Hodge-Optimized Triangulations

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Joint work with

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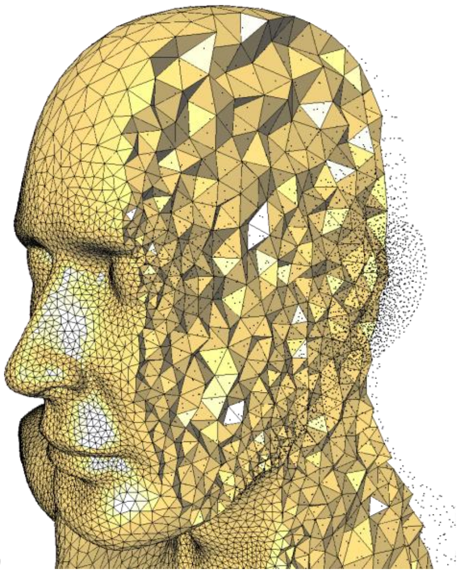
NSF workshop on barycentric coordinates
in geometry processing and finite/boundary element methods
July 25-27, 2012, Columbia University

Problem Statement

Discretization:

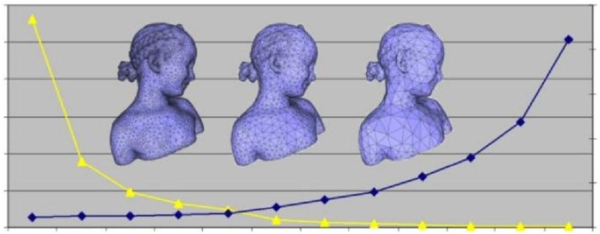
- Efficiency (linearity and sparsity)
- Accuracy



  - Subdivide mesh: more points
  - Higher order operator: denser system
  - Optimize mesh: precomputation

Given discrete operators, what types of meshes should be used?

Mesh Optimization



accuracy ↑ ↓ efficiency

Trade off

Variational method: $\min_{M \in Tr} \text{Energy}(M)$

Energy ≈ Approximation Error

Tight bounds for specific operators

Expand the space of triangulations

Generalization of primal dual complexes

Improve the minimization procedure

Different representation of space Tr

Computations on Meshes (DEC)

Continuous differential forms (in 3D)

- 0-form: scalar field (function at a point)
- 1-form: vector field (circulation along a curve)
- 2-form: flux through a surface
- 3-form: density field (volume)

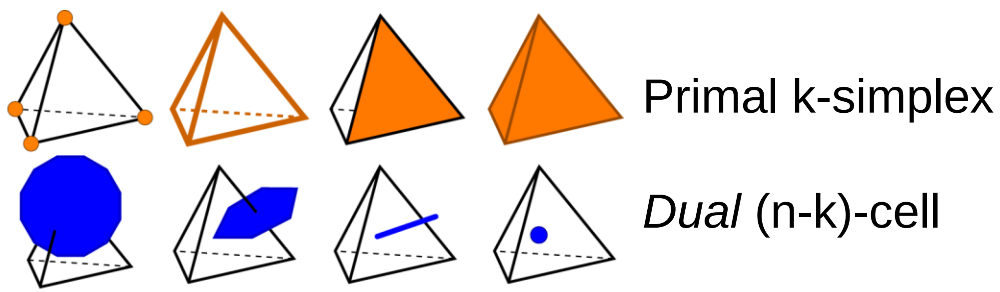
Discrete forms (integrated quantities)

- 0-forms on vertices
- 1-forms on edges
- 2-forms on faces
- 3-forms on cells…

Dual Mesh

Associate to each k-simplex a (n-k)-cell

- connectivity of triangulation *induces* another mesh

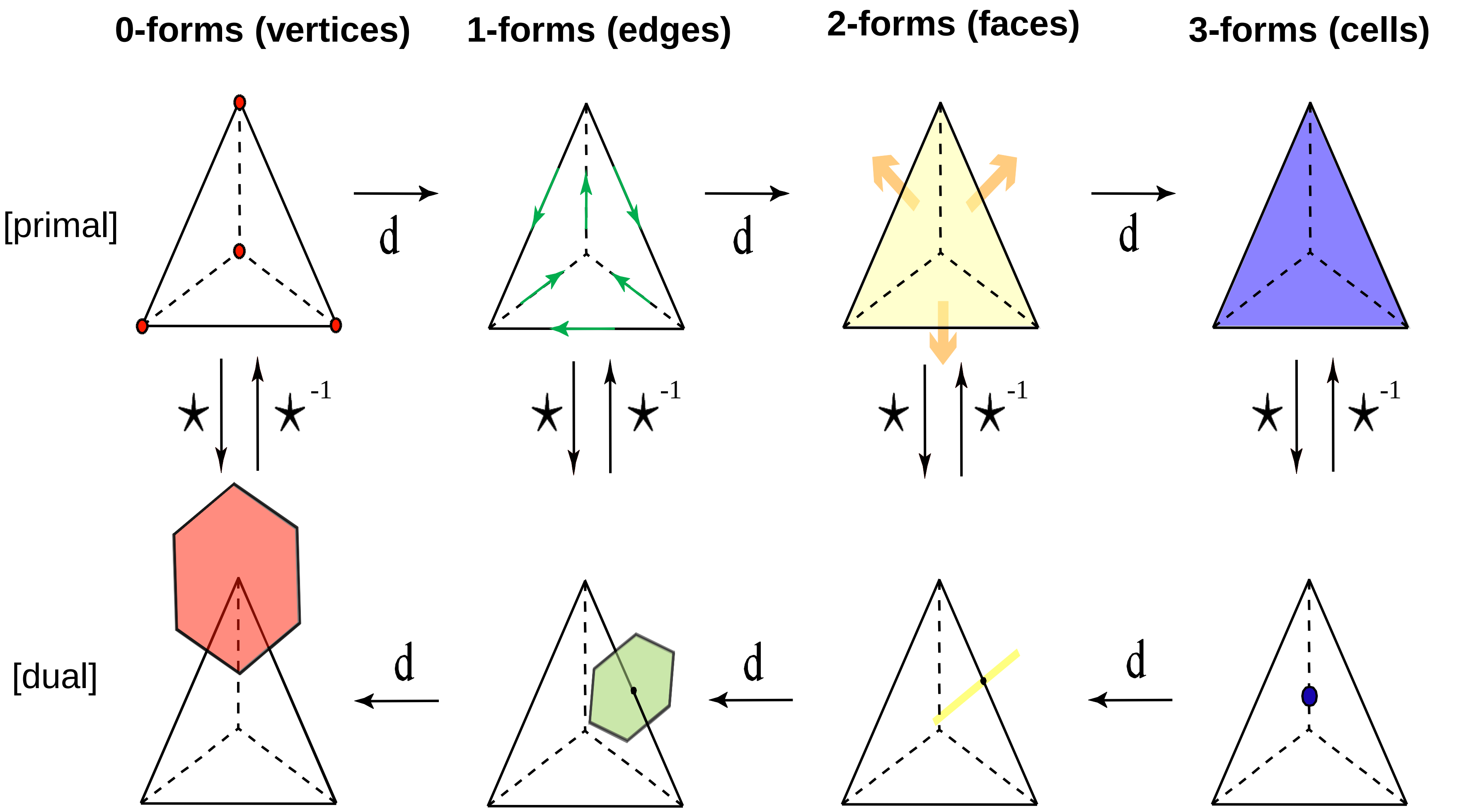


Orthogonality (leads to sparse operators)

Computations can be on dual mesh as well

DEC: Discrete Exterior Calculus

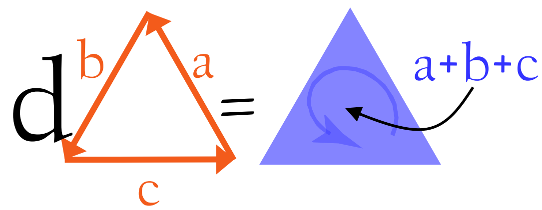
discrete differential forms = integration on mesh elements [Desbrun et al. 06]



Exterior Derivative

Generalize gradient to differential forms

d: k-form!(k+1)-form



Linear and Sparse

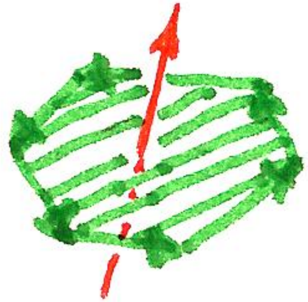
$$\int_{\sigma} d\omega = \int_{\partial\sigma} \omega$$

discrete d is exact !

Hodge Star

Relationship between primal and dual values

k-form => (n-k)-form $\quad \star : \Omega^k \rightarrow \Omega_\star^{n-k}$



e.g.: $\star dx = dy \wedge dz$, Circulation => Flux

The discrete k^{th} Hodge star is a diagonal matrix $\star^k$ with

Linear and sparse

$$\int_{*\sigma} \star\alpha \equiv \frac{|*\sigma|}{|\sigma|} \int_{\sigma} \alpha, \quad \forall i, \quad (\star^k)_{ii} := \frac{|*\sigma_i^k|}{|\sigma_i^k|},$$

Discrete $\star$ depends on the metric (generally only exact for constant forms)... error of approximation.

Hodge-Optimized Triangulations

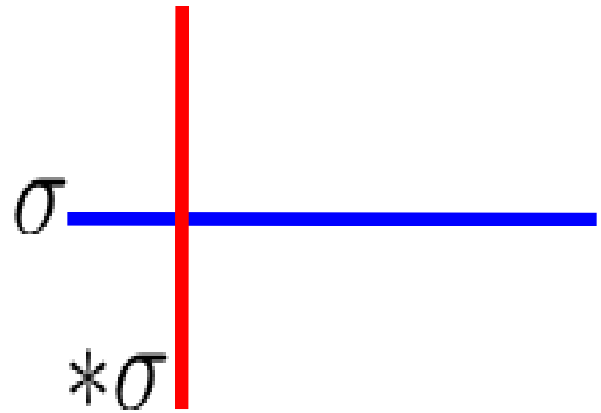
- Diagonal Hodge star is very convenient
  - as sparse as can be
  - as easy to inverse as can be

- HOT idea:
  - let's make meshes for which $\star$ is "good"
    - with good error bounds for Lipschitz forms
  - multiple stars means multiple notions of "good"
    - HOT meshes

DEC Accuracy

Discrete Hodge Star operator $\star$

$$\alpha(\sigma) := \int_\sigma \alpha \longrightarrow \star\alpha(*\sigma) := \int_{*\sigma} \star\alpha$$



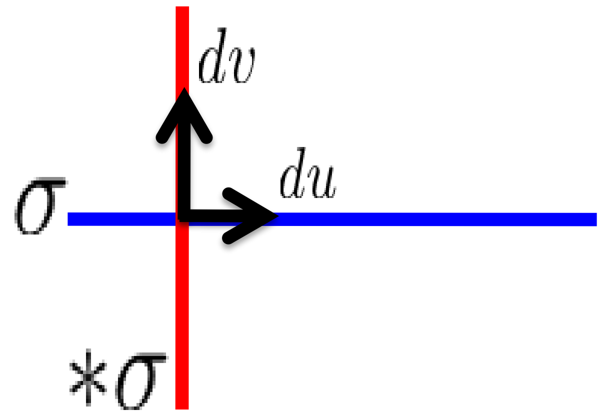
$$\frac{1}{|*\sigma|}\int_{*\sigma} \star\alpha \approx \frac{1}{|\sigma|}\int_\sigma \alpha$$

$$\star\alpha(*\sigma) \approx \overset{\star}{\boxed{\frac{|*\sigma|}{|\sigma|}}}\,\alpha(\sigma)$$

Tight Error Bound

$$\int_\sigma \alpha = \int_\sigma f(u,v)\,du$$

$$\int_{*\sigma} \star\alpha = \int_{*\sigma} f(u,v)\,dv$$



$$e := \left| \frac{1}{|\sigma|}\int_\sigma \alpha - \frac{1}{|*\sigma|}\int_{*\sigma} \star\alpha \right|$$

$$e = \left| \int_{\sigma\cup*\sigma} f(u,v) \left(\frac{du}{|\sigma|} - \frac{dv}{|*\sigma|} \right) \right|$$

Optimal Transport

Probability Distributions

Tight Error Bound

$$e = \left| \int_{\sigma \cup * \sigma} f(u,v) \left(\frac{du}{|\sigma|} - \frac{dv}{| * \sigma|} \right) \right|$$

λ-Lipschitz

$$W_q(\mu,\nu) = \left(\inf_{\pi \in \mathcal{P}(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^q \, d\pi(x,y) \right)^{1/q}.$$

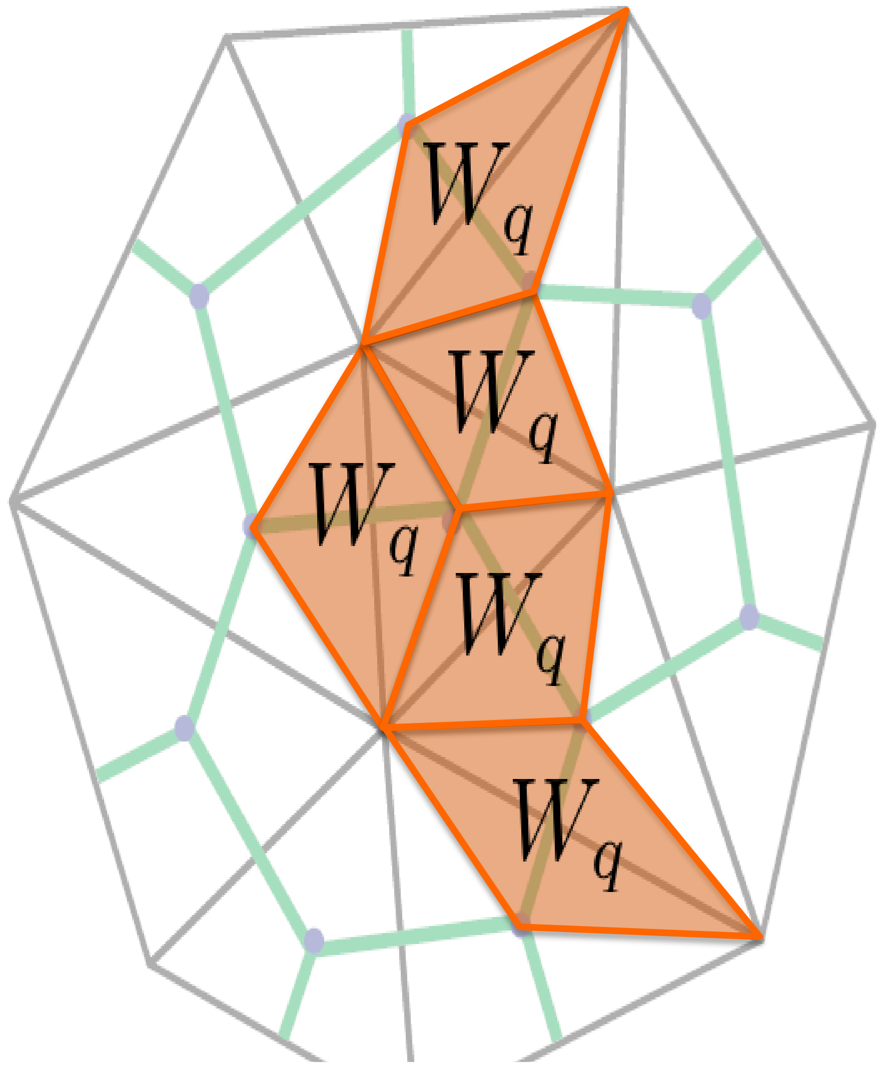
Kantorovich Theorem

$$e \leq \lambda W_1(\sigma, *\sigma)$$

Hölder Inequality

$$W_1(\sigma, *\sigma) \leq W_2(\sigma, *\sigma)$$

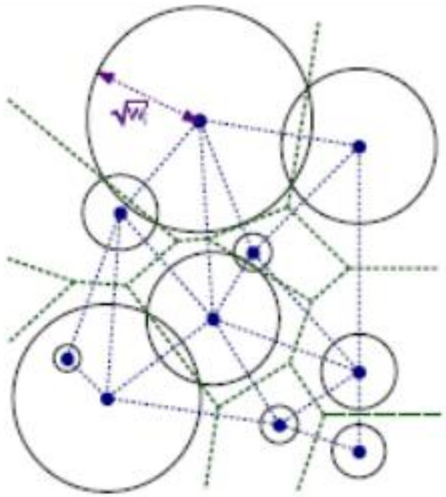
HOT Energies



$$\star^k\text{-}\,\mathrm{HOT}_{p,q}(\mathcal{M}) \equiv \sum_{\sigma_i \in \Sigma^k} |*\sigma_i||\sigma_i|\, W_q(\sigma_i, *\sigma_i)^p$$

Expand the space of triangulations

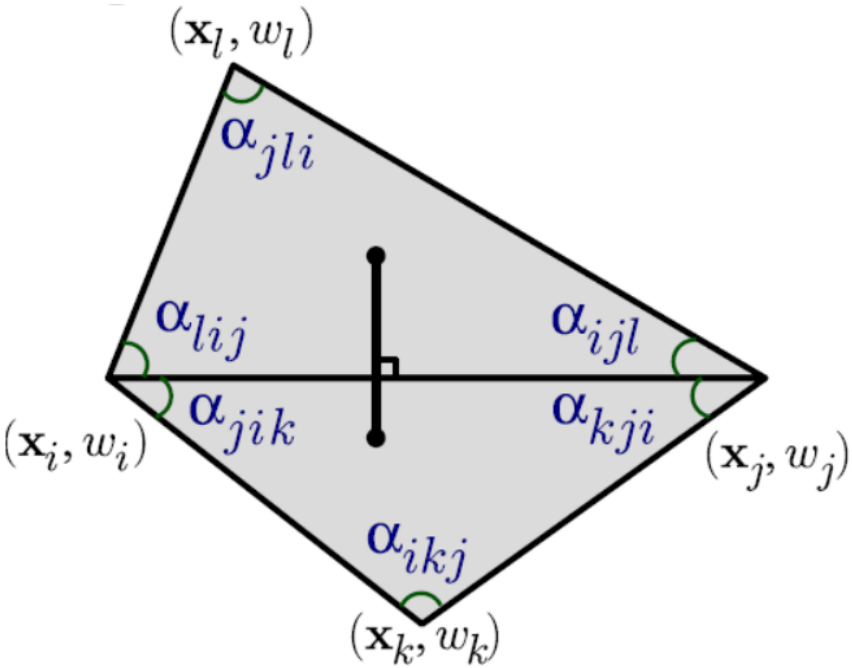
- We want orthogonal duals edges.
- Regular/Power duality obviously relevant
  - one scalar per vertex



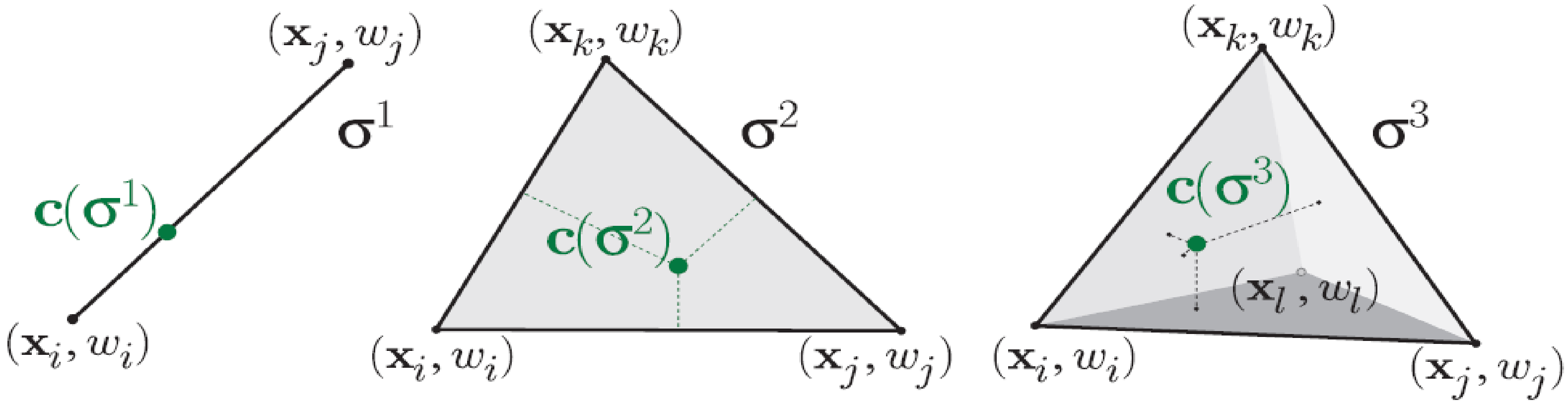
Power diagram: $V(\mathbf{x_i}, w_i) = \{\mathbf{x} | d^2(\mathbf{x}, \mathbf{x_i}) - w_i \leq d^2(\mathbf{x}, \mathbf{x_j}) - w_j, \forall j\}.$

- spans the space of orthogonal duals with convex cells

Weighted Delaunay Triangulation



$$
\begin{aligned}
(\star^1)_{ij} =& \frac{1}{2}\Big(\cot\alpha_{ikj} + \cot\alpha_{jli} \\
&+ (w_i - w_k)\frac{\cot\alpha_{kji}}{||\mathbf{x}_i - \mathbf{x}_j||^2} + (w_j - w_k)\frac{\cot\alpha_{jik}}{||\mathbf{x}_i - \mathbf{x}_j||^2} \\
&+ (w_i - w_l)\frac{\cot\alpha_{ijl}}{||\mathbf{x}_i - \mathbf{x}_j||^2} + (w_j - w_l)\frac{\cot\alpha_{lij}}{||\mathbf{x}_i - \mathbf{x}_j||^2}\Big)
\end{aligned}
$$



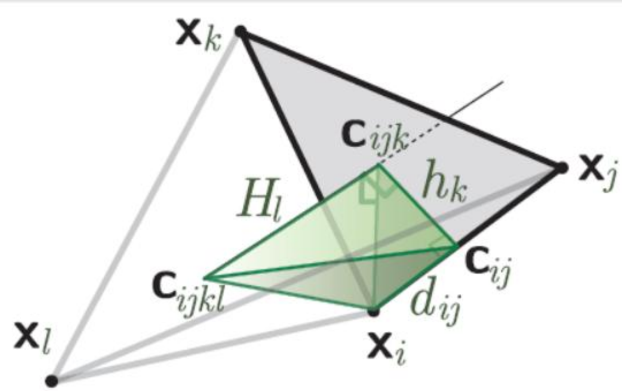
3D $\text{HOT}_{2,2}$ Formulas:

$$\star^0\text{- HOT}_{2,2}(T_{ijkl}) = \sum_{i,j,k,l} \frac{1}{5}\left(\frac{H_l^3 h_k d_{ij}}{12} + \frac{H_l h_k^3 d_{ij}}{4} + \frac{H_l h_k d_{ij}^3}{2}\right).$$

$$\star^1\text{- HOT}_{2,2}(T_{ijkl}) = \sum_{i,j,k,l} \frac{1}{3}\left(\frac{H_l^3 h_k d_{ij}}{12} + \frac{H_l h_k^3 d_{ij}}{4} + \frac{H_l h_k d_{ij}^3}{6}\right).$$

$$\star^2\text{- HOT}_{2,2}(T_{ijkl}) = \sum_{i,j,k,l} \frac{1}{3}\left(\frac{H_l^3 h_k d_{ij}}{6} + \frac{H_l h_k^3 d_{ij}}{4} + \frac{H_l h_k d_{ij}^3}{12}\right).$$

$$\star^3\text{- HOT}_{2,2}(T_{ijkl}) = \sum_{i,j,k,l} \frac{1}{5}\left(\frac{H_l^3 h_k d_{ij}}{2} + \frac{H_l h_k^3 d_{ij}}{4} + \frac{H_l h_k d_{ij}^3}{12}\right).$$



Algorithm

MESH OPTIMIZATION

Input: vertices $\mathbf{x}^0 = \{\mathbf{x}_i\}$ *and weights* $w^0 = \{w_i\}$*,*
and a HOT functional $E(\mathbf{x}, w)$*.*

$n \leftarrow 0$

repeat

Compute $E(\mathbf{x}^n, w^n)$

Optimize for $\mathbf{x}$

Pick step direction $\mathbf{d}^x$ for $E(\mathbf{x}^n, w^n)$

Find β satisfying Wolfe's condition(s)

$\mathbf{x}^{n+1} \leftarrow \mathbf{x}^n + \beta\, \mathbf{d}^x$ *Vertex updates*

Optimize for w

Pick step direction $\mathbf{d}^w$ for $E(\mathbf{x}^{n+1}, w^n)$

Find α satisfying Wolfe's condition(s)

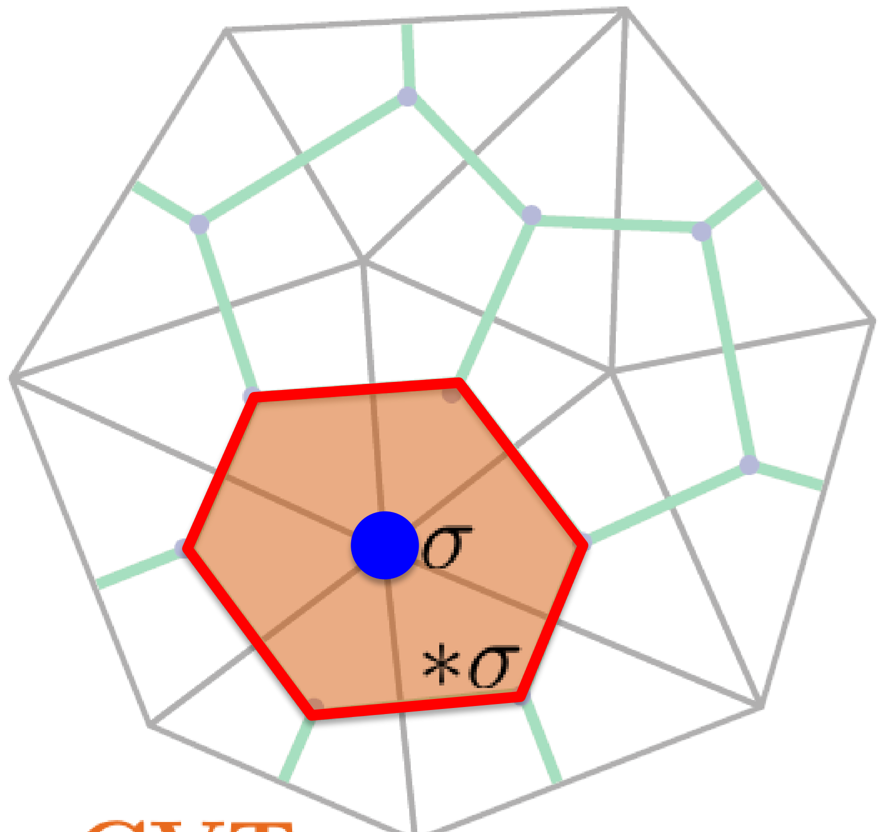
$w^{n+1} \leftarrow w^n + \alpha\, \mathbf{d}^w$ *Weight updates*

$n \leftarrow n + 1$

until (convergence criterion met)

Centroidal Voronoi Tessellation (CVT)

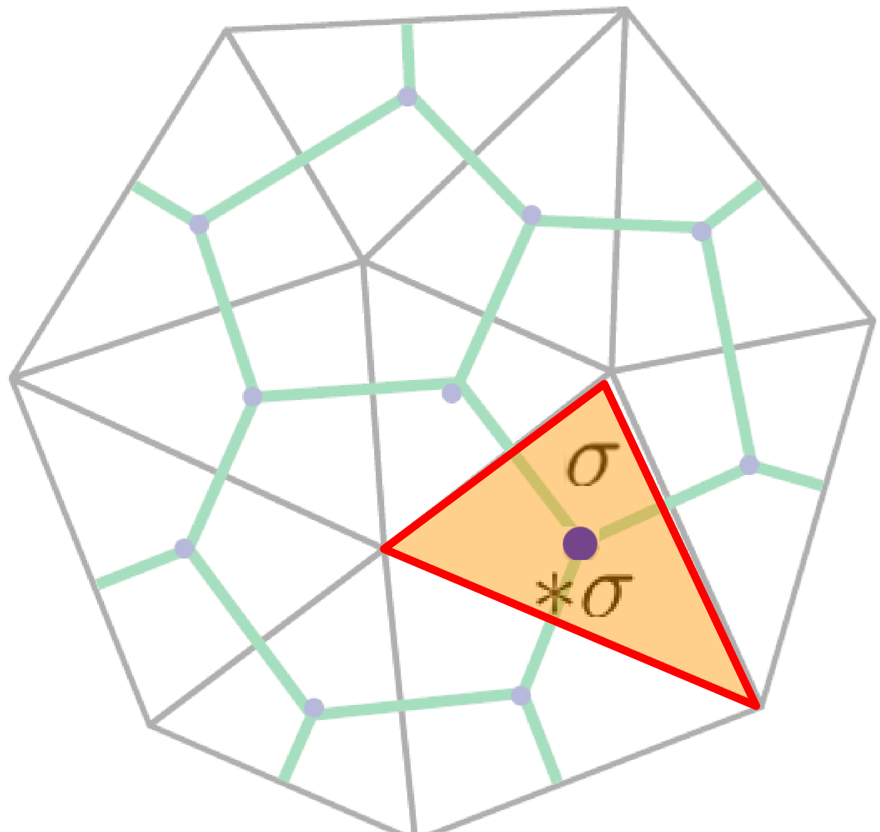
$$\sum_i \int_{V_i} \|\mathbf{x} - \mathbf{x}_i\|^2 dV$$



$\star^0$-$\mathbf{HOT}_{2,2} = \mathbf{CVT}$

Nearly-barycentric orthogonal duals

$$\sum_{\mathbf{t}} \int_{\mathbf{t}} \|\mathbf{x} - \mathbf{c}_{\mathbf{t}}\|^2 dV$$

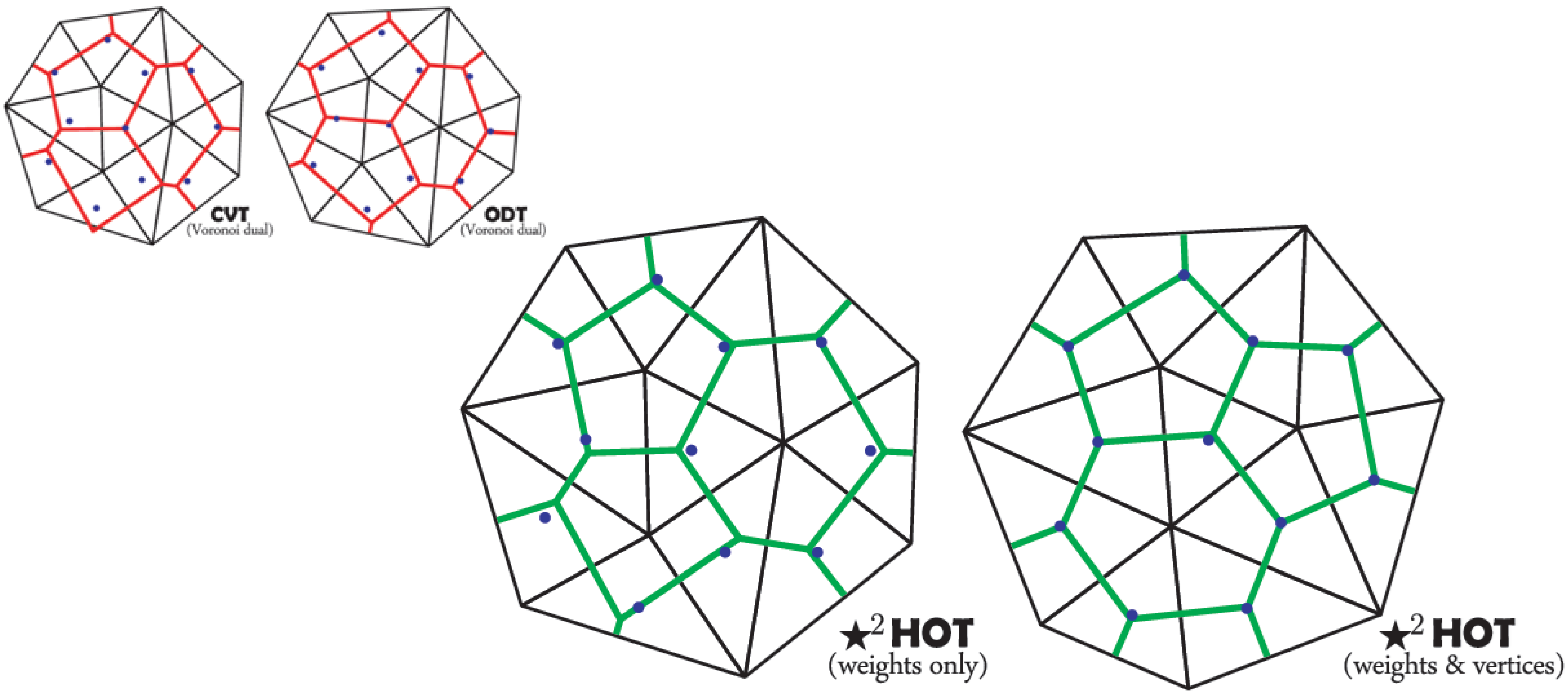


$\star^2$-$\mathbf{HOT}_{2,2}$ = "Well-centered" meshes

Results and Applications

Nearly-barycentric orthogonal duals

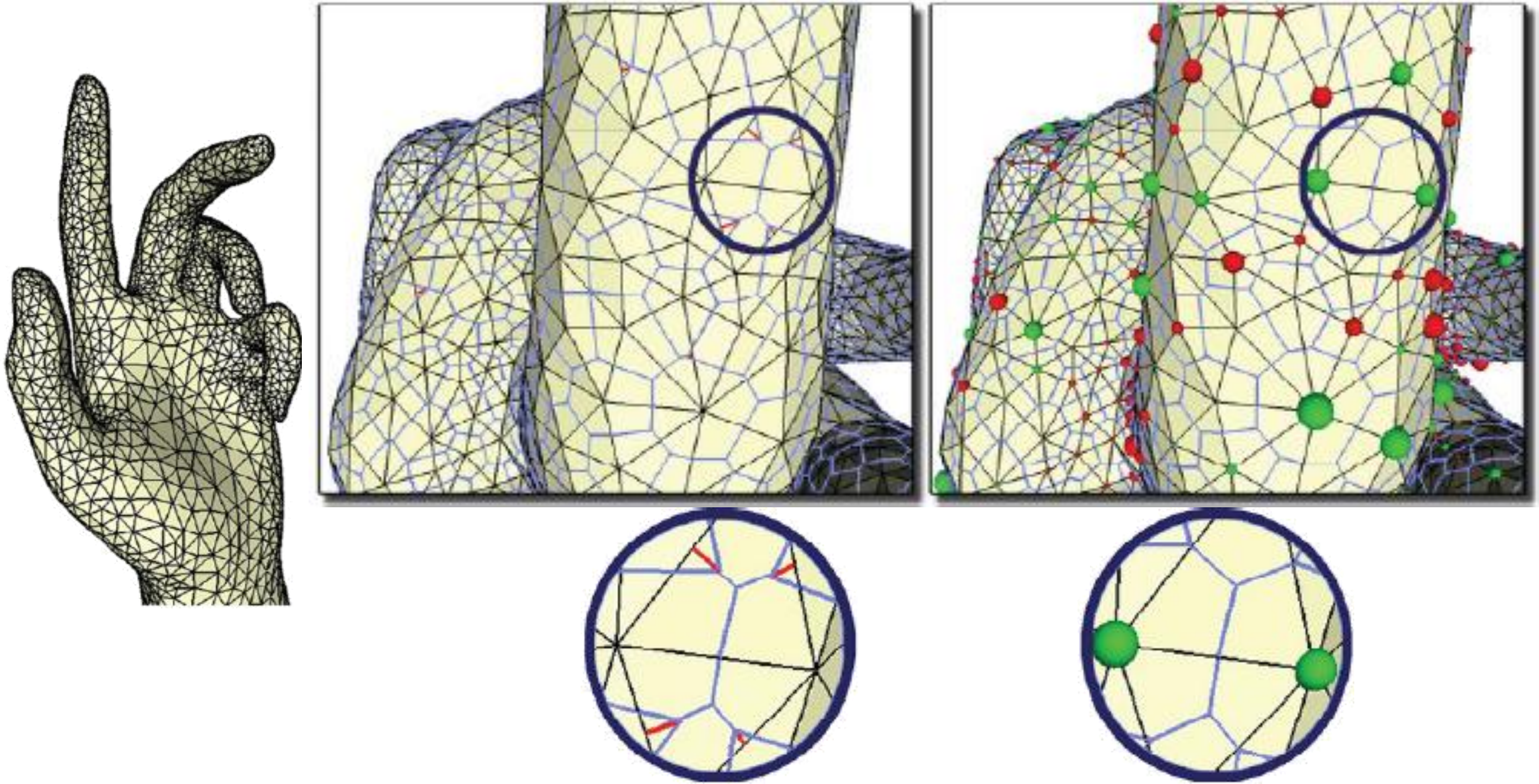
- "well-centered" meshes



Results and Applications

Nearly-barycentric orthogonal duals

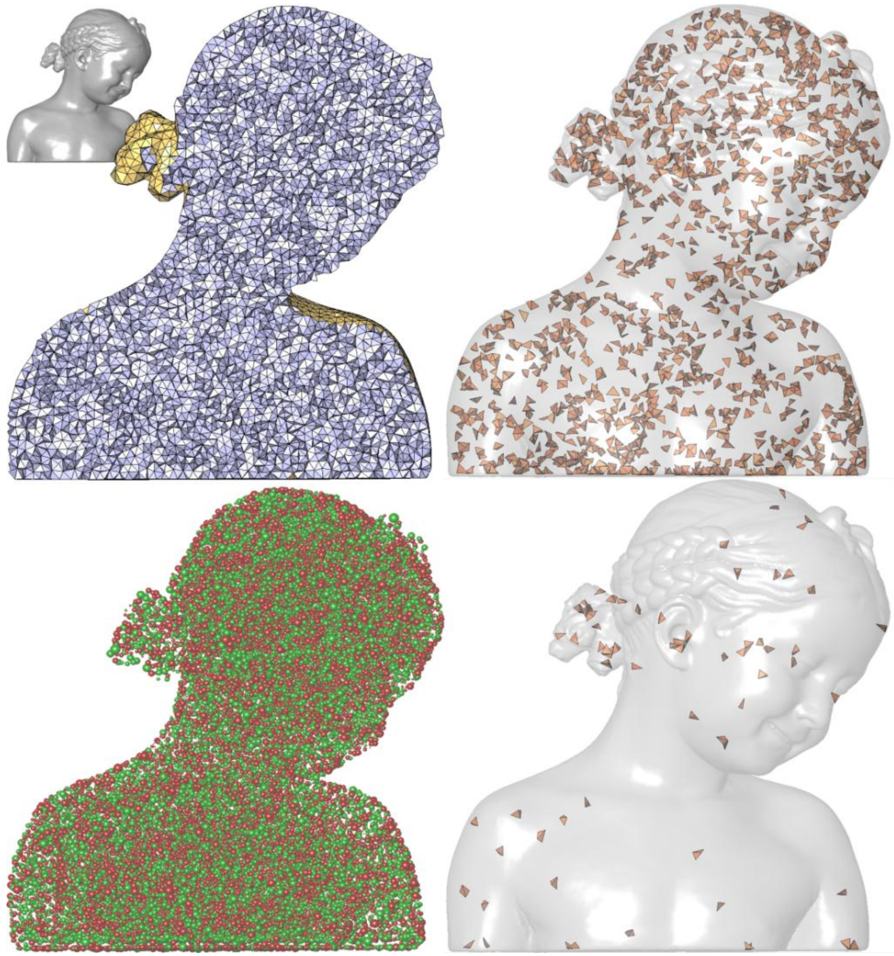
- “well-centered” meshes ($\star_2$-HOT$_{2,2}$)



Results and Applications

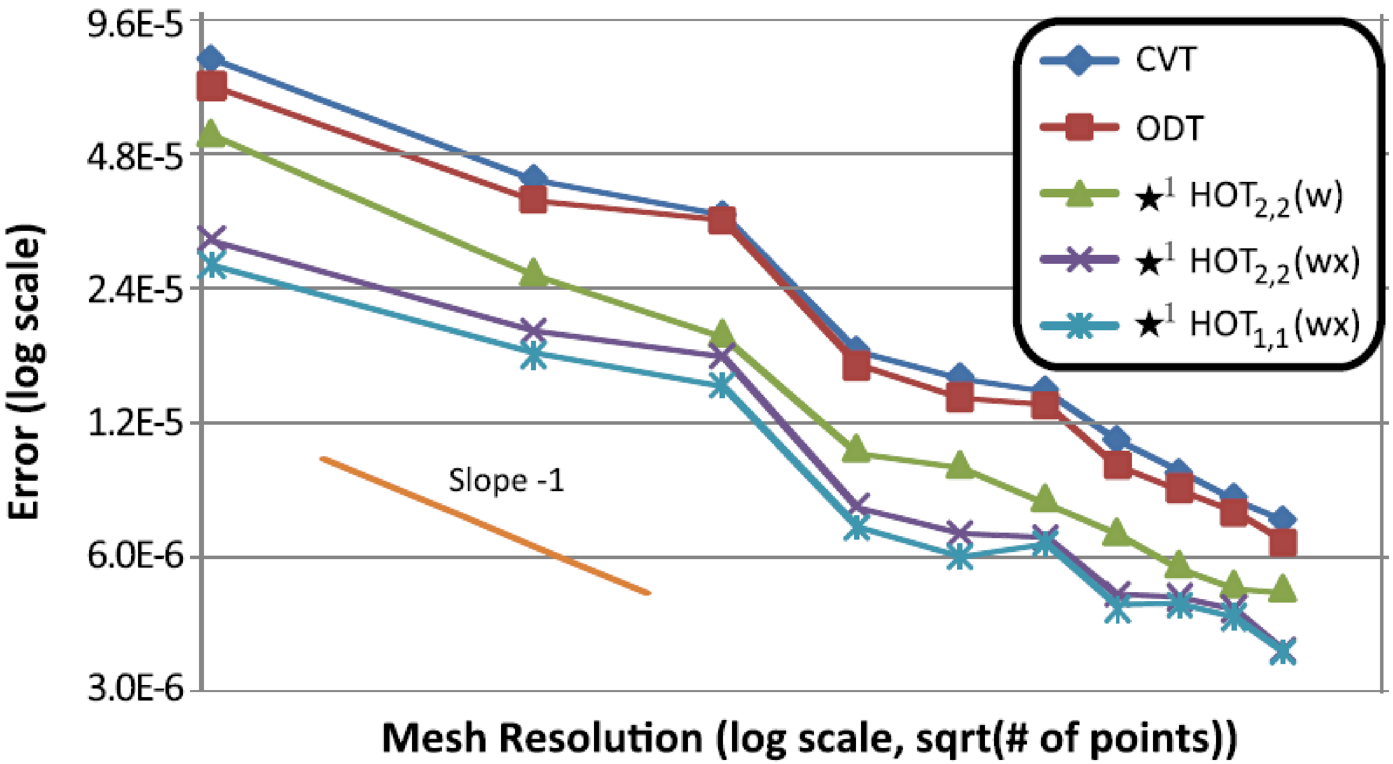
Nearly-barycentric orthogonal duals

- "well-centered" meshes ($\star_3$-HOT$_{2,2}$)



Results and Applications

Solving a (node-based) Laplace equation



Future Work

- Optimize directly for general operators
    - optimal linear combination of Hodge stars?
- Better solver
- Better boundary handling
- Other primal/dual criteria
    - weights are powerful
    - weights + vectors even more!
        - parameterizing *every* primal/dual structure

Thank you!