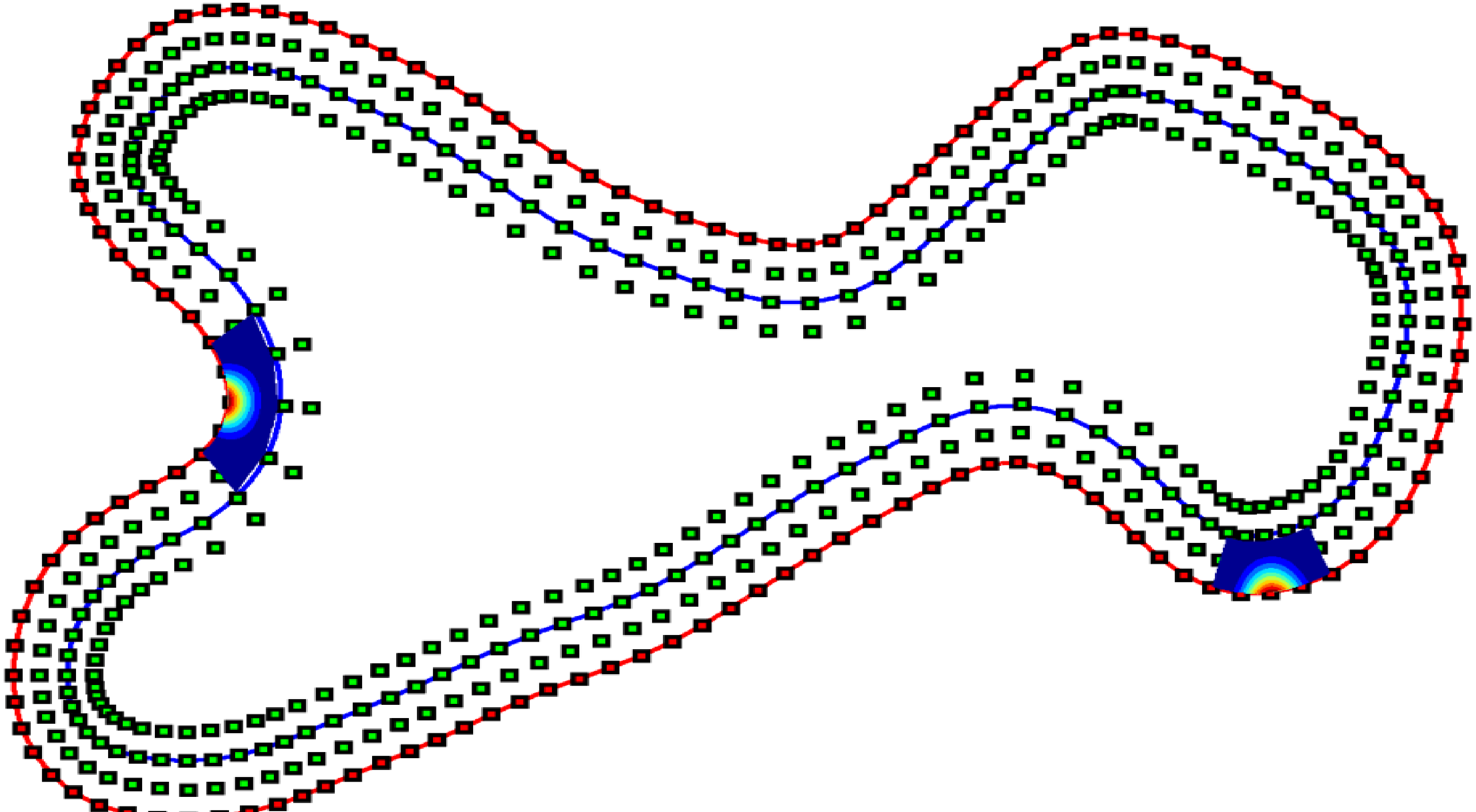
Blending B-Splines and max-ent

- CAD description of the boundary (B-Spline curve)
- Single layer of Isogeometric (B-Spline + isoparametric mapping) basis functions



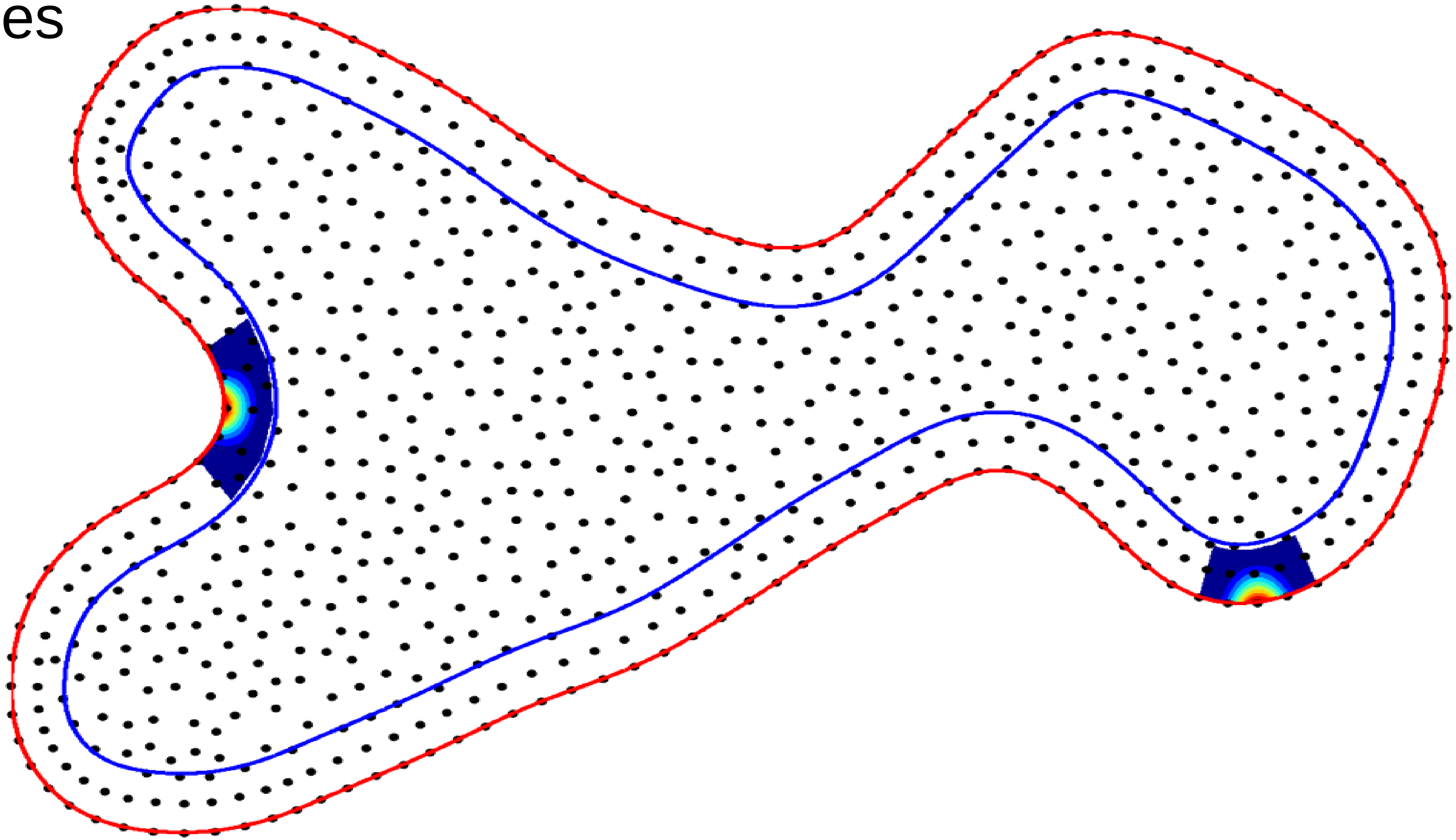
Blending B-Splines and max-ent

- CAD description of the boundary (B-Spline curve)
- Single layer of Isogeometric (B-Spline + isoparametric mapping) basis functions
- Isoparametric mapping (control points involved shown in green)



Blending B-Splines and max-ent

- The distribution of the interior points is unstructured
- The basis functions are purely isogeometric for boundary nodes



Blending B-Splines and max-ent

- Blending scheme: convex optimization problem
  - Imposition of reproducibility conditions
  - The interior basis functions are unknown

Iso/LME basis function

$$\text{For fixed } \boldsymbol{x} \text{ minimize} \quad \sum_{a \in \mathfrak{I}_{ME}} m_a \ln m_a + \sum_{a \in \mathfrak{I}_{ME}} \beta_a m_a |\boldsymbol{x} - \boldsymbol{x}_a|^2$$

$$\text{subject to} \quad m_a \geq 0, \quad a \in \mathfrak{I}_{ME}$$

B-Splines

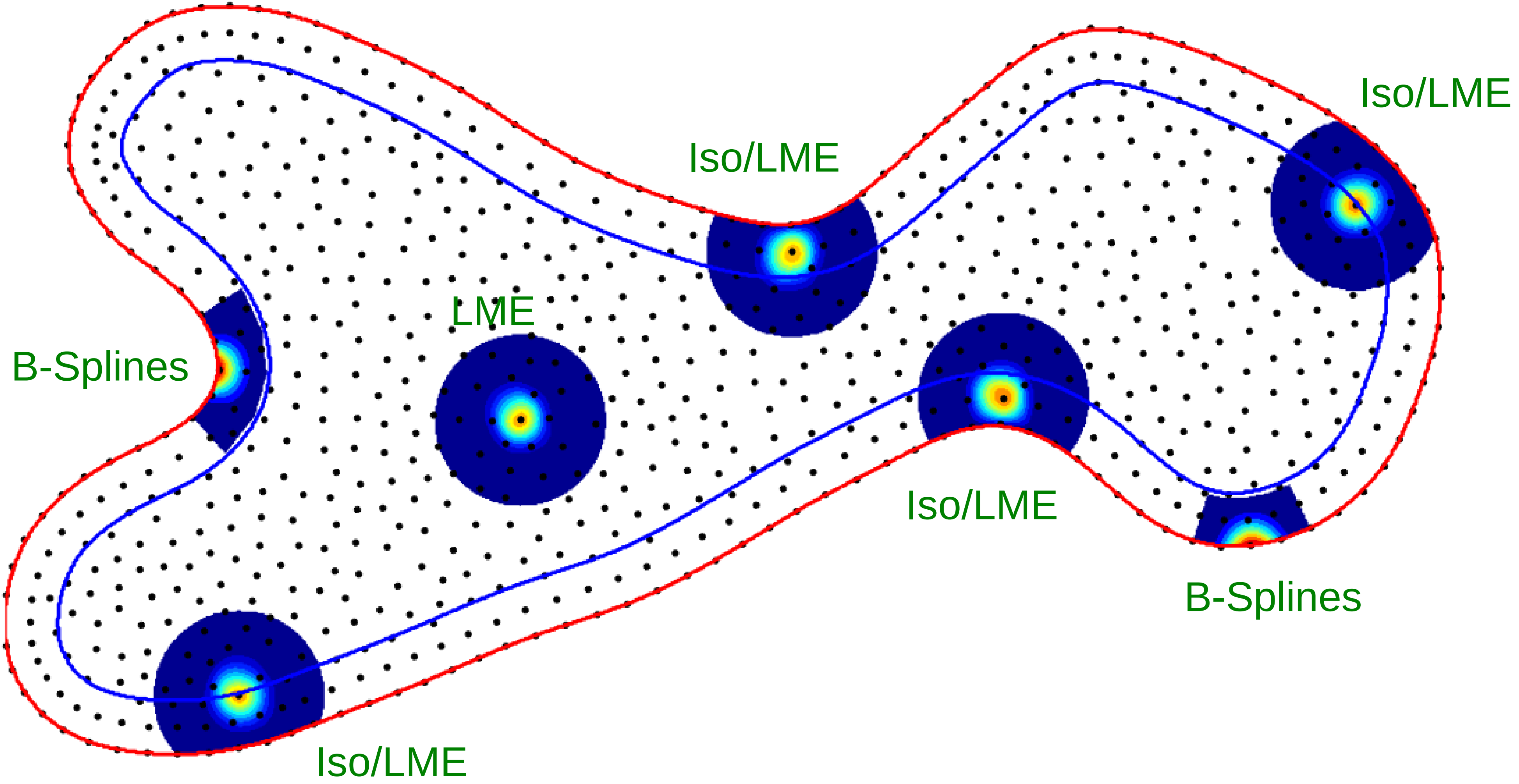
$$\sum_{a \in \mathfrak{I}_{ME}} m_a + \sum_{b \in \mathfrak{I}_{BS}} N_b(\boldsymbol{x}) = 1$$

Interior Nodes — Boundary nodes

$$\sum_{a \in \mathfrak{I}_{ME}} m_a \, \boldsymbol{x}_a + \sum_{b \in \mathfrak{I}_{BS}} N_b(\boldsymbol{x}) \, \boldsymbol{x}_b = \boldsymbol{x}.$$

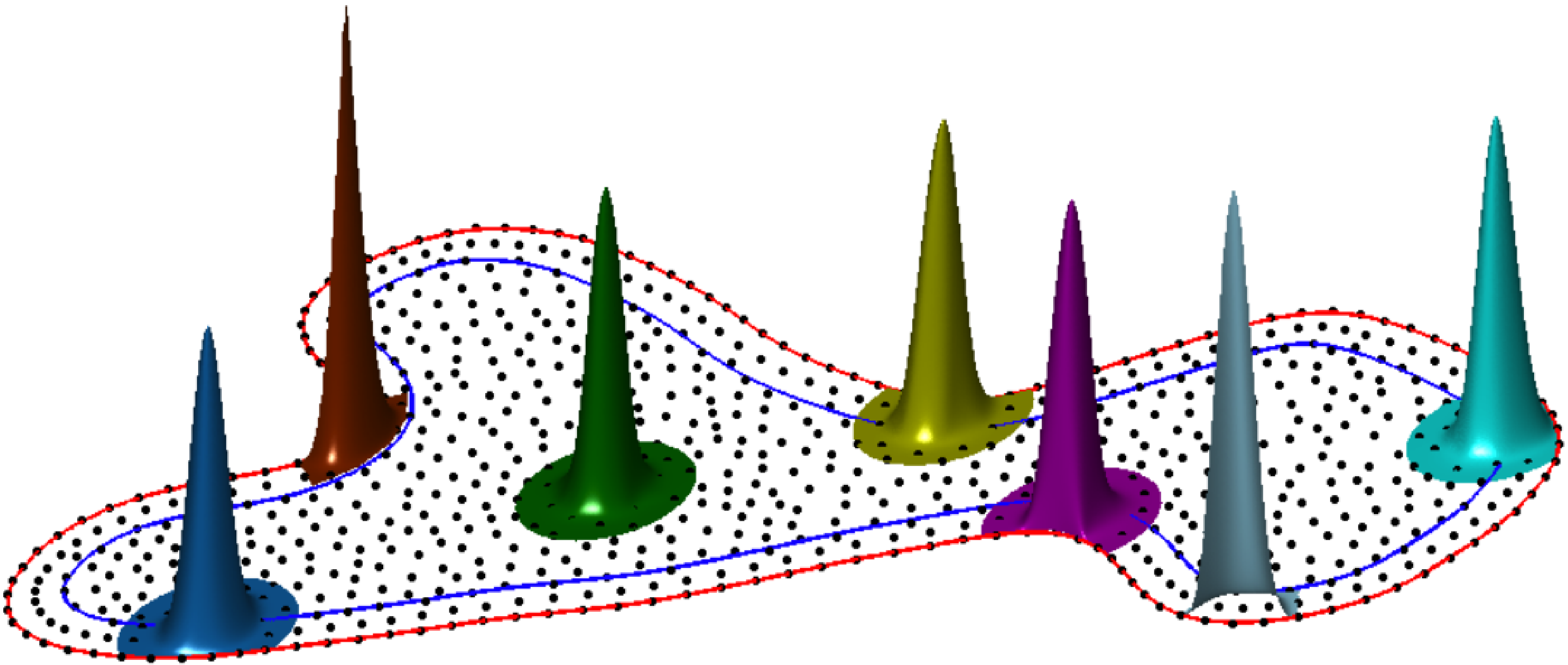
Blending B-Splines and max-ent

- Top view of B-Splines, local maximum-entropy (LME) approximants, and isogeometric/LME basis functions

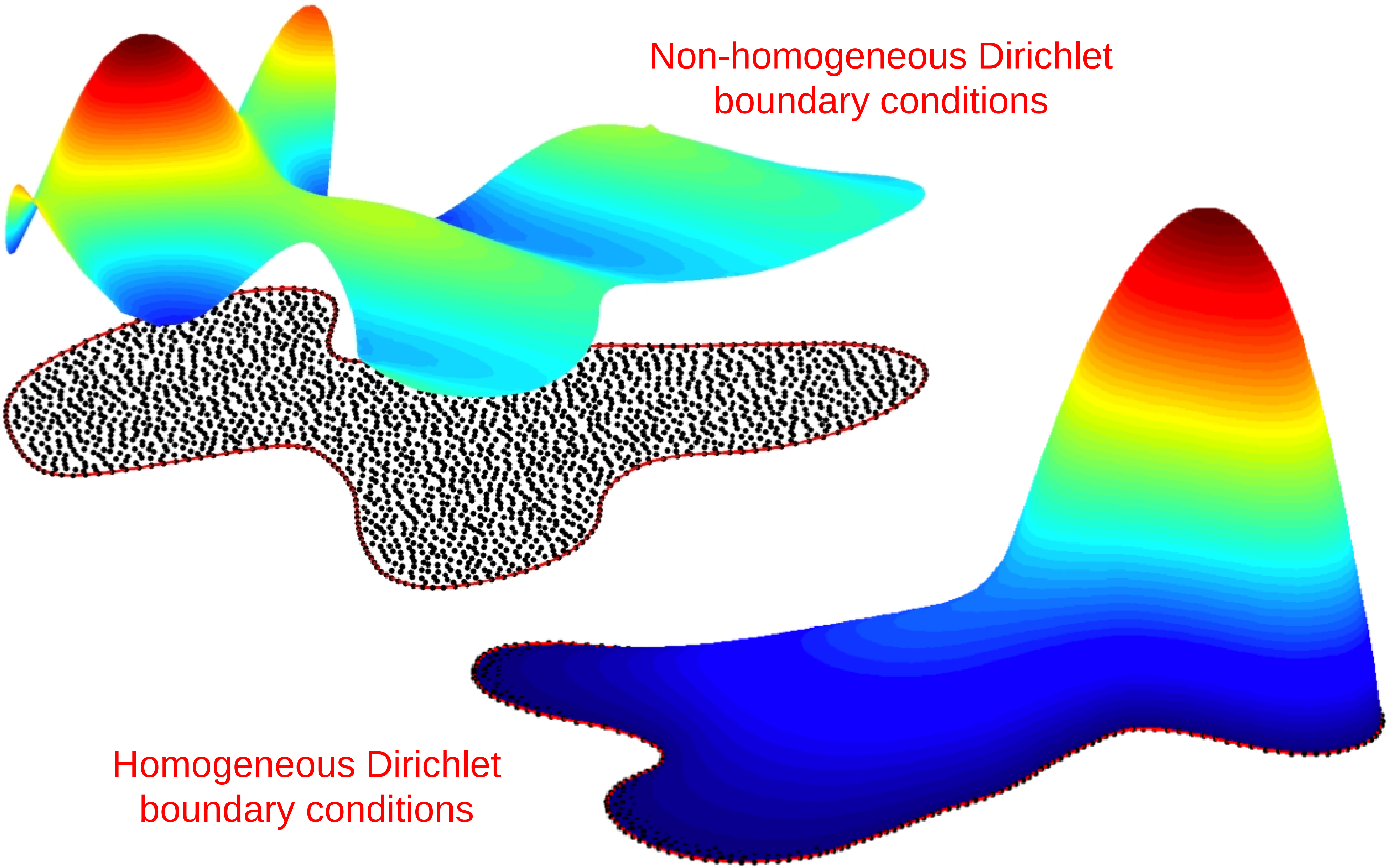


Blending B-Splines and max-ent

- 3D view of B-Splines, local maximum-entropy (LME) approximants, and isogeometric/LME basis functions

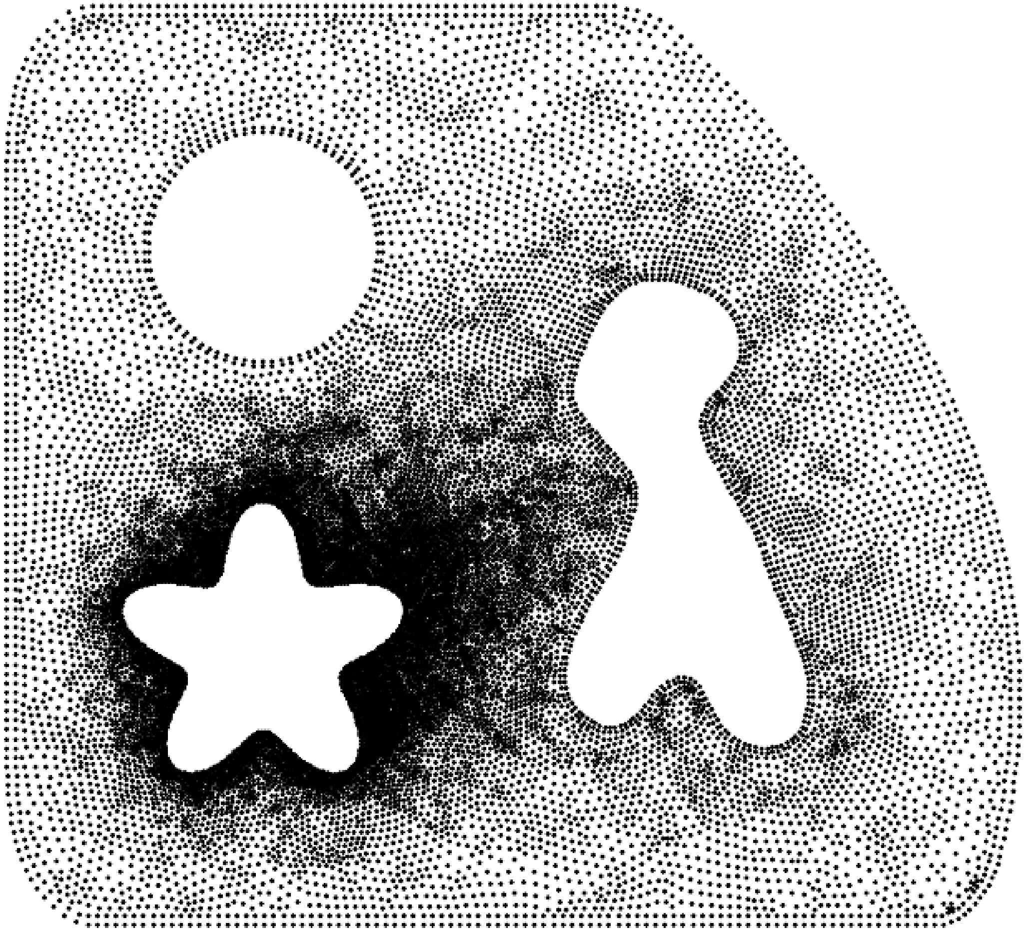


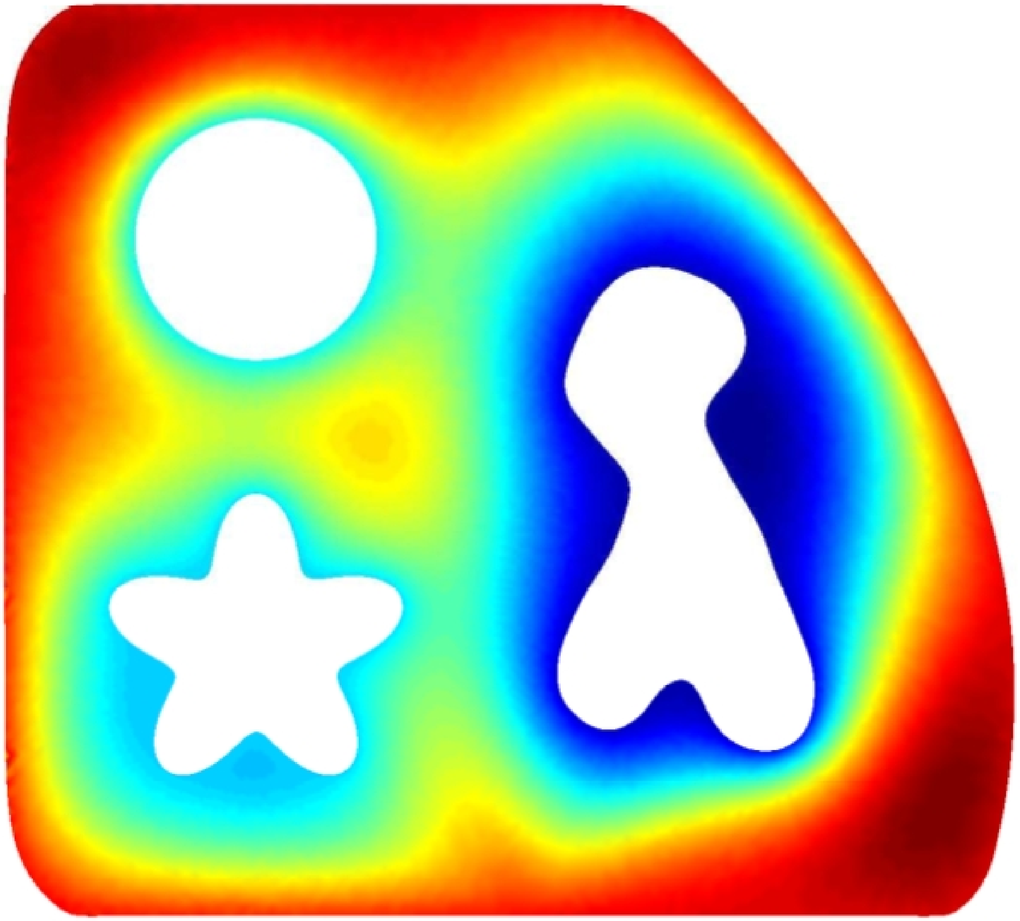
Blending B-Splines and max-ent



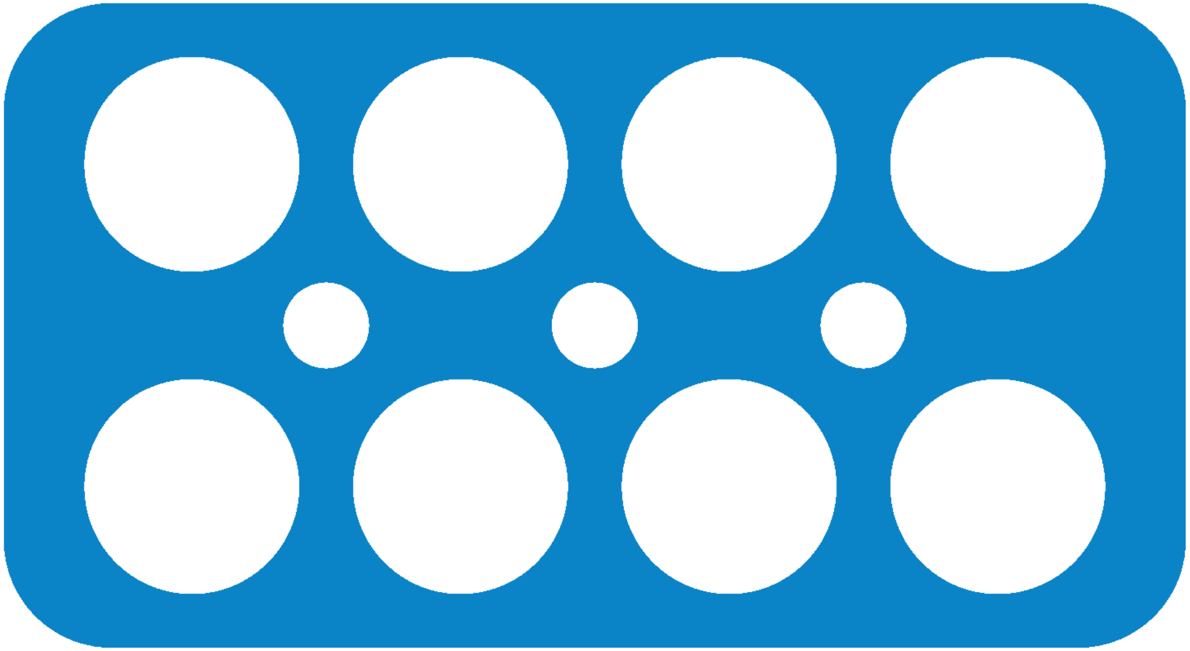
Blending B-Splines and max-ent

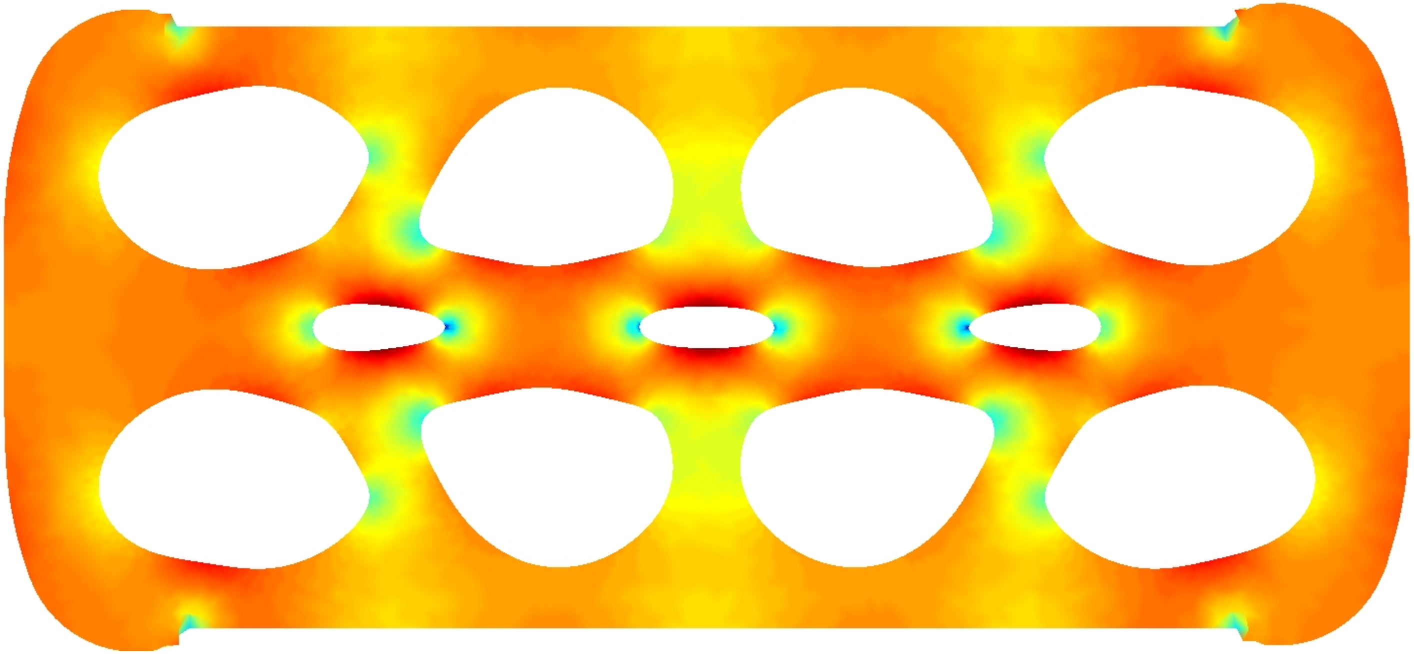
- Three holes, prescribed data on the boundaries, non uniform mesh
- Isogeometric requires significant preprocessing, partitioning of the domain in patches, specialized techniques to have global smoothness





Blending B-Splines and max-ent





Blending B-Splines and max-ent

- Conclusions
  - We present a method to blend maximum-entropy approximants and B-Splines
  - The proposed method exploits the best features and overcomes the main drawbacks of isogeometric analysis and local maximum approximation schemes
- Open line of research
  - Developments of appropriate tools to facilitate the preprocessing work for 3D problems

Thank you for your attention