High-quality weight functions via constrained optimization

Alec Jacobson

ETH Zurich

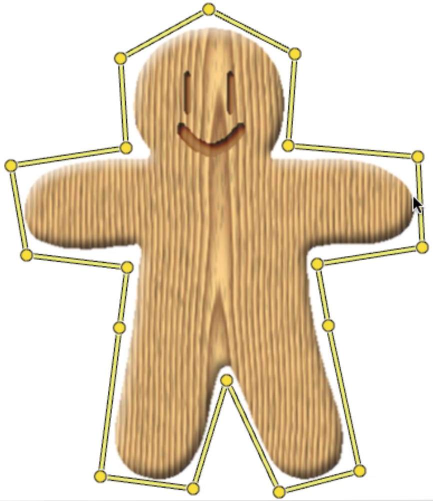
INTERACTIVE GEOMETRY LAB

August 1, 2012

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Coordinate-based deformation is a special case of Linear Blend Skinning

$$\mathbf{x}' = \sum_{j \in H} w_j(\mathbf{x})\mathbf{h}'_j$$



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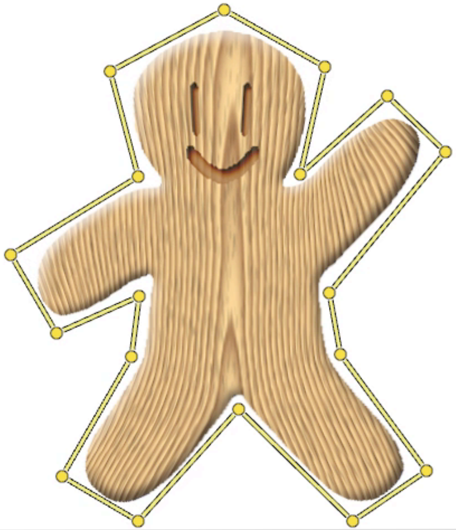

linear precision (reproduction)

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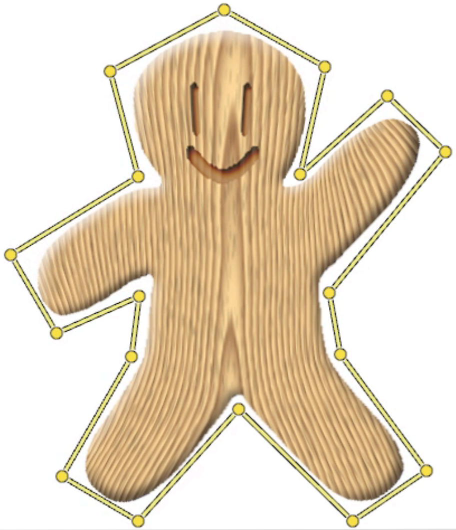
$$\mathbf{x}' = \sum_{j \in H} w_j(\mathbf{x}) T_j \mathbf{x}$$

restricted to translation



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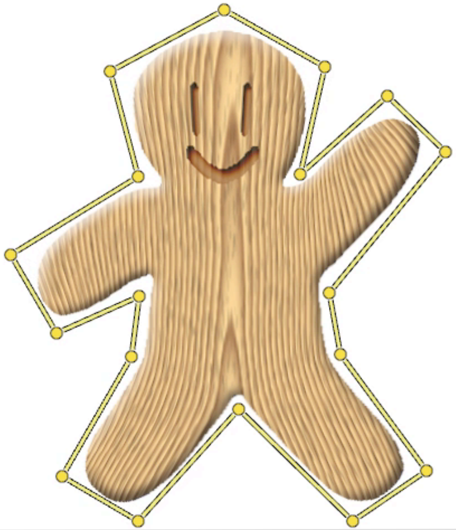
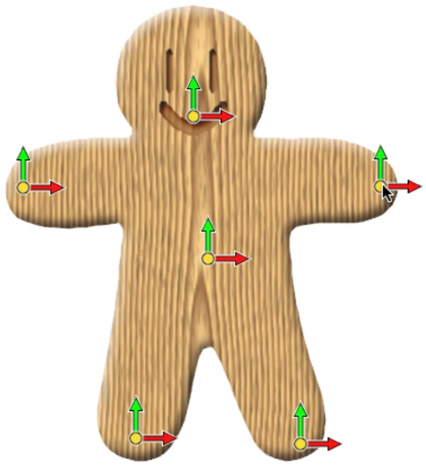
any affine





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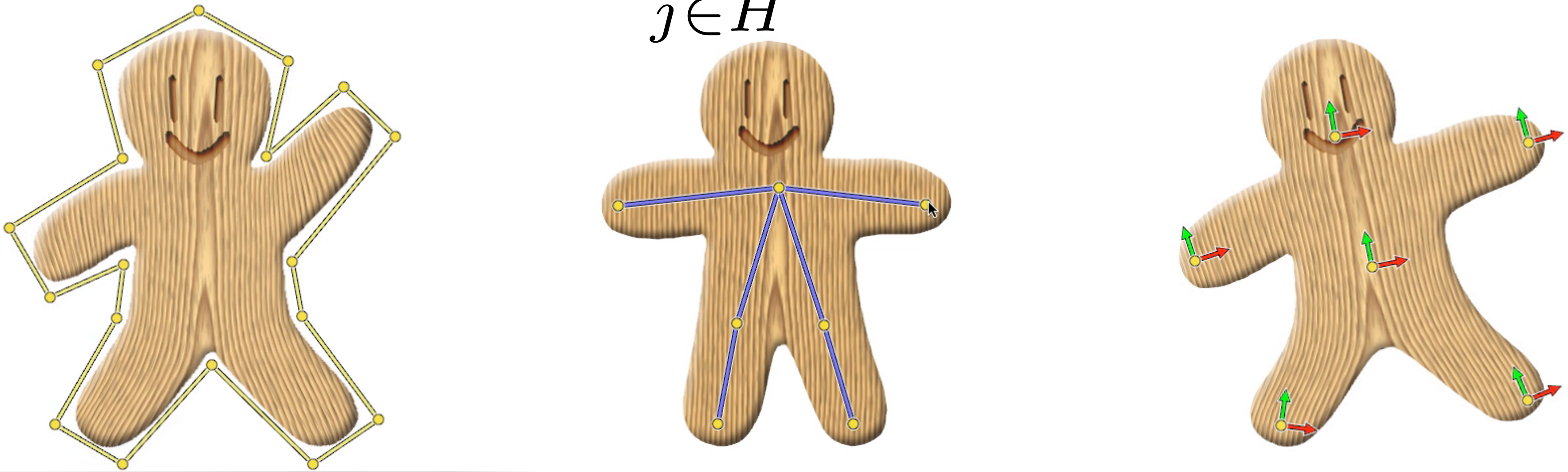

Bounded Biharmonic Weights for Real-Time Deformation

SIGGRAPH 2011

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Olga Sorkine[1,2]

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[4]Adobe Systems

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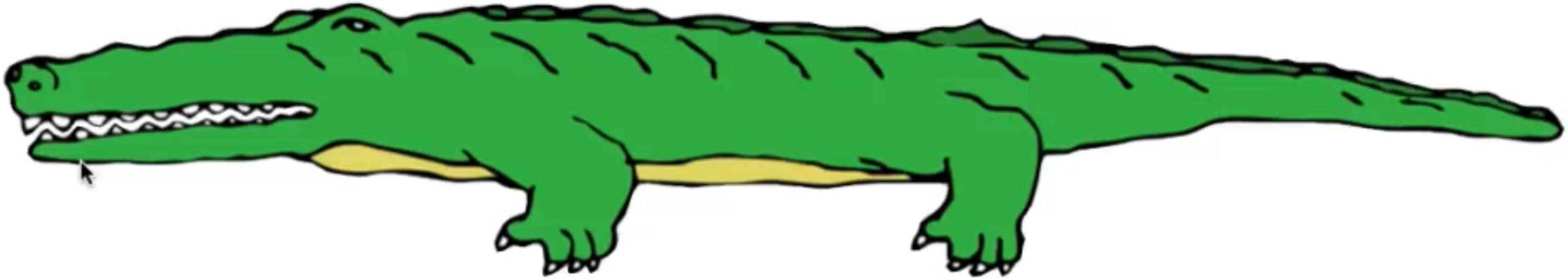
Each handle type has a specific task, more than just *different modeling metaphor*



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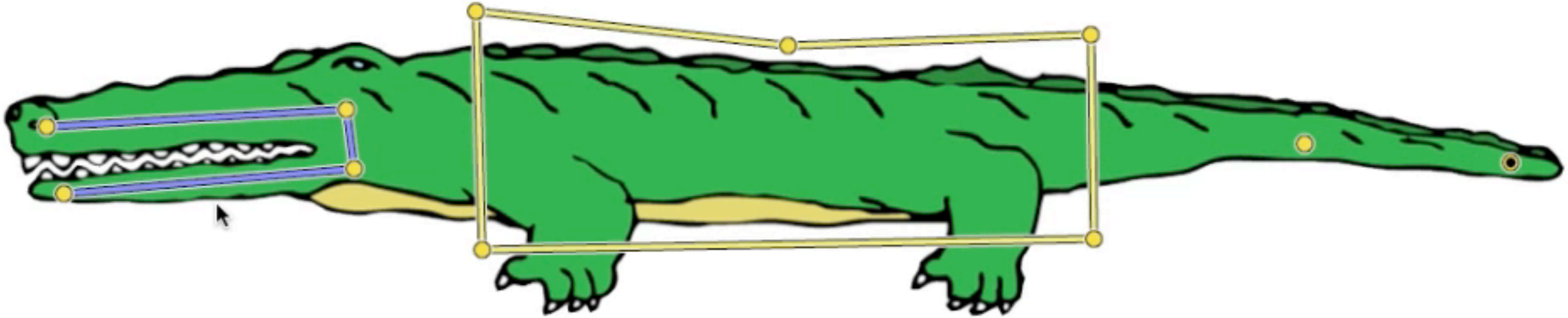

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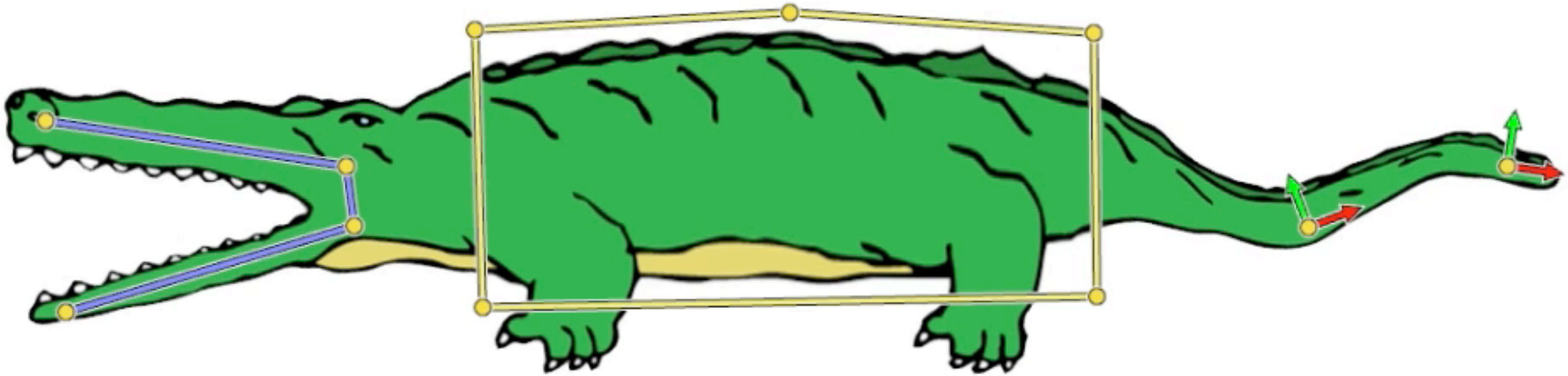
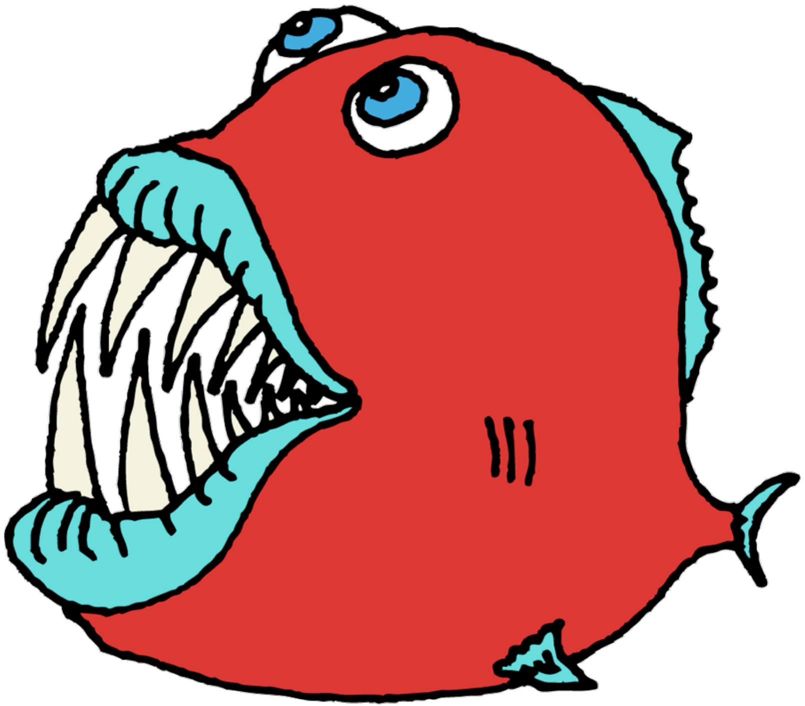


Cages can often be tedious to build and control



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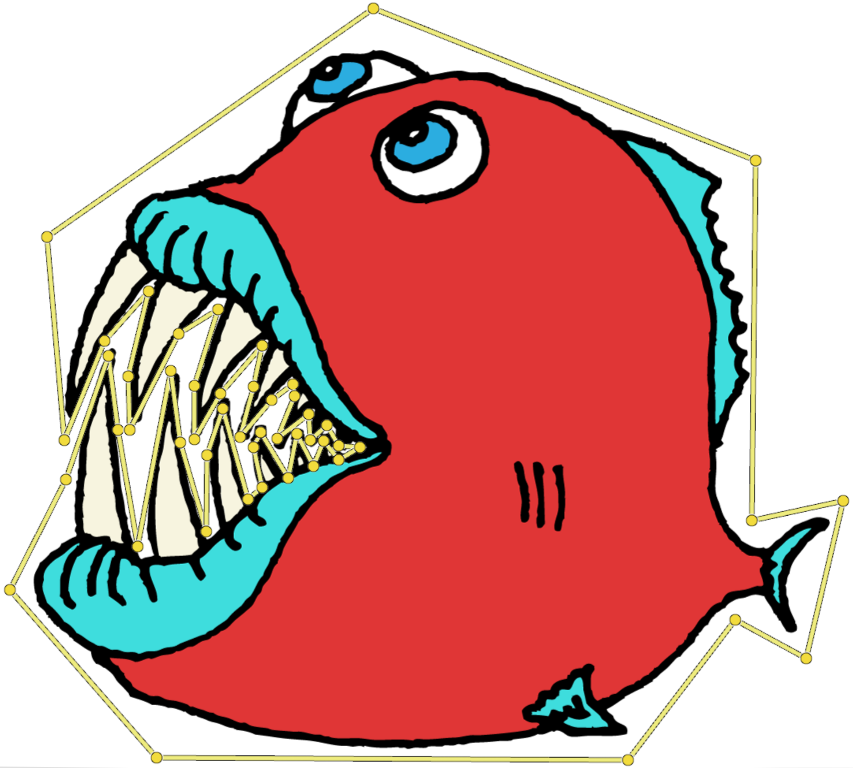
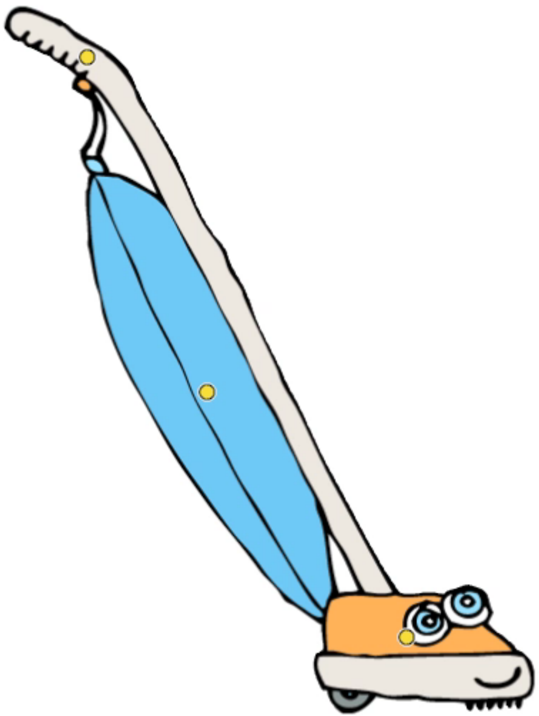
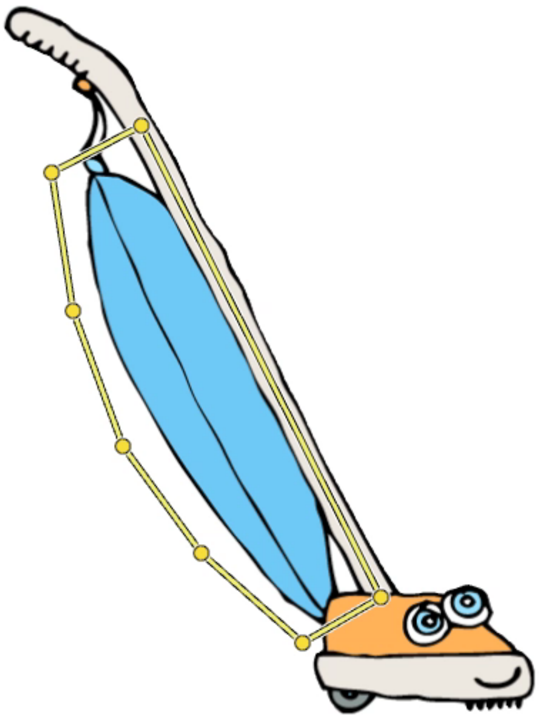


Points can only provide crude scaling





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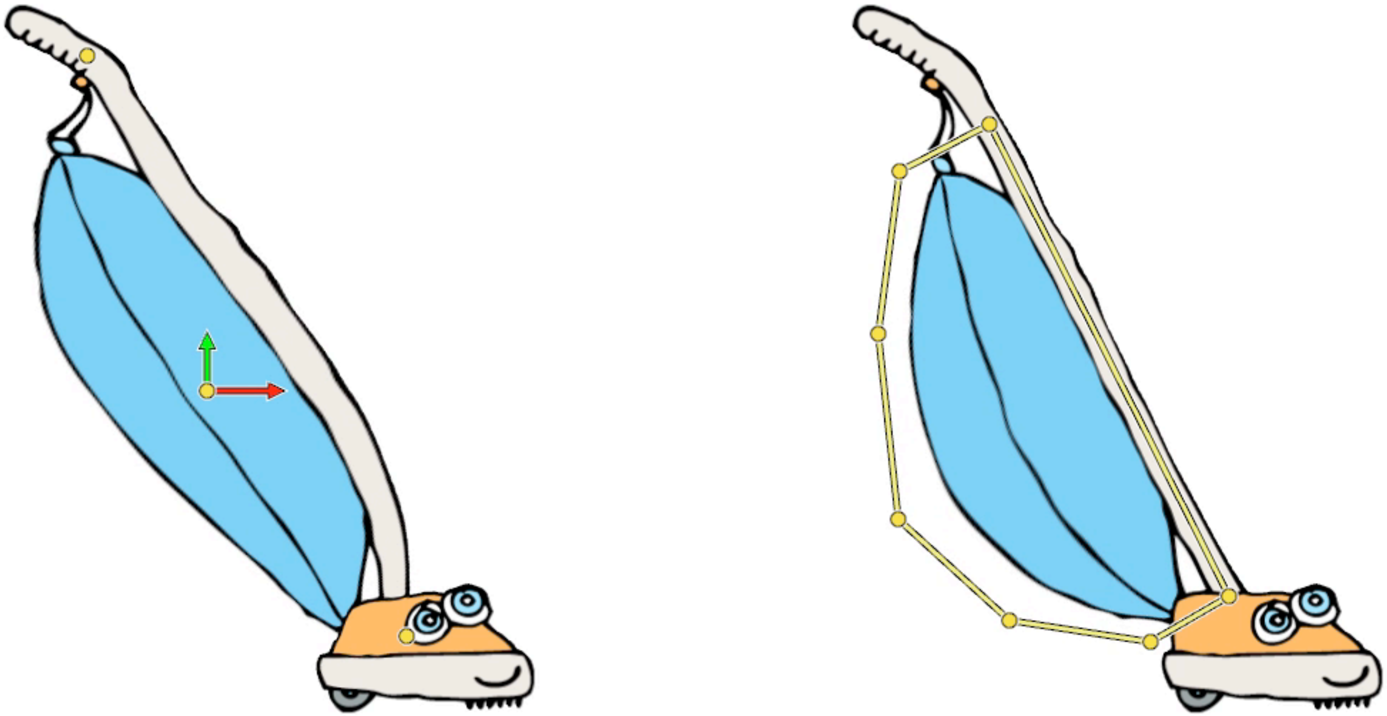

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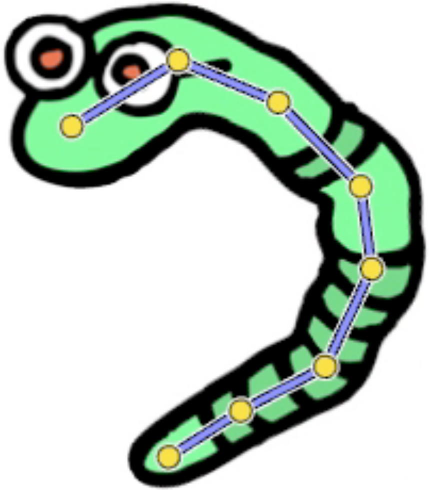
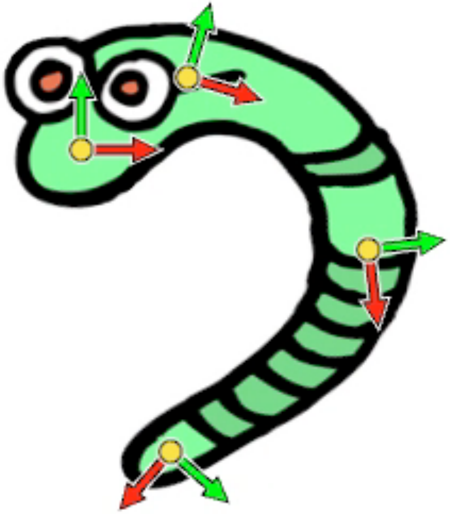
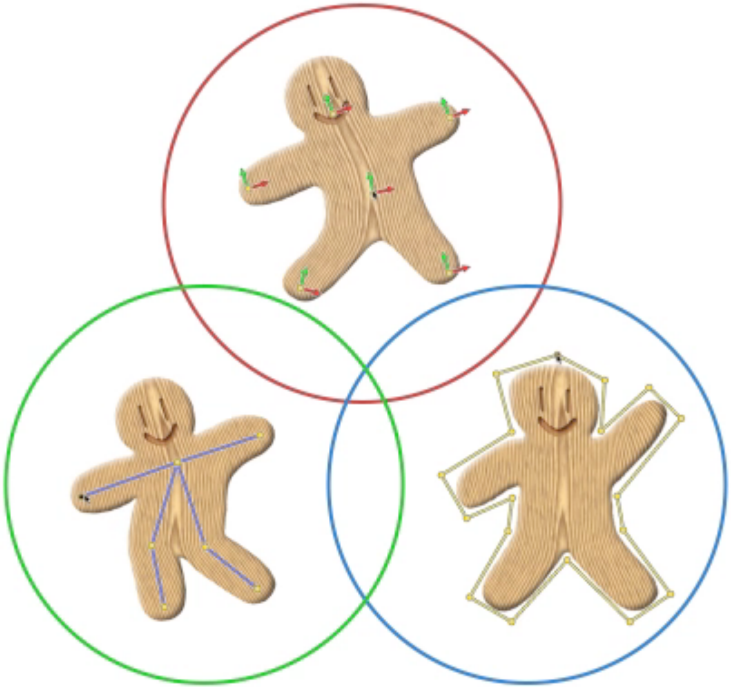
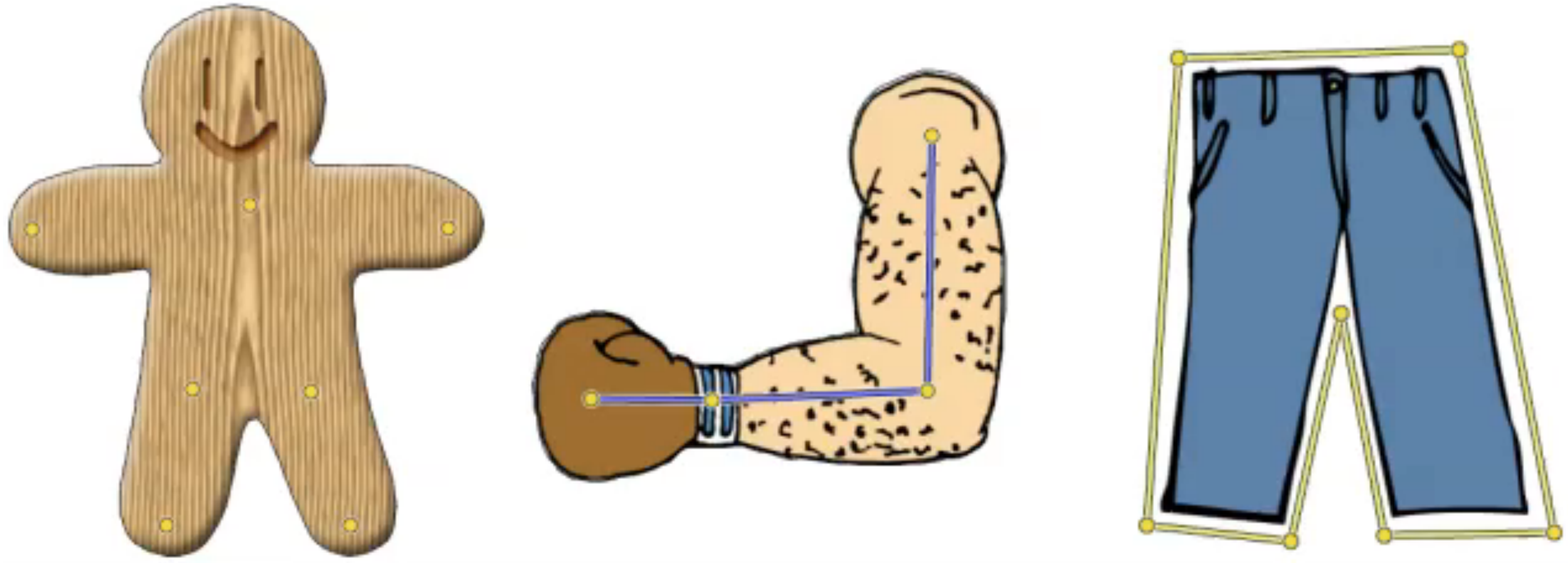




We want to compute weights that unify points, skeletons and cages



Weights should be smooth, shape-aware, positive and *intuitive*



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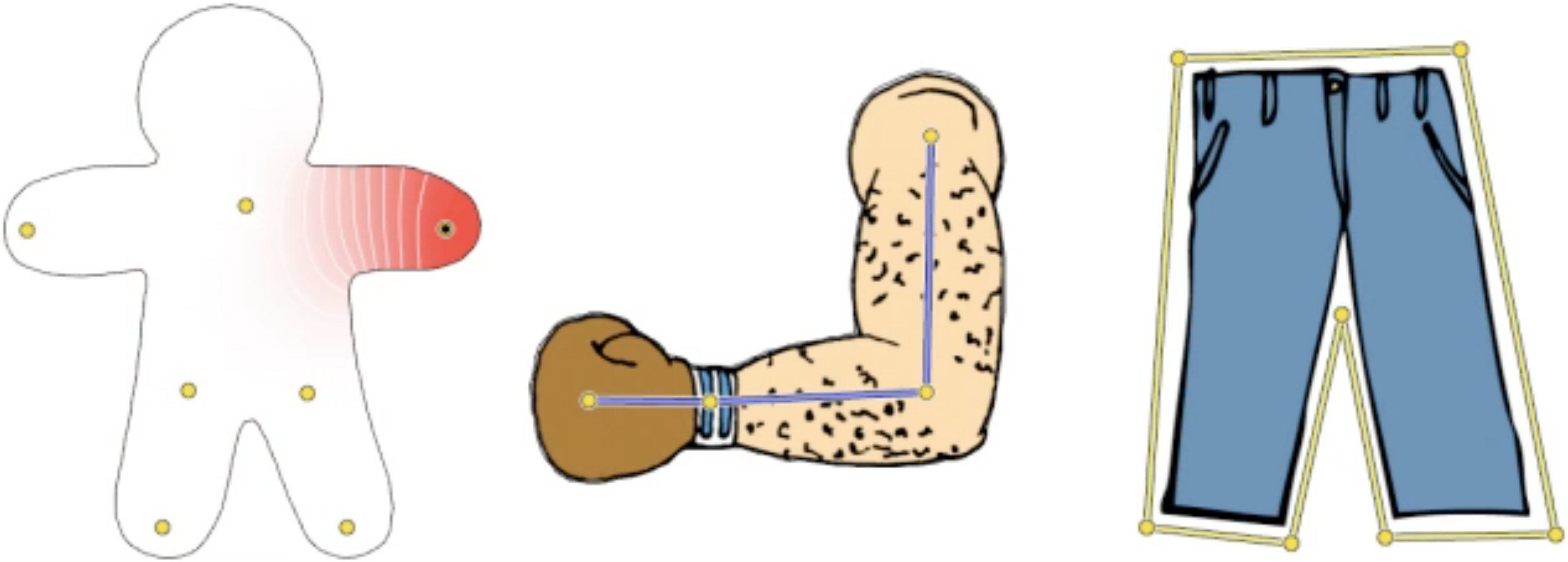

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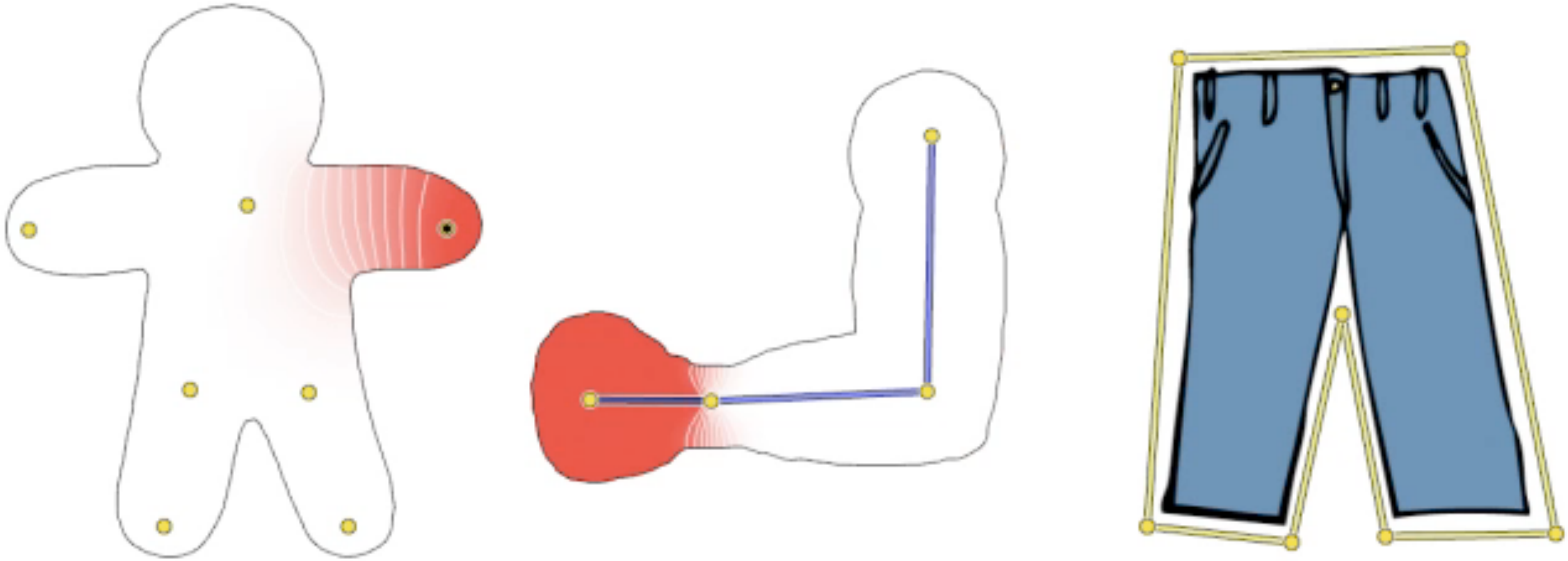

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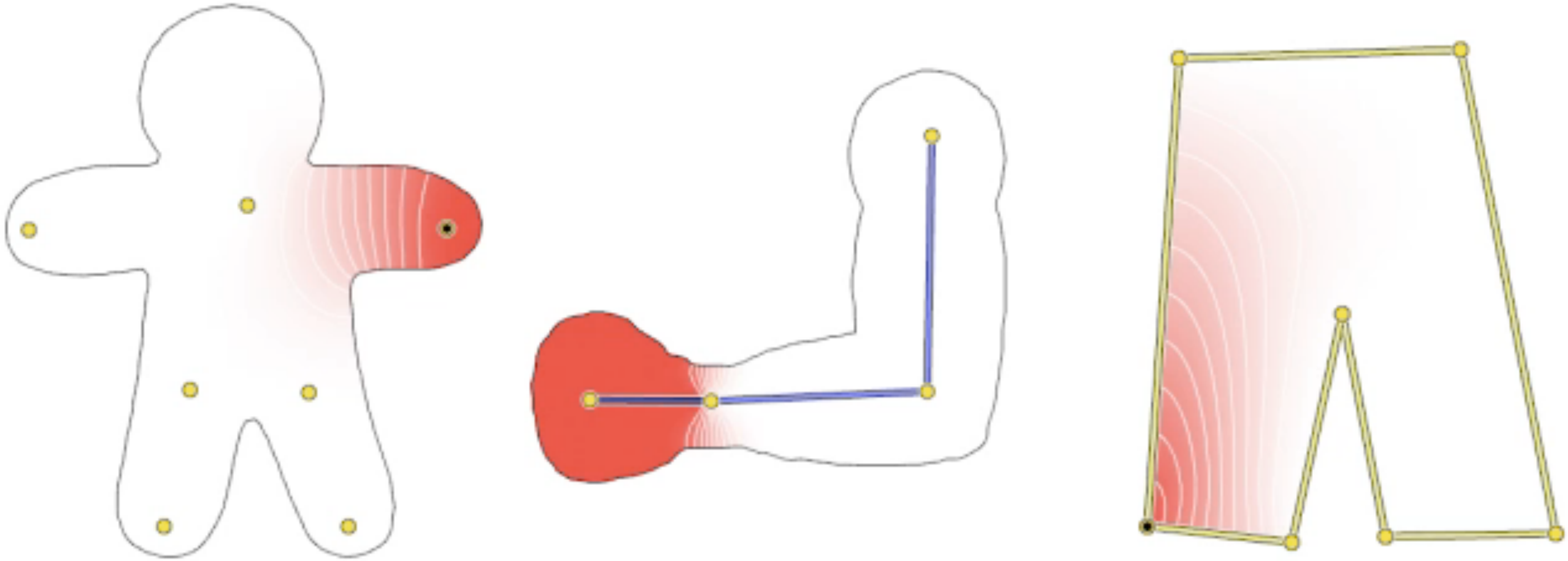

Weights must be smooth everywhere, *especially* at handles

Our method

Extension of Harmonic Coordinates
[Joshi et al. 2005]

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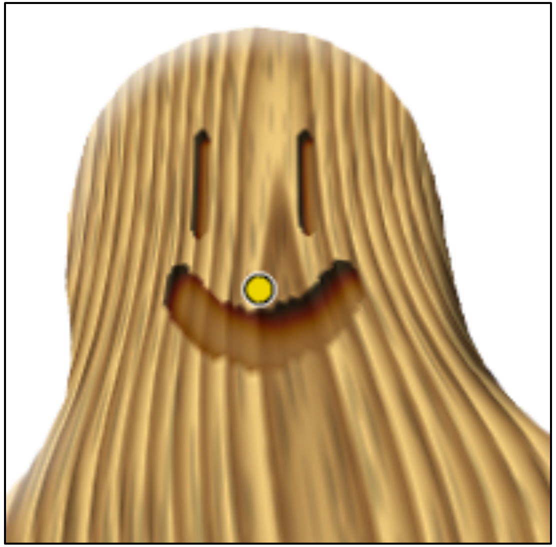

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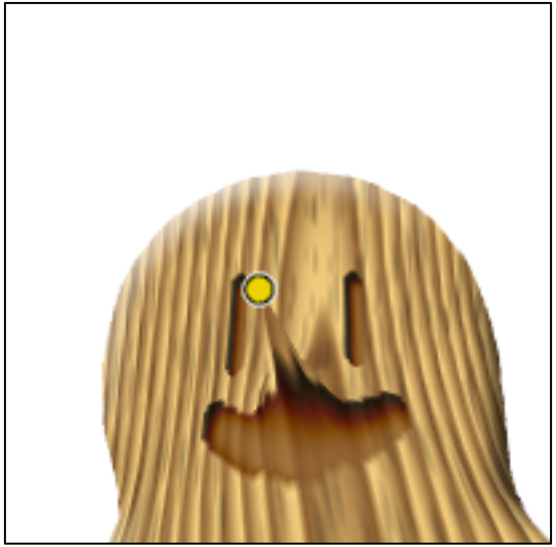

Extension of Harmonic Coordinates [Joshi et al. 2005]

Shape-awareness ensures respect of domain's features

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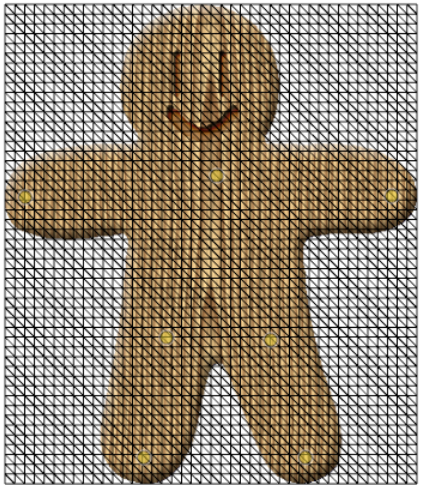

Non-shape-aware methods
e.g. [Schaefer et al. 2006]

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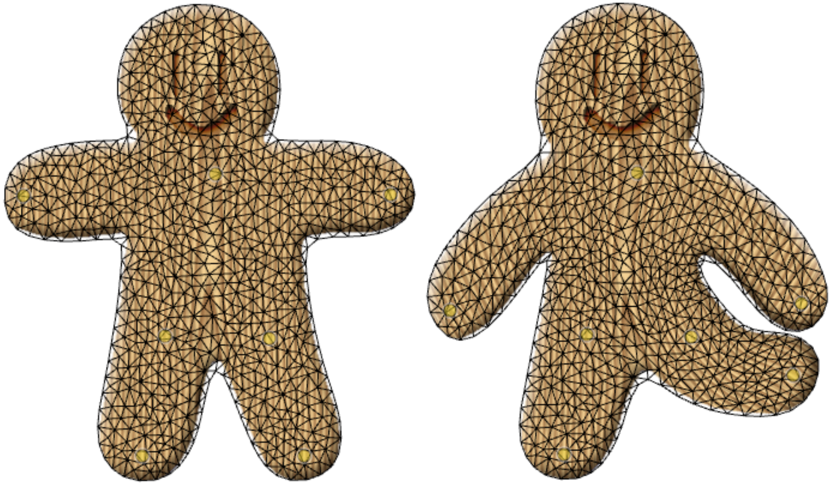

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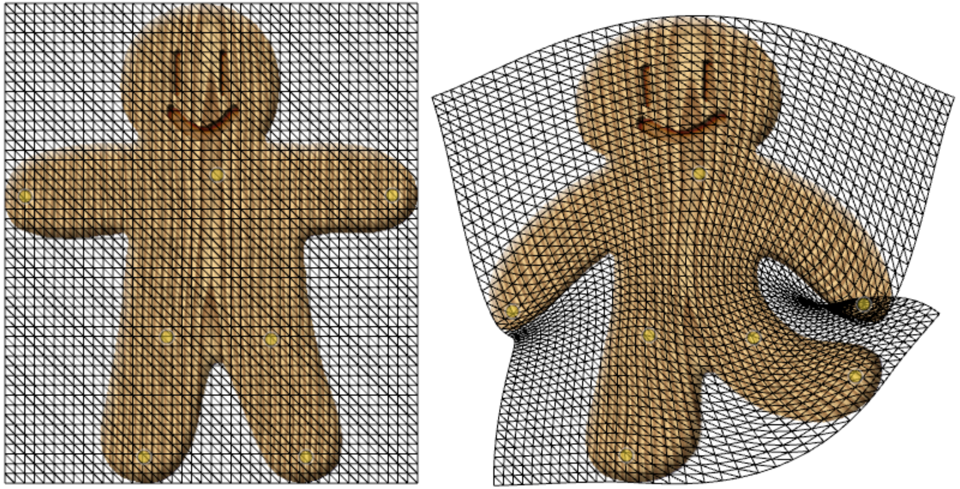

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Non-negative weights are mandatory

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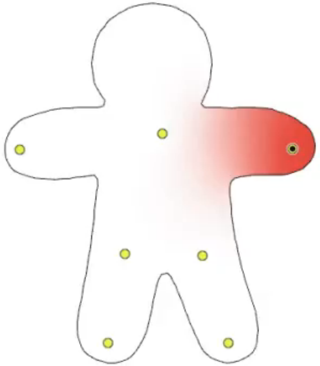
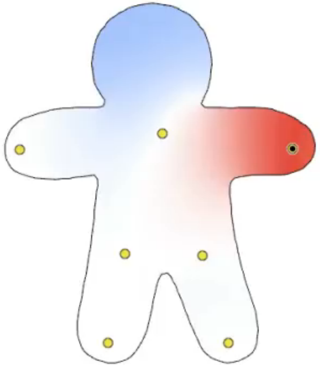


Unconstrained biharmonic
[Botsch and Kobbelt 2004]



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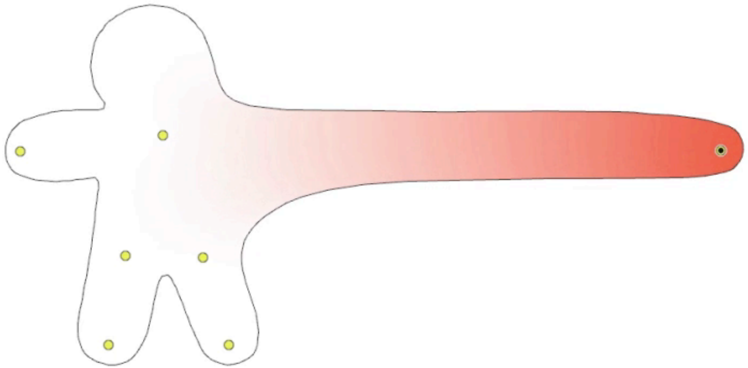

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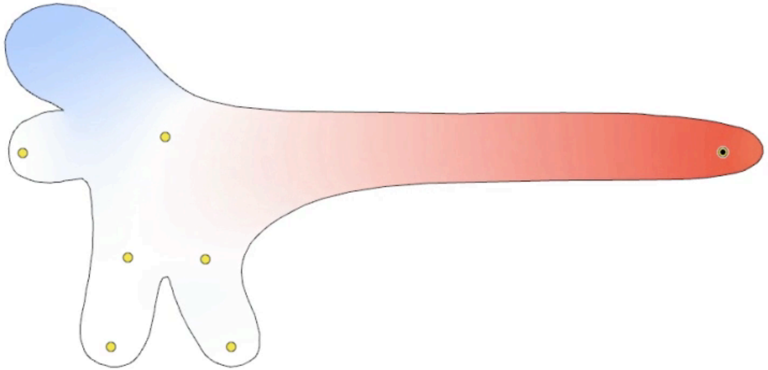

Weights must maintain other simple, but important properties

$$\sum_{j \in H} w_j(\mathbf{x}) = 1$$

$$w_j \Big|_{H_k} = \delta_{jk}$$

w_j is linear along cage faces

Partition of unity

Interpolation of handles

Weights must maintain other simple, but important properties

Set of handles, aka mesh vertices under handles

Kronecker's delta

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Previous methods only partially satisfactory

	Harmonic Coordinates [Joshi et al. 2005]	Unconstrained biharmonic [Botsch and Kobbelt 2004]	Shepard interpolation [Shepard 1968]	Natural neighbors [Sibson 1981]
Smoothness	-	✓	✓	-
Non-negativity	✓	-	✓	✓
Shape-aware	✓	✓	-	-
Locality, sparsity	-	-	-	✓
No local extrema	✓	-	-	✓

$$\Delta w_j = 0$$

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Inverse distance,
weighted least-squares

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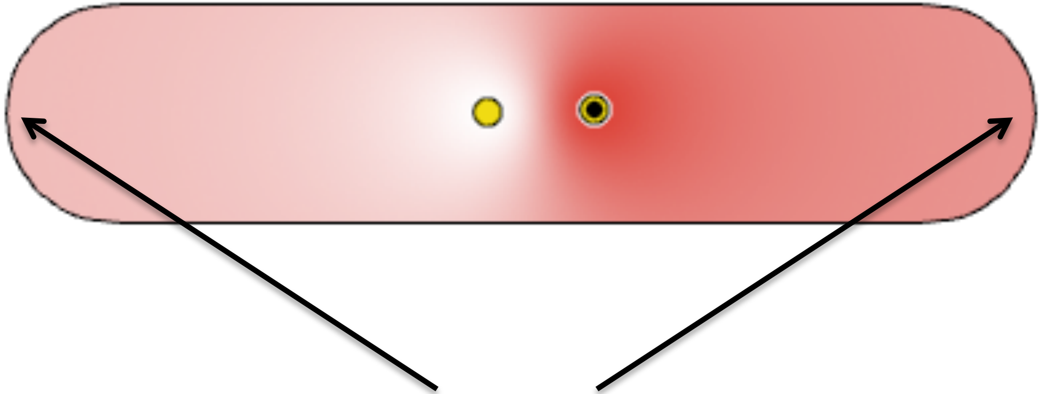
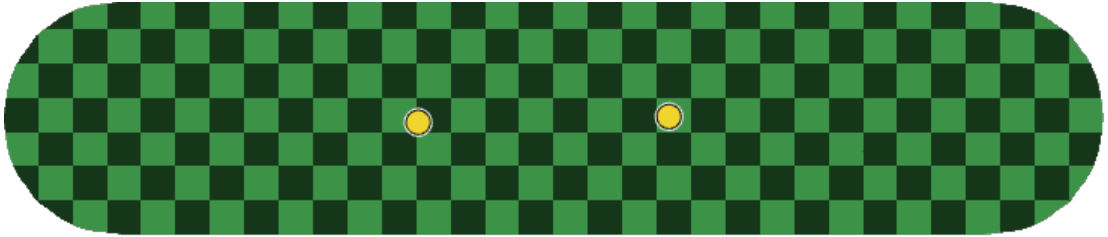


Approaching 0.5

Inverse distance methods inherently suffer from *fall-off effect*

Shepard



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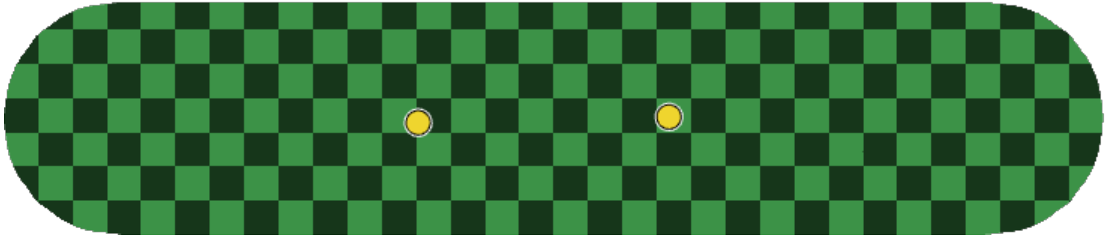

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Support bones and cages?
Shape-aware?

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$$\Delta^2 w_j = 0$$

Bounded biharmonic weights enforce properties as constraints to minimization

$$\arg\min_{w_j} \frac{1}{2}\int_{\Omega} \|\Delta w_j\|^2 dV$$

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$$0 \leq w_j(\mathbf{x}) \leq 1$$

Partition of unity

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Solve independently, normalize

$$w_j(\mathbf{x}) = \frac{w_j(\mathbf{x})}{\sum_{i \in H_k} w_i(\mathbf{x})}$$

Weights optimized as precomputation at bind-time

$$\arg\min_{w_j} \frac{1}{2}\int_{\Omega} \|\Delta w_j\|^2 dV$$

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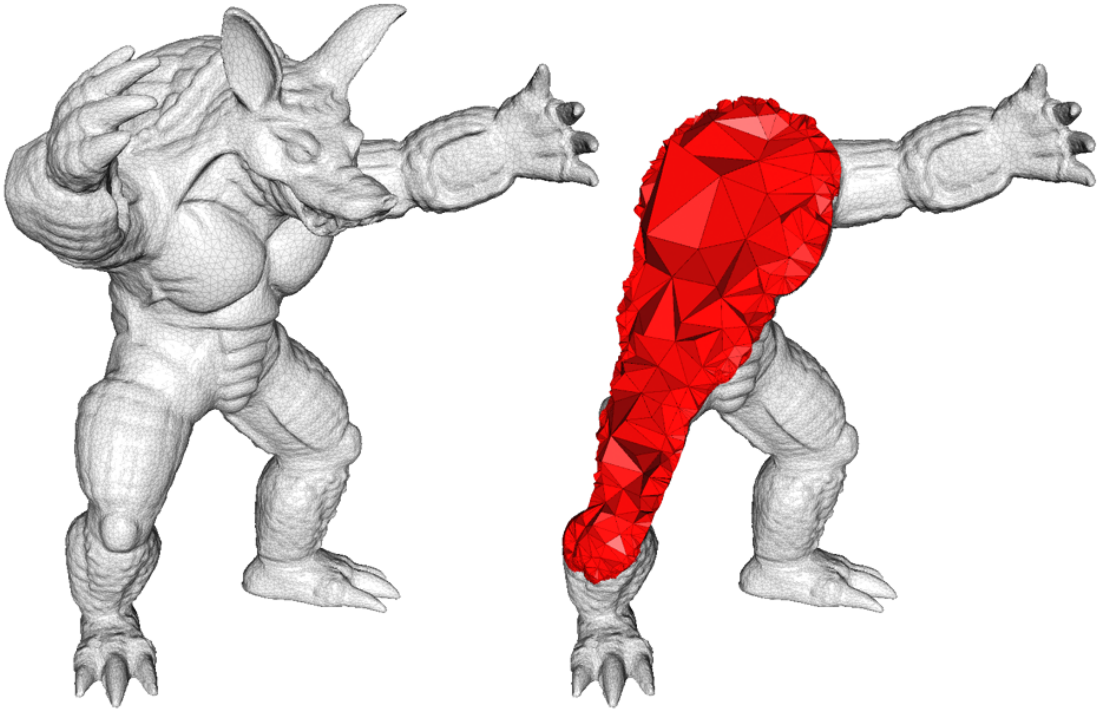
FEM discretization

- 2D → Triangle mesh
- 3D → Tet mesh



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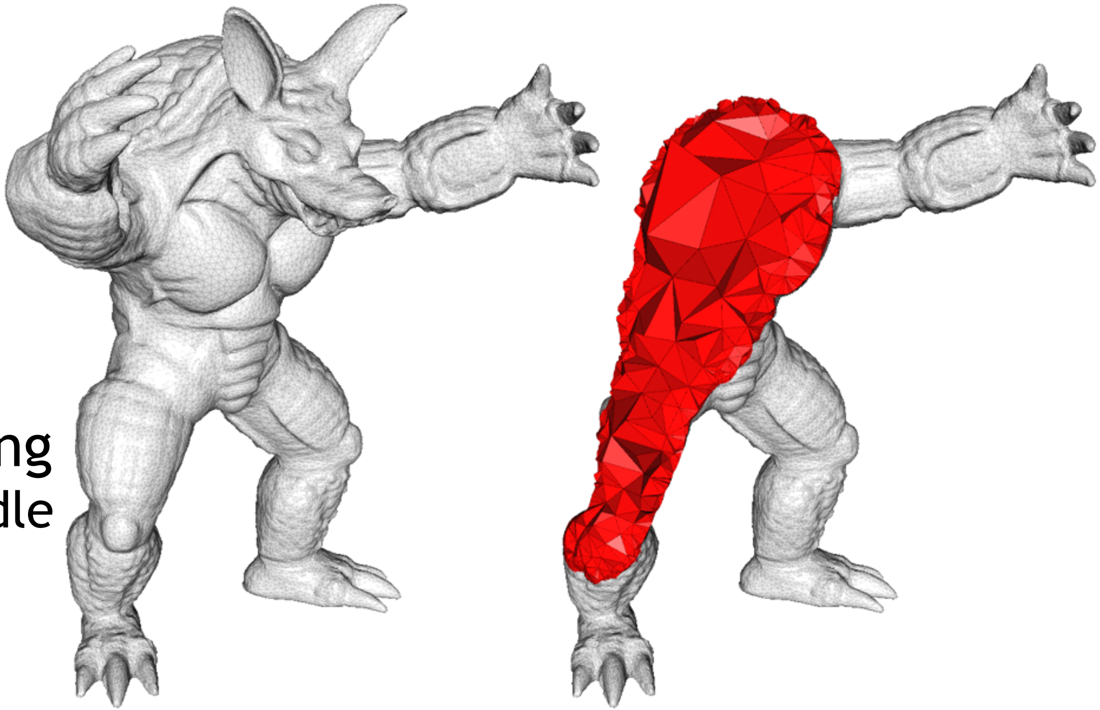
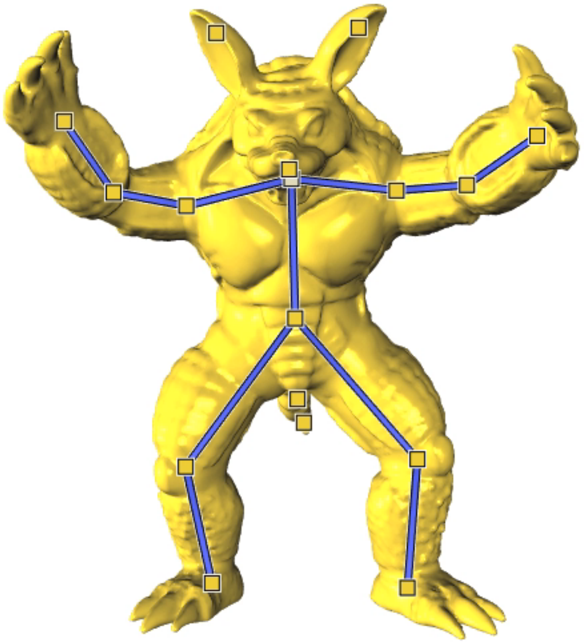


Sparse quadratic programming

2D ~O(milliseconds) per handle

3D ~O(seconds) per handle

Weights in 3D also retain nice properties



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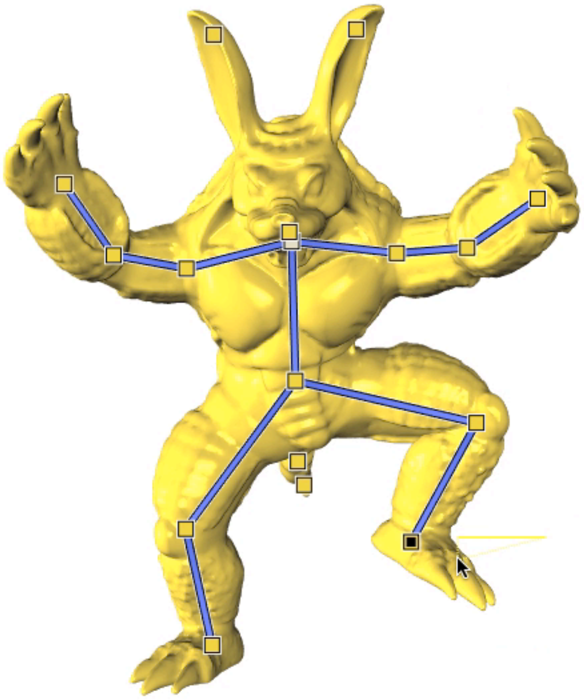

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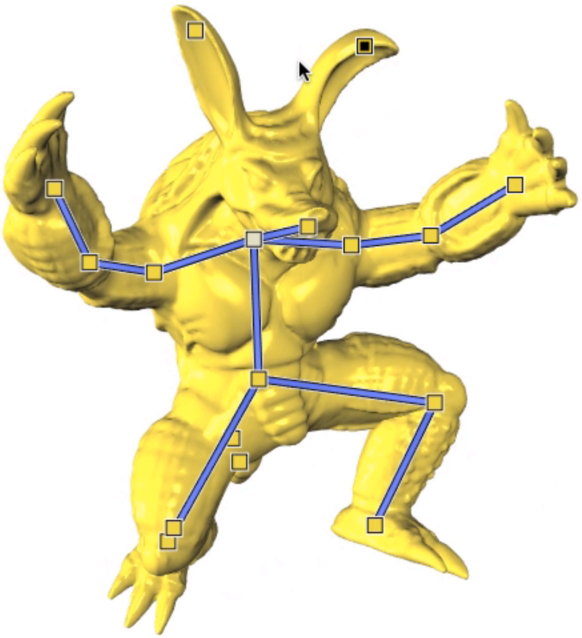
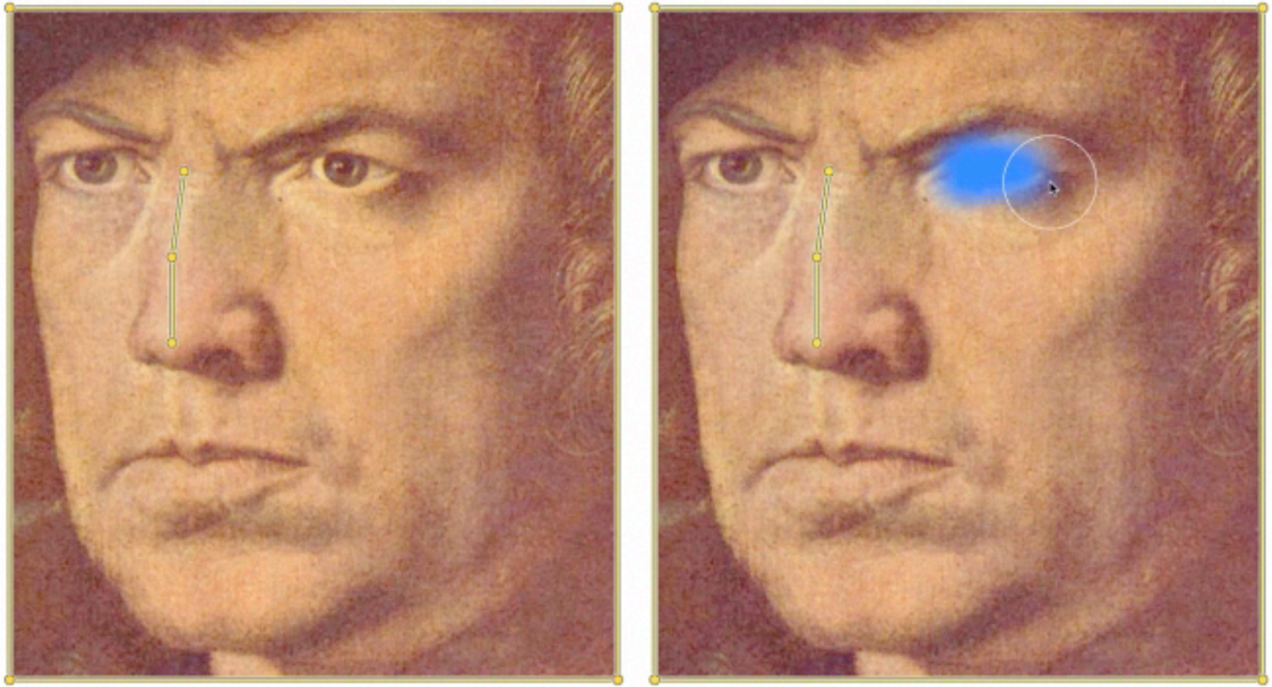


Variational formulation allows additional, problem-specific constraints



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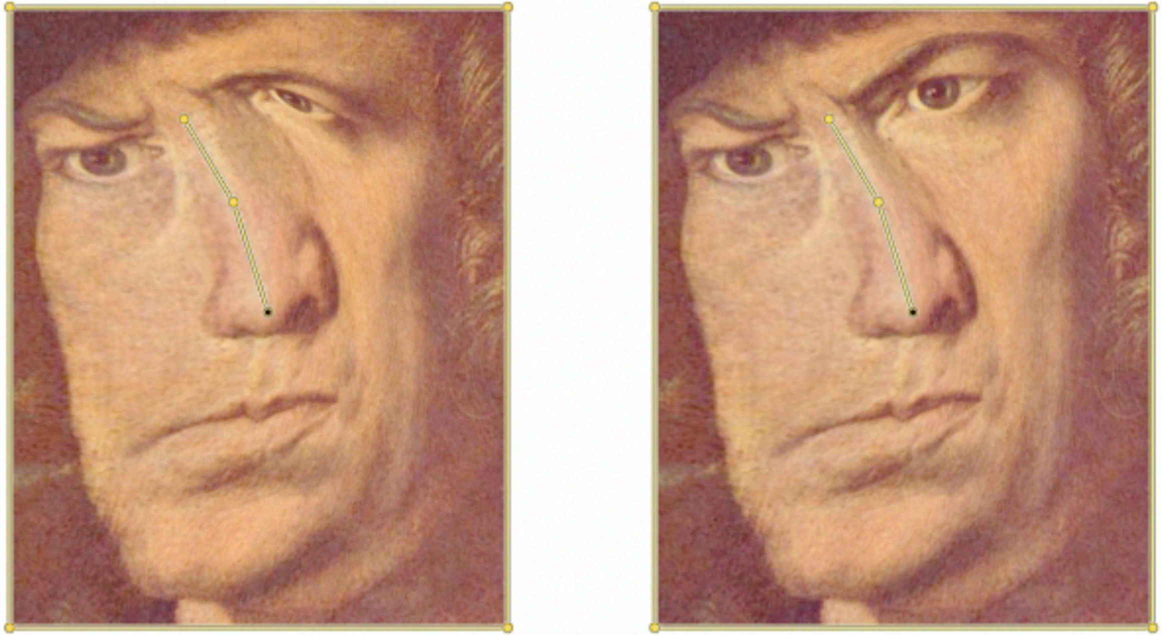

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Smoothness	-	✓	✓	-
Non-negativity	✓	-	✓	✓
Shape-aware	✓	✓	-	-
Locality, sparsity	-	-	-	✓
No local extrema	✓	-	-	✓

$$\Delta^2 w_j = 0$$

Our weights obtain all properties…

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*Empirically confirmed

… or so we thought

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Shape-aware	✓	✓	-	-
Locality, sparsity	-	✓*	-	✓
No local extrema	✓	-	-	✓

Smooth Shape-Aware Functions with Controlled Extrema

SGP 2012

Alec Jacobson[1]
Tino Weinkauf[2]
Olga Sorkine[1]

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August 1, 2012

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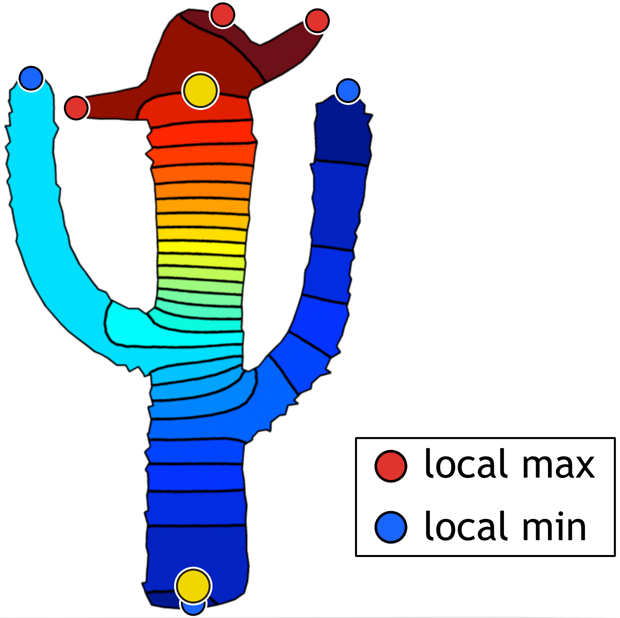
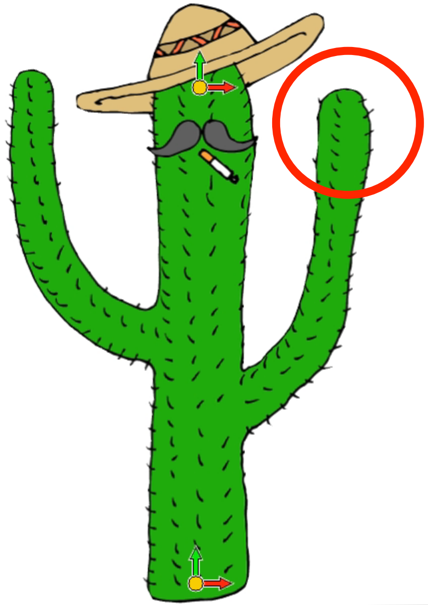
Swiss Federal Institute of Technology Zurich

Spurious extrema cause distracting artifacts

unconstrained Δ^2
[Botsch & Kobbelt 2004]

$$\mathbf{x}_i' = \sum_{j=1}^{H} f_j(\mathbf{x}_i)\, T_j\, \mathbf{x}_i$$





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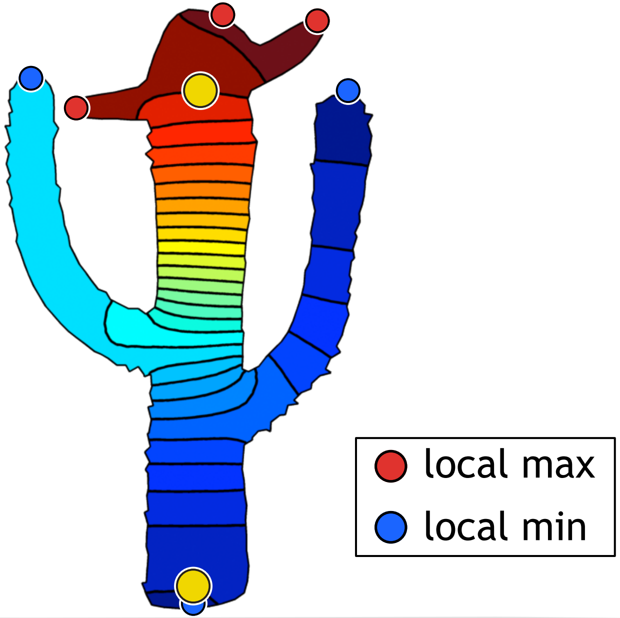
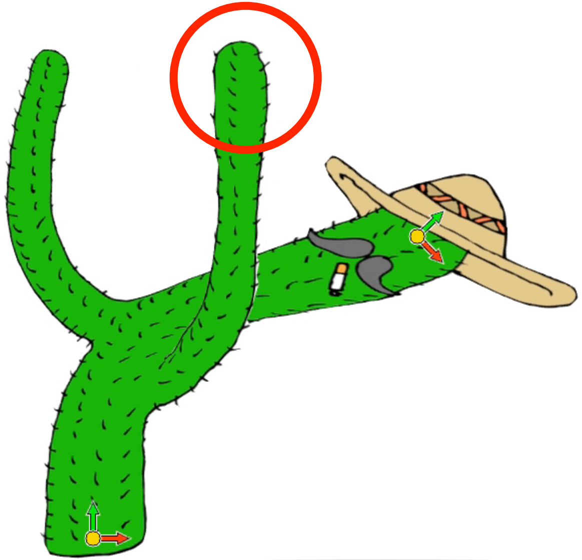
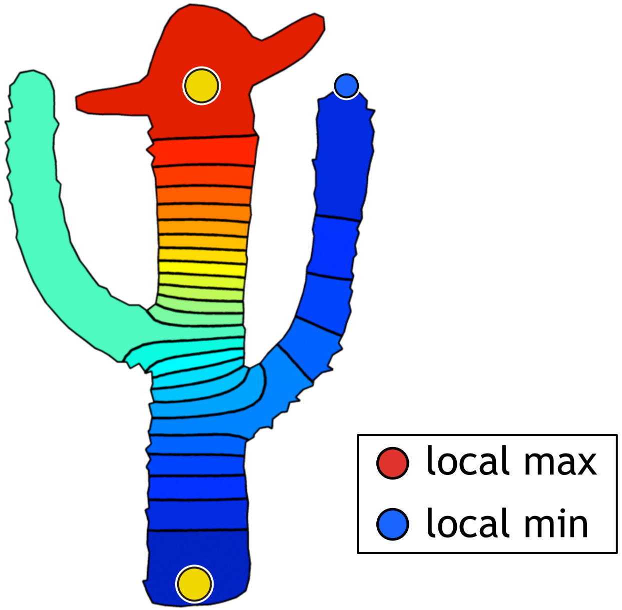


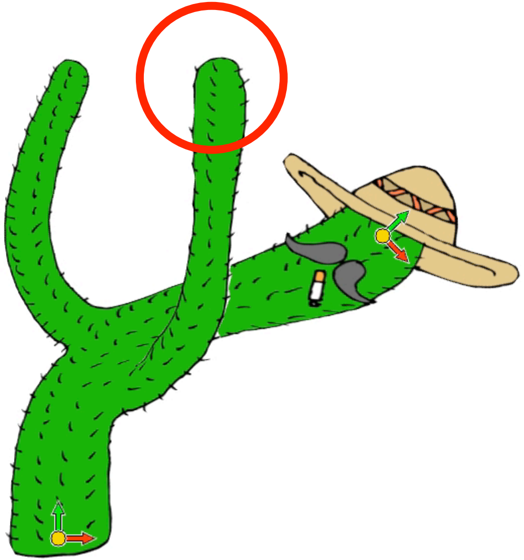


Bounds help, but don't solve problem

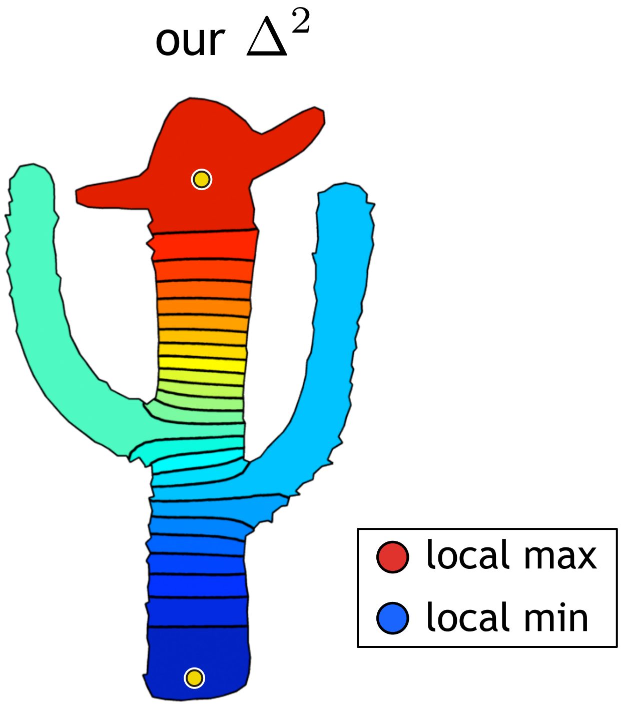
bounded Δ^2
[Jacobson et al. 2011]

$$\mathbf{x}'_i = \sum_{j=1}^{H} f_j(\mathbf{x}_i)\, T_j\, \mathbf{x}_i$$

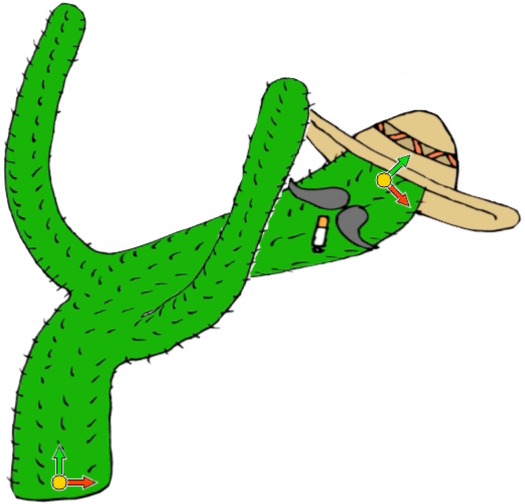




We explicitly prohibit spurious extrema



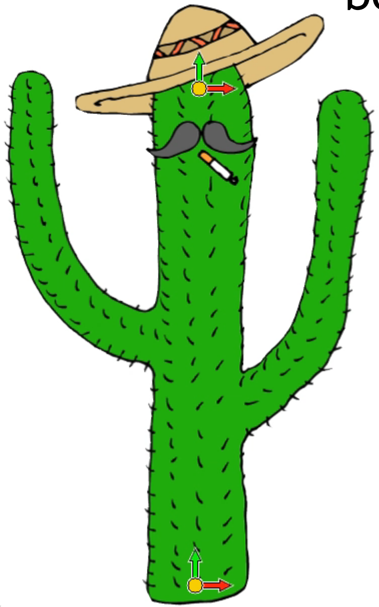
$$\mathbf{x}_i' = \sum_{j=1}^{H} f_j(\mathbf{x}_i)\, T_j\, \mathbf{x}_i$$



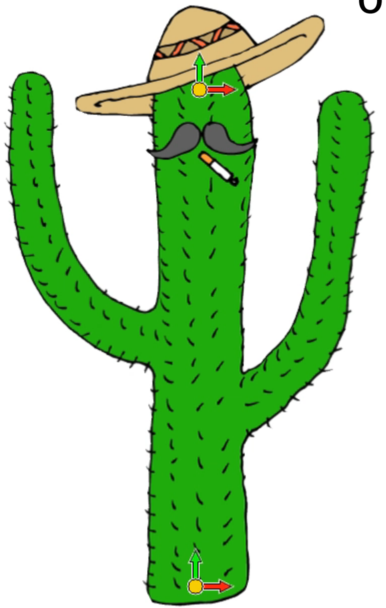
We *explicitly* prohibit spurious extrema

$$\mathbf{x}'_i = \sum_{j=1}^{H} f_j(\mathbf{x}_i)\, T_j\, \mathbf{x}_i$$

bounded Δ^4

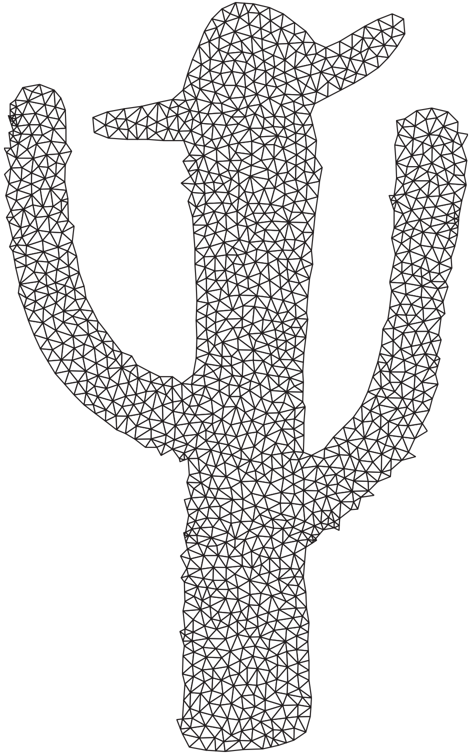


our Δ^4



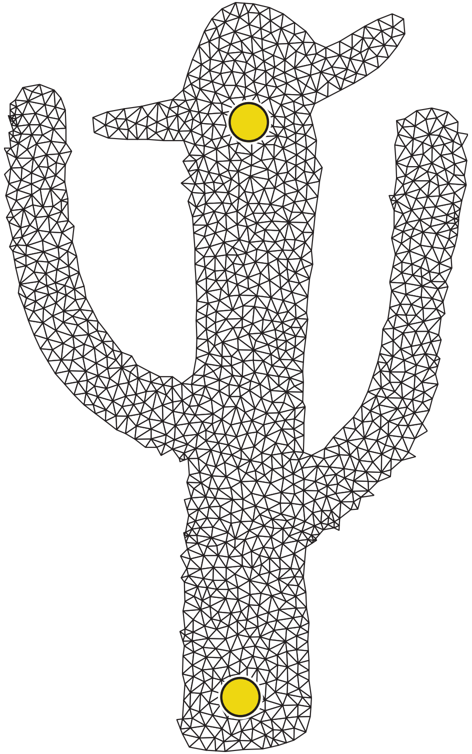
Ideal discrete problem is intractable

$$\arg\min_{f} \; E(f)$$



Ideal discrete problem is intractable

$$\arg\min_{f} \; E(f)$$
$$\text{s.t.} \quad f_{\max} = \mathit{known}$$
$$f_{\min} = \mathit{known}$$



Ideal discrete problem is intractable

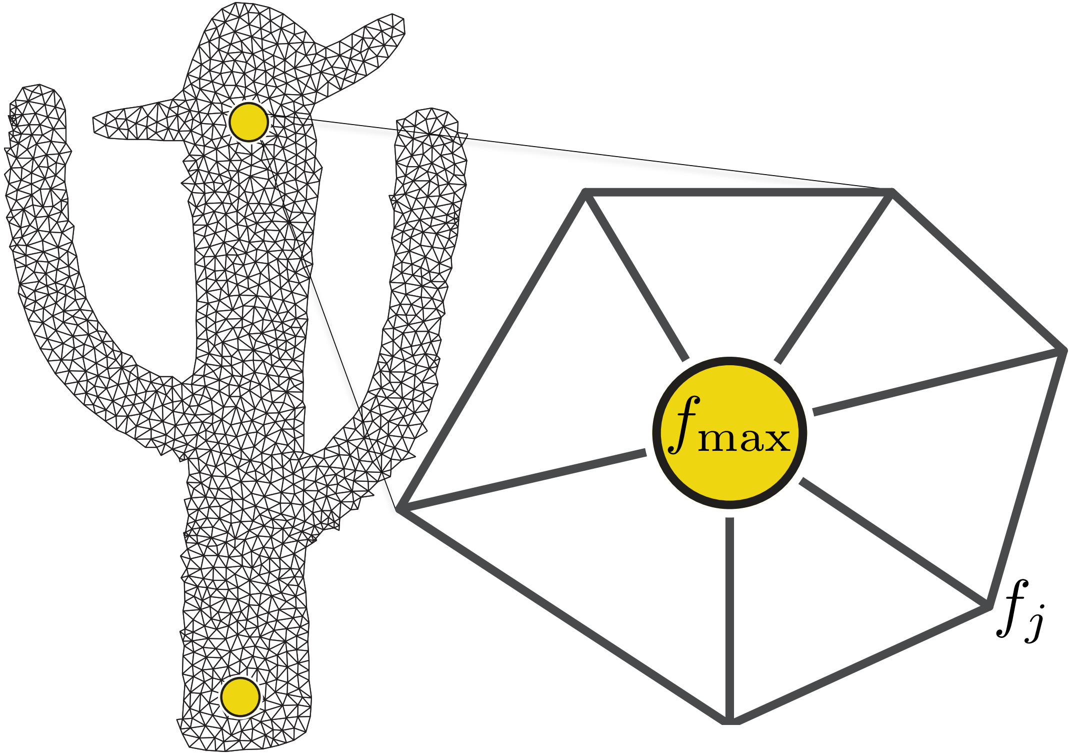
$$\arg\min_{f} \; E(f)$$

s.t. $\quad f_{\max} = known$

$f_{\min} = known$

linear $\quad f_j < f_{\max}$

$f_j > f_{\min}$



Ideal discrete problem is intractable

$$\arg\min_{f} \; E(f)$$

s.t. $f_{\max} = known$

$f_{\min} = known$

linear

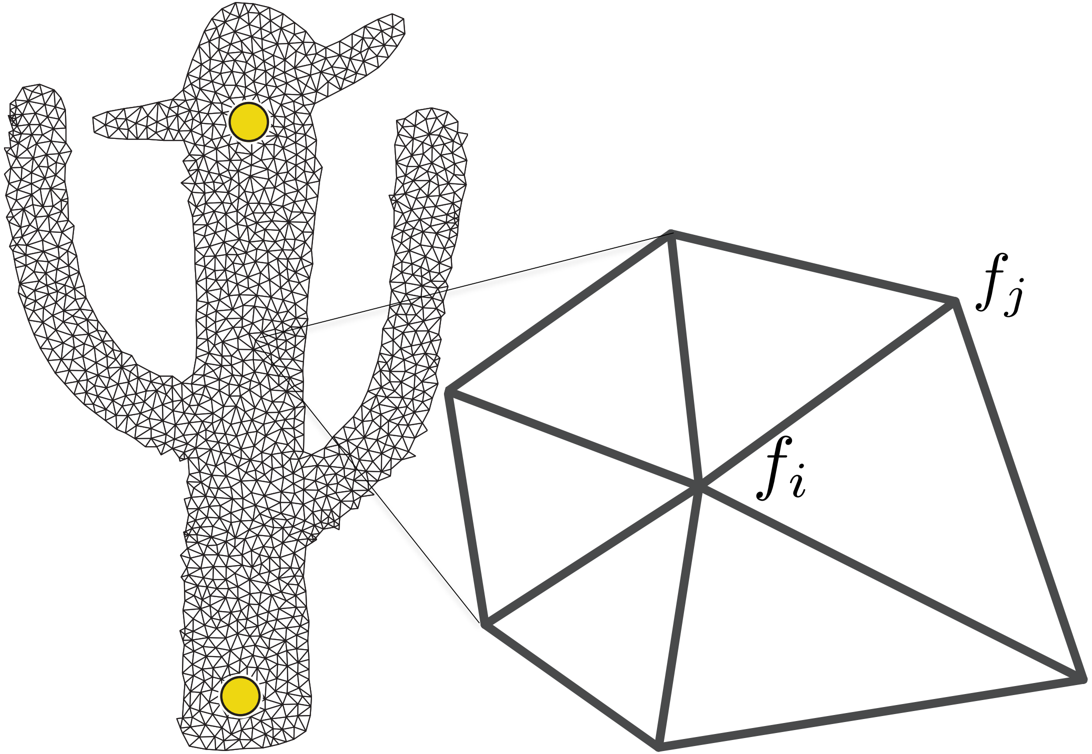
$f_j < f_{\max}$

$f_j > f_{\min}$

nonlinear

$f_i > \min_{j \in \mathcal{N}(i)} f_j$

$f_i < \max_{j \in \mathcal{N}(i)} f_j$



Assume we have a feasible solution

$$\arg\min_f \; E(f)$$

s.t.	$f_{\max} = known$
	$f_{\min} = known$
linear	$f_j < f_{\max}$
	$f_j > f_{\min}$
nonlinear	$f_i > \min_{j \in \mathcal{N}(i)} f_j$
	$f_i < \max_{j \in \mathcal{N}(i)} f_j$

"Representative function" u

$u_j < u_{\max}$	handles
$u_j > u_{\min}$	
$u_i > \min_{j \in \mathcal{N}(i)} u_j$	interior
$u_i < \max_{j \in \mathcal{N}(i)} u_j$	

Copy "monotonicity" of representative

$$\arg\min_{f} \; E(f)$$

$$\text{s.t.} \quad f_{\max} = known$$

$$f_{\min} = known$$

$$(f_i - f_j)(u_i - u_j) > 0 \qquad \text{linear} \qquad \forall (i,j) \in \mathcal{E}$$

At least one edge in either direction per vertex

Rewrite as conic optimization

QP

$$\begin{aligned} \underset{\mathbf{f}}{\text{minimize}} \quad & \frac{1}{2}\|\mathbf{F}\,\mathbf{f}\|^2 + \mathbf{c}^\intercal \mathbf{f} + \text{const} \\ \text{subject to} \quad & \mathbf{A}_{leq}^\intercal \mathbf{f} \leq \mathbf{b}_{leq}, \\ & \mathbf{f} \leq \mathbf{u}_f, \quad \mathbf{f} \geq \mathbf{l}_f \end{aligned}$$

Conic

$$\begin{aligned} \underset{[\mathbf{f}^\intercal \ \mathbf{t}^\intercal \ v]}{\text{minimize}} \quad & \begin{bmatrix} \mathbf{c}^\intercal & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \mathbf{t} \\ v \end{bmatrix} + \text{const} \\ \text{subject to} \quad & \begin{bmatrix} \mathbf{F} & -\mathbf{I} & 0 \\ \mathbf{A}_{leq}^\intercal & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \mathbf{t} \\ v \end{bmatrix} \geq \begin{bmatrix} 0 \\ -\infty \end{bmatrix} \\ & \begin{bmatrix} \mathbf{F} & -\mathbf{I} & 0 \\ \mathbf{A}_{leq}^\intercal & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \mathbf{t} \\ v \end{bmatrix} \leq \begin{bmatrix} 0 \\ \mathbf{b}_{leq} \end{bmatrix} \\ & \begin{bmatrix} \mathbf{f} \\ \mathbf{t} \\ v \end{bmatrix} \leq \begin{bmatrix} \mathbf{u}_f \\ \infty \\ \infty \end{bmatrix} \\ & \begin{bmatrix} \mathbf{f} \\ \mathbf{t} \\ v \end{bmatrix} \geq \begin{bmatrix} \mathbf{l}_f \\ -\infty \\ 0 \end{bmatrix} \\ & 2v \geq \sum_i t_i^2 \end{aligned}$$

Optimize with MOSEK

Harmonic functions obey maximum principle

	Harmonic Coordinates [Joshi et al. 2005]	Unconstrained biharmonic [Botsch and Kobbelt 2004]	Shepard interpolation [Shepard 1968]	Natural neighbors [Sibson 1981]
Smoothness	-	✓	✓	-
Non-negativity	✓	-	✓	✓
Shape-aware	✓	✓	-	-
Locality, sparsity	-	-	-	✓
No local extrema	✓	-	-	✓

$$\Delta u = 0$$

Final algorithm is simple and efficient

- Harmonic representative
  - Linear solve ~O(milliseconds)

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  - 2D ~O(milliseconds), 3D ~O(seconds)

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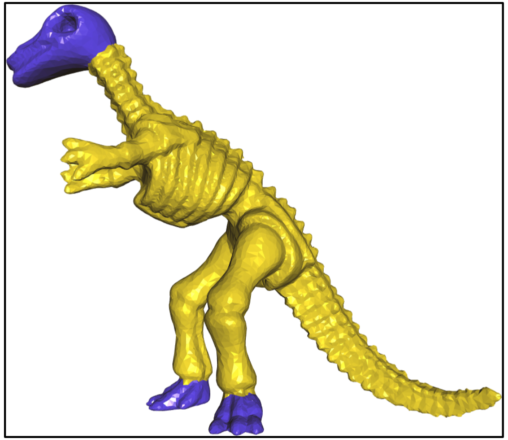
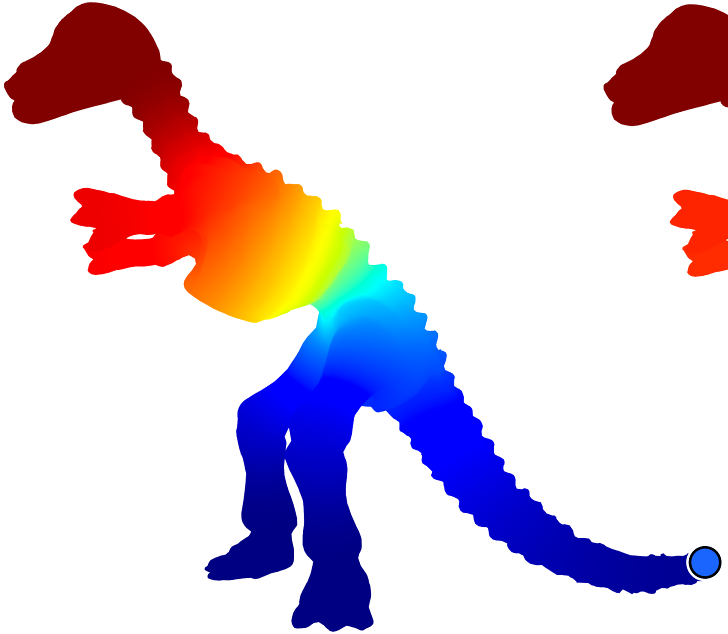
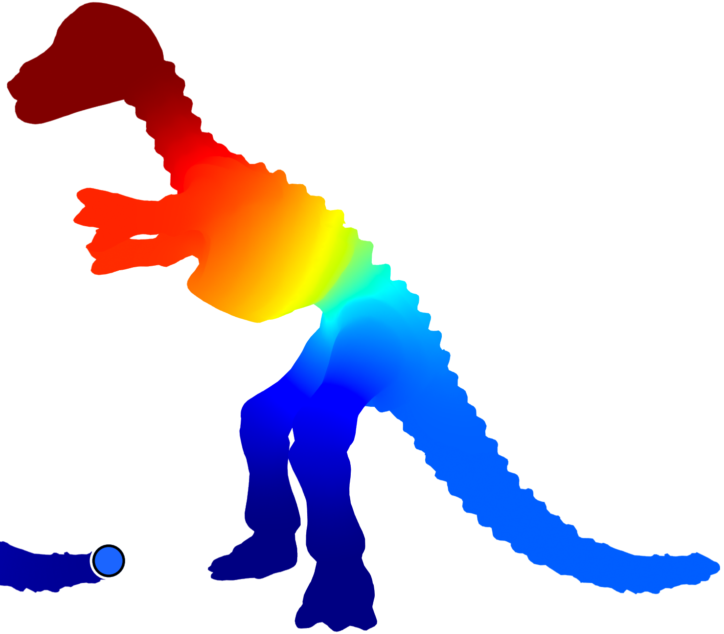
Again, functions are **precomputed**

Our weights attach appendages to body



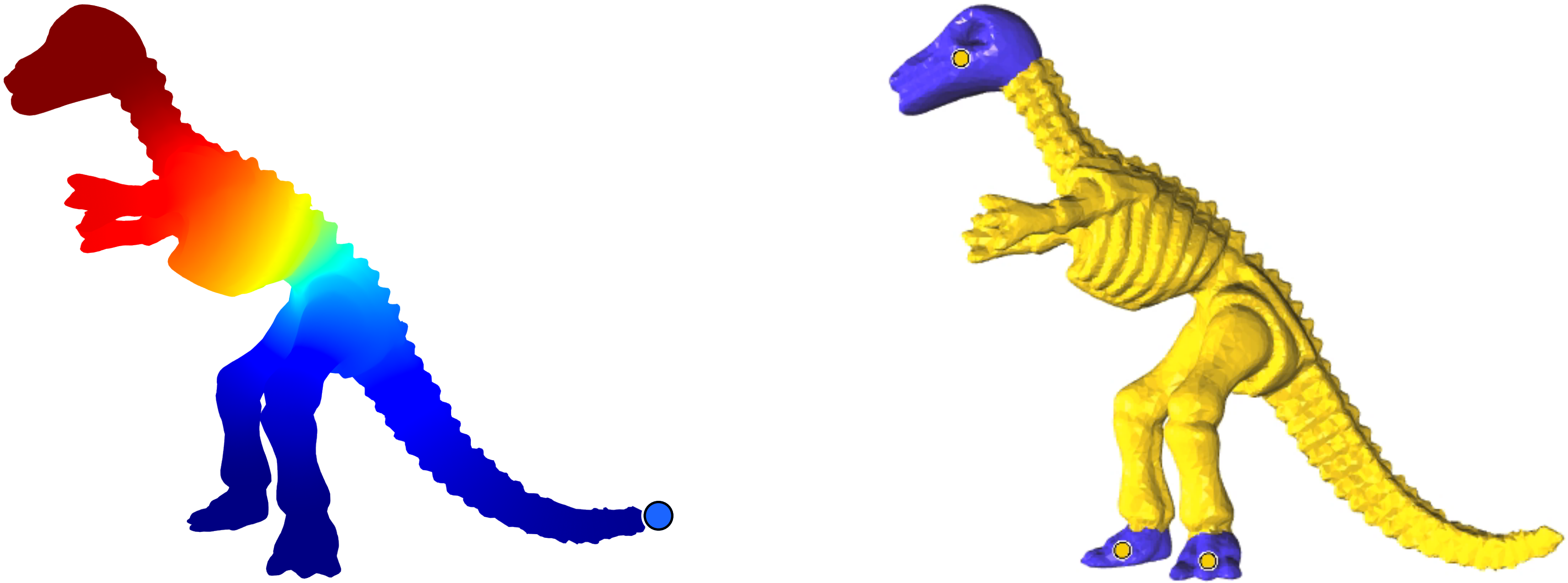


[Botsch & Kobbelt 2004, Jacobson et al. 2011]



Our method

Extrema glue appendages to far-away handles



[Botsch & Kobbelt 2004, Jacobson et al. 2011]

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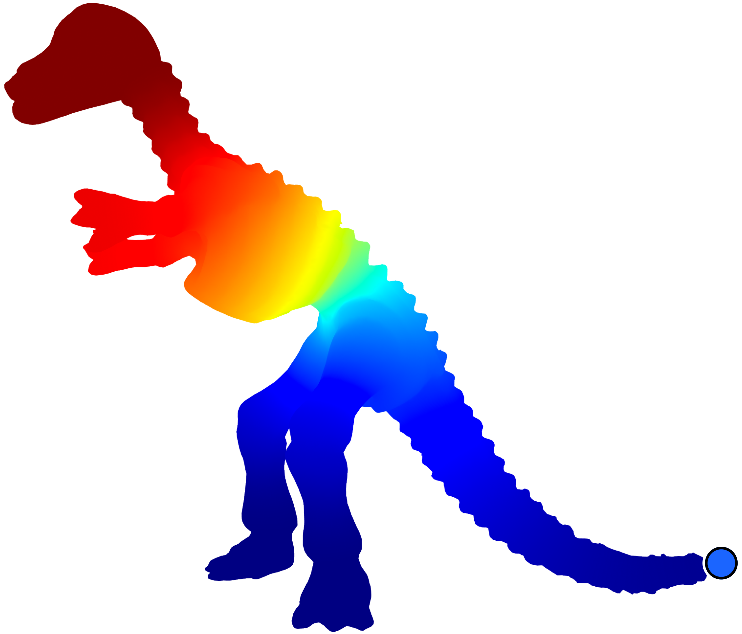

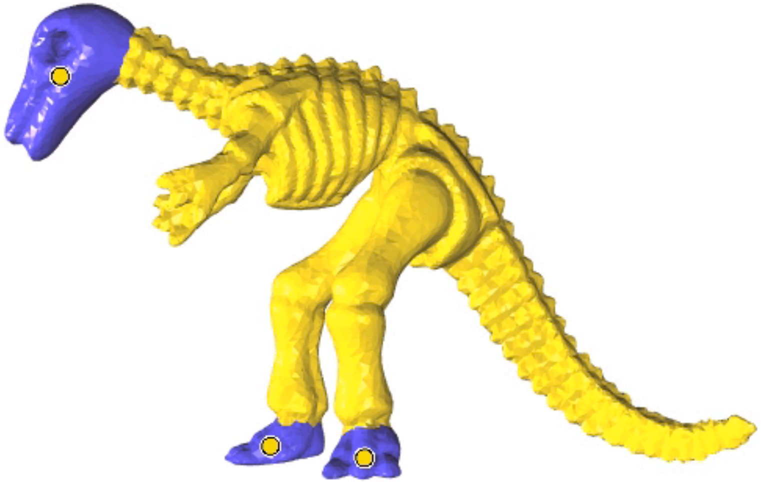

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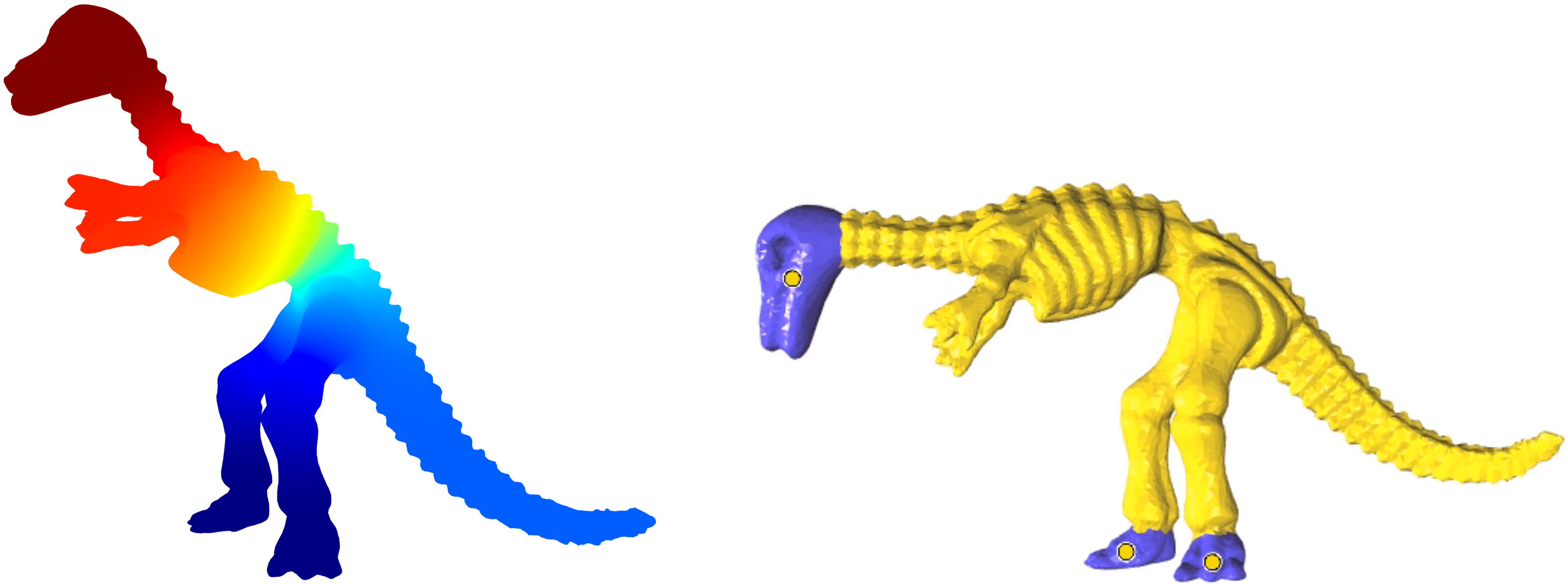
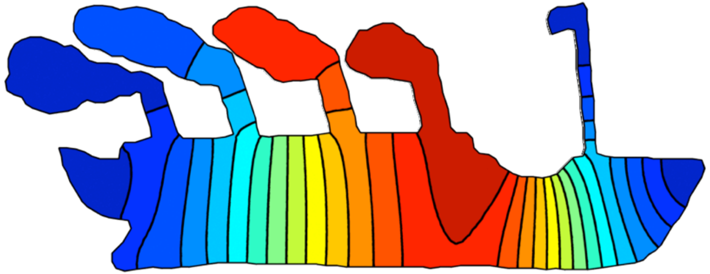


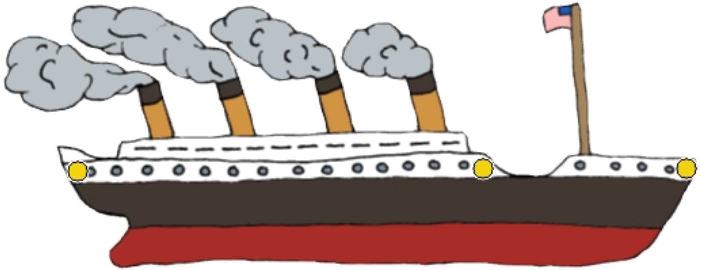
Our method

Extrema distort small features

Bounded Δ^2 [Jacobson et al. 2011]



weight of middle point



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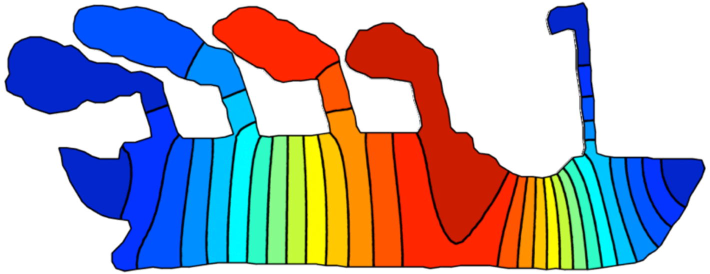

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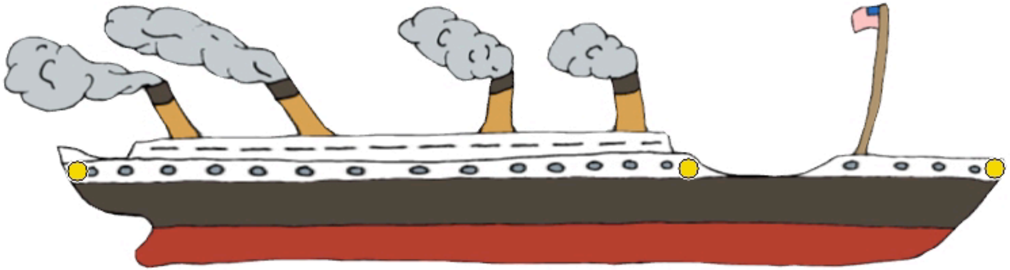
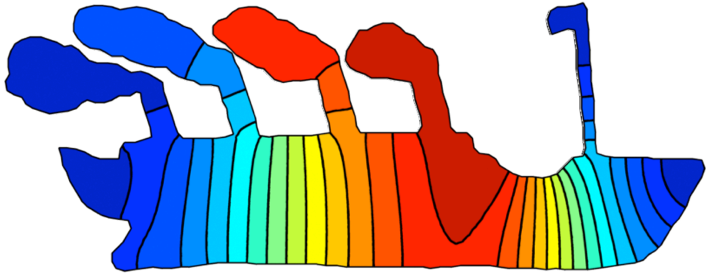
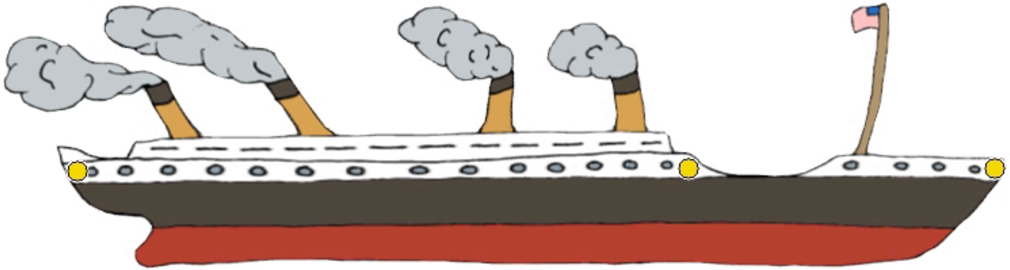


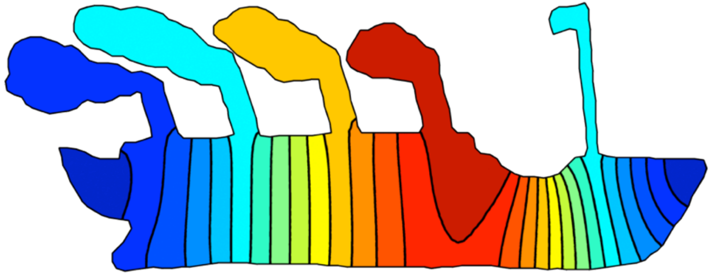
“Monotonicity” helps preserve small features

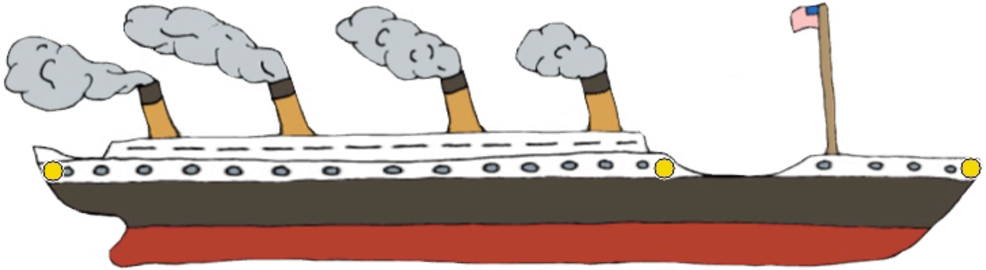
Bounded Δ^2 [Jacobson et al. 2011]





Our Δ^2





Conclusion: variational framework allows explicit control over desired properties

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Future work and discussion

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Acknowledgements

We thank Kenshi Takayama, Yang Song, Jaakko Lehtinen, Bob Sumner and Denis Zorin.

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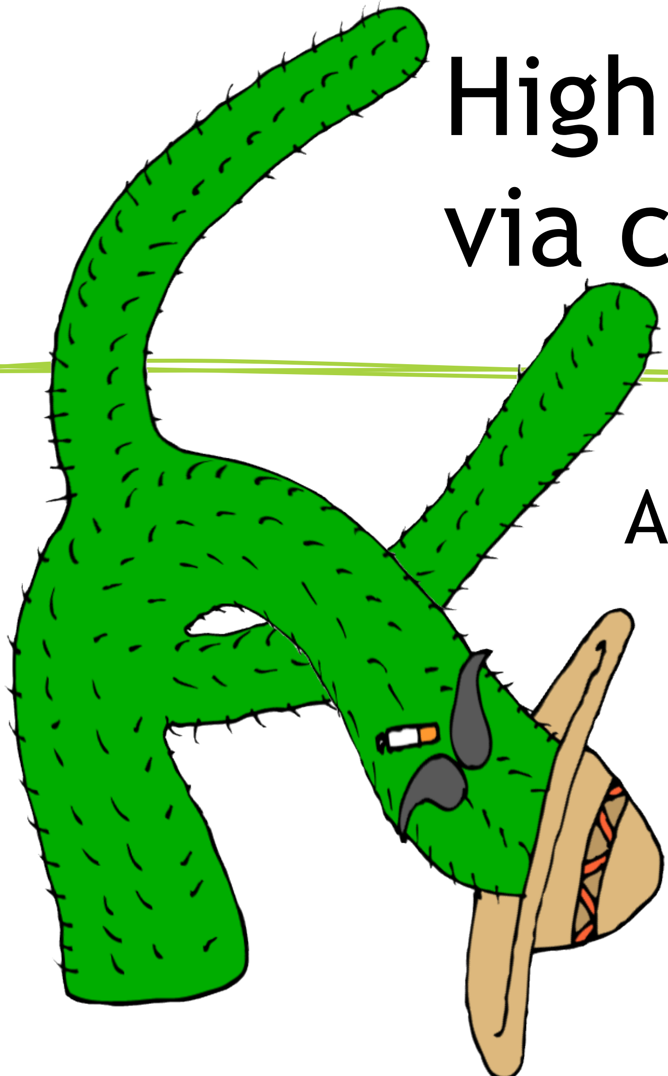

High-quality weight functions via constrained optimization

Alec Jacobson jacobson@inf.ethz.ch

MATLAB Demos and more:
http://igl.ethz.ch/projects/bbw/
http://igl.ethz.ch/projects/monotonic/