Discrete Laplace operators: No free lunch

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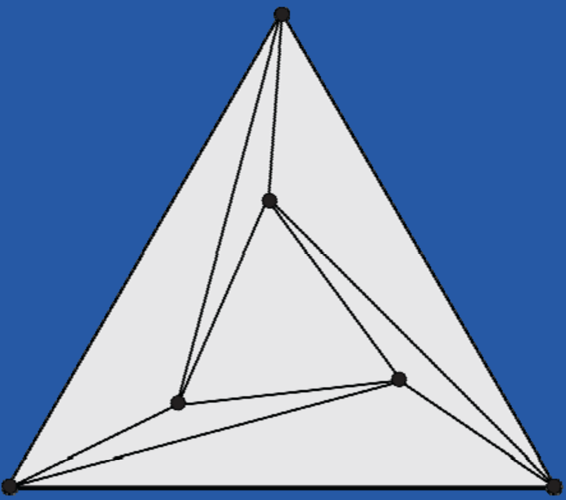

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Key properties of smooth Laplacians (Sym) (Loc) (Lin) (Psd) (Max)

- (Sym) Symmetry: $(\Delta u, v)_{L^2} = (u, \Delta v)_{L^2}$

- (Loc) Locality: changing $u(q)$ does not change $(\Delta u)(p)$

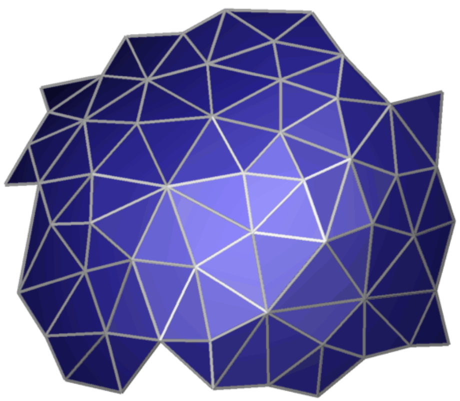
What about the discrete case?

- (Lin) Linear precision: $(\partial_x^2 + \partial_y^2)(ax + by + c) = 0$

- (Psd) Laplacians are positive (semi)definite

- (Max) Maximum principle

Discrete Laplace operators:



Input:

$u = (u_i)$ function on mesh vertices

Output:

$$(Lu)_i = \sum_j \omega_{ij}(u_i - u_j)$$

Properties of L are encoded by $\omega = (\omega_{ij})$

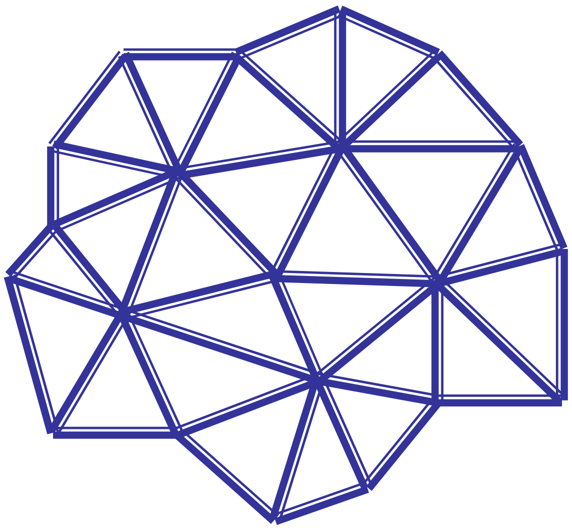
Desired properties

1. (Sym) Symmetry: $\omega_{ij} = \omega_{ji}$

Motivation:

- smooth symmetry
- real eigenvalues & orthogonal eigenvectors

2. (Loc) Locality: $\omega_{ij} = 0$ if (ij) is not an edge



Motivation:

- smooth locality
- diffusion: $u_t = -\Delta u$

  discrete: ω_{ij} random walk 'probabilities' along edges

Desired properties

3. (Lin) Linear precision: $(Lu)_i = 0$

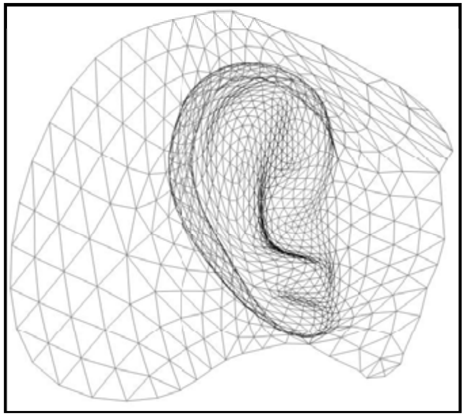
   if mesh is in the plane and u is linear

Motivation:

- smooth linear precision
- mesh denoising: no tangential vertex drift
- mesh parameterization: planar vertices don’t move

4. (Pos) Positivity: $\omega_{ij} \geq 0$

$\Longrightarrow$ (Psd) + (Max)



[Gortler/Gotsman/Thurston ‘05]

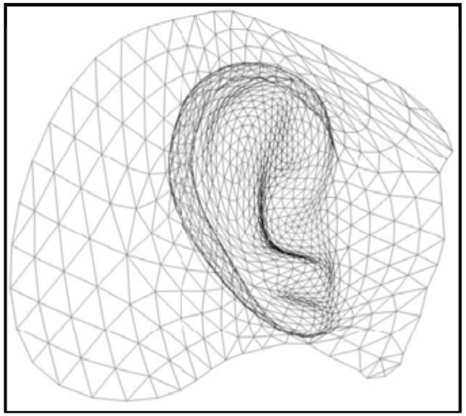
Motivation:

- positive (semi)-definiteness
- parameterization: no flipped triangles (locally)
- barycentric coordinates (maximum principle)

$$\lambda_{ij} = \frac{\omega_{ij}}{\sum_{j \neq i} \omega_{ij}} \Longrightarrow \sum_{j \neq i} \lambda_{ij} = 1$$

Four desired properties

(Sym) (Loc) (Lin) (Pos)



[Gortler/Gotsman/Thurston '05]

4. (Pos) Positivity: $\omega_{ij} \geq 0$

$\Longrightarrow$ (Psd) + (Max)

Motivation:

Barycentric coordinates —> Laplacians

Barycentric coordinates that satisfy (Lin) give a Laplacian that satisfies (Loc), (Lin), (Pos), but usually not (Sym).

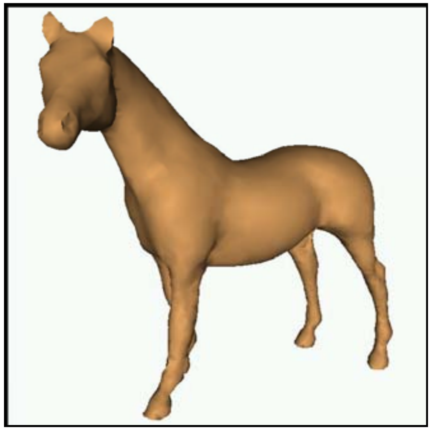
$$\lambda_{ij} = \frac{\omega_{ij}}{\sum_{j \neq i} \omega_{ij}}$$

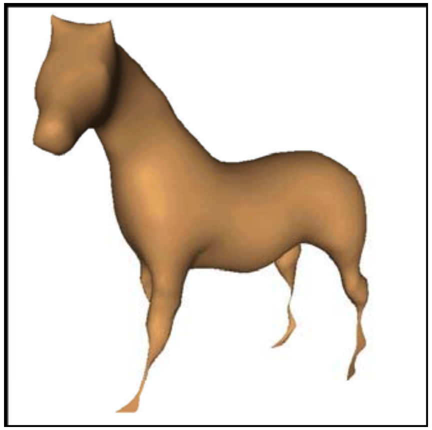
Four examples

(Sym) (Loc) (Lin) (Pos)

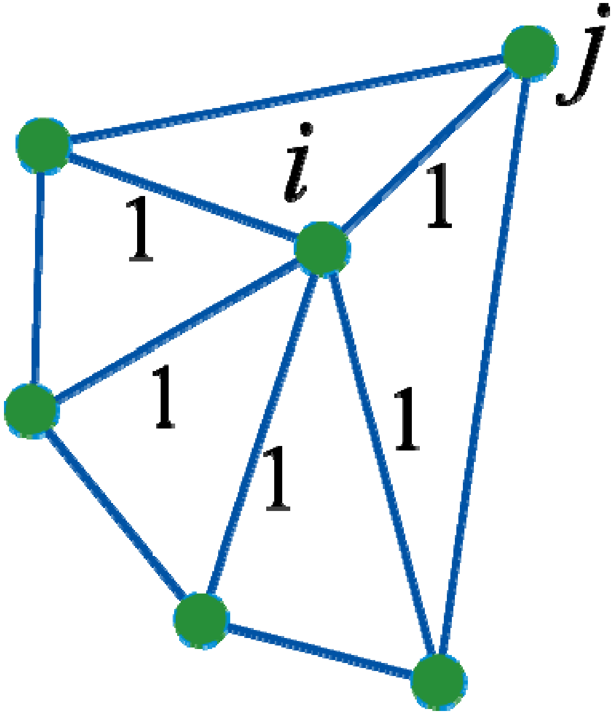
1. Combinatorial Laplacians [Tutte ’63, ...]

$$\omega_{ij} = 1$$



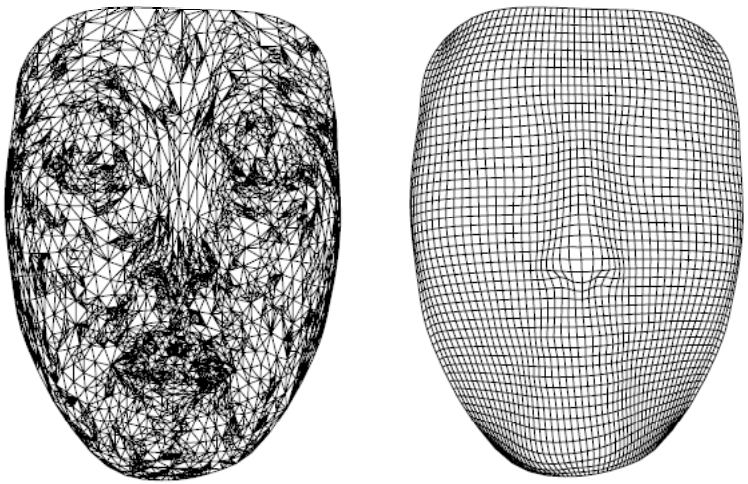


[Karni/Gotsman ’00]

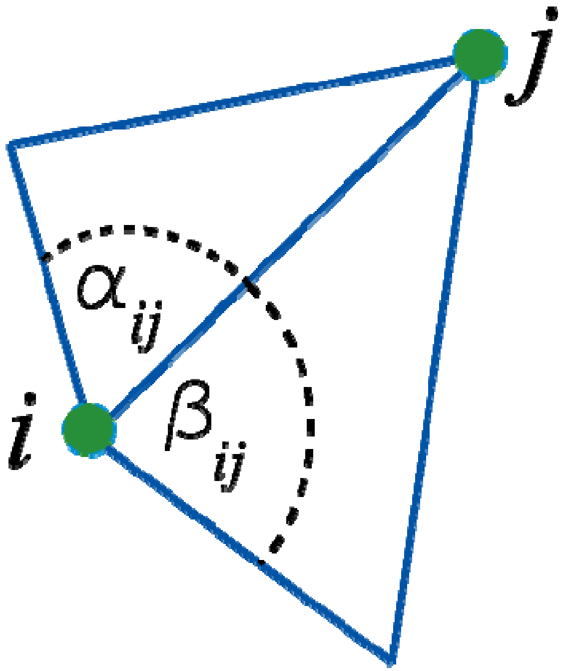


2. Mean-value coordinates [Floater ’03, ...]

$$\omega_{ij} = \frac{\tan\frac{\alpha_{ij}}{2} + \tan\frac{\beta_{ij}}{2}}{|e_{ij}|}$$



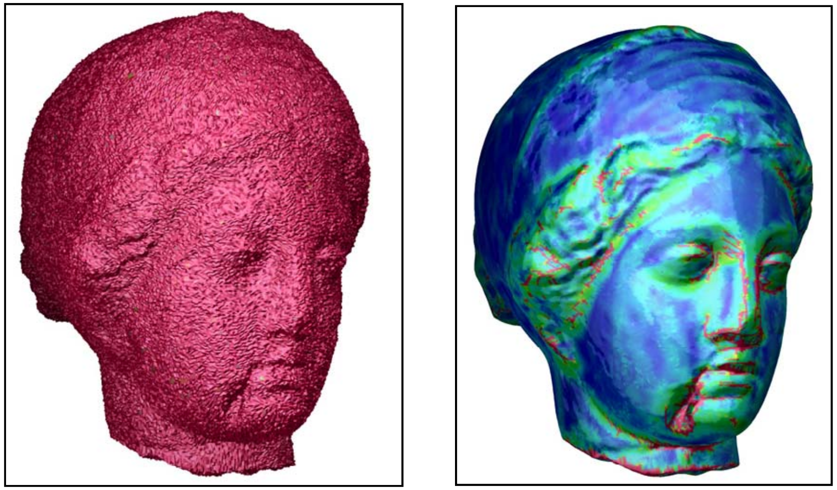
[Floater ’03]



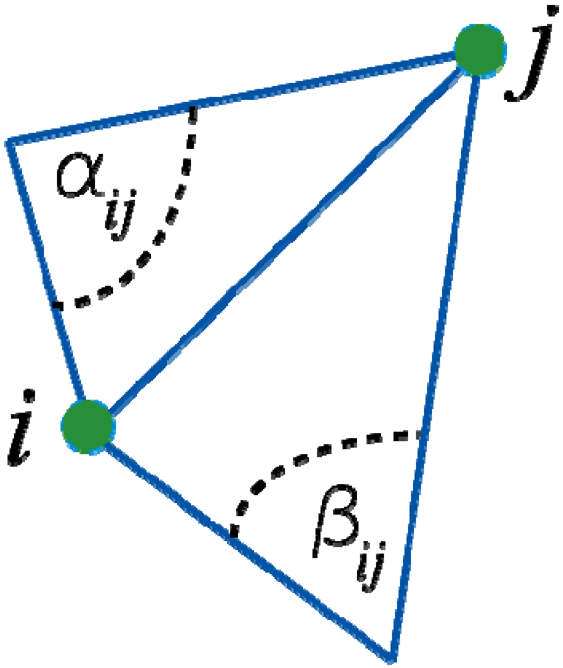
3. cotan weights [Pinkall/Poltier ’93, ...]

$$\omega_{ij} = \cot \alpha_{ij} + \cot \beta_{ij}$$

$$\alpha_{ij} + \beta_{ij} > \pi \iff \omega_{ij} < 0$$

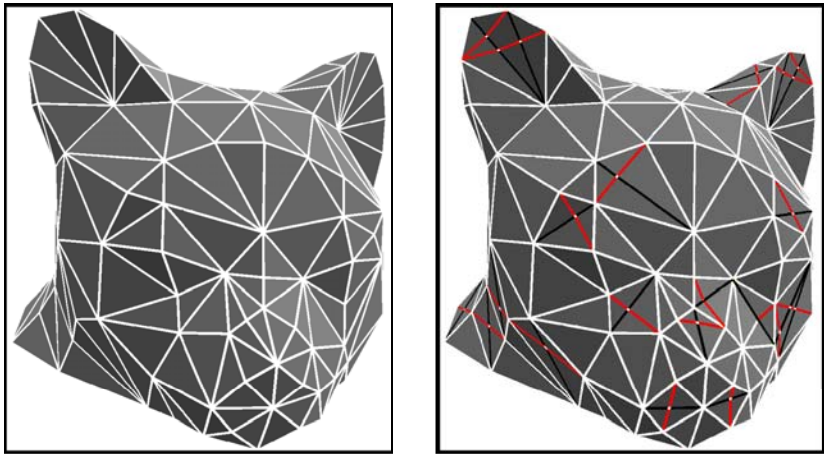


[Hildebrandt/Polthier ’05]

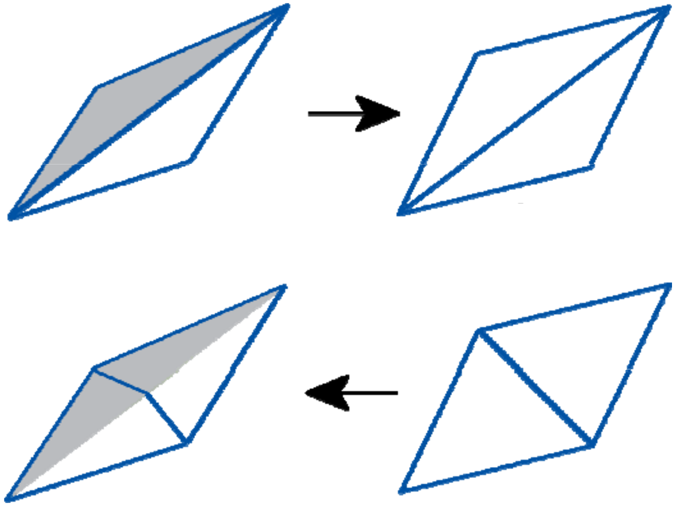


4. intrinsic Delaunay [Bobenko/Sprinborn ’05, ...]

$$\omega_{ij} = \cot\alpha_{ij} + \cot\beta_{ij}$$
$$\geq 0$$



[Fisher et al. ’06]



[intrinsic edge flips]

Putting four things together

	(Sym)	(Loc)	(Lin)	(Pos)
mean value	ø	✓	✓	✓
intrinsic Delaunay	✓	ø	✓	✓
combinatorial	✓	✓	ø	✓
cotan	✓	✓	✓	ø

... on general irregular meshes!

Putting four things together

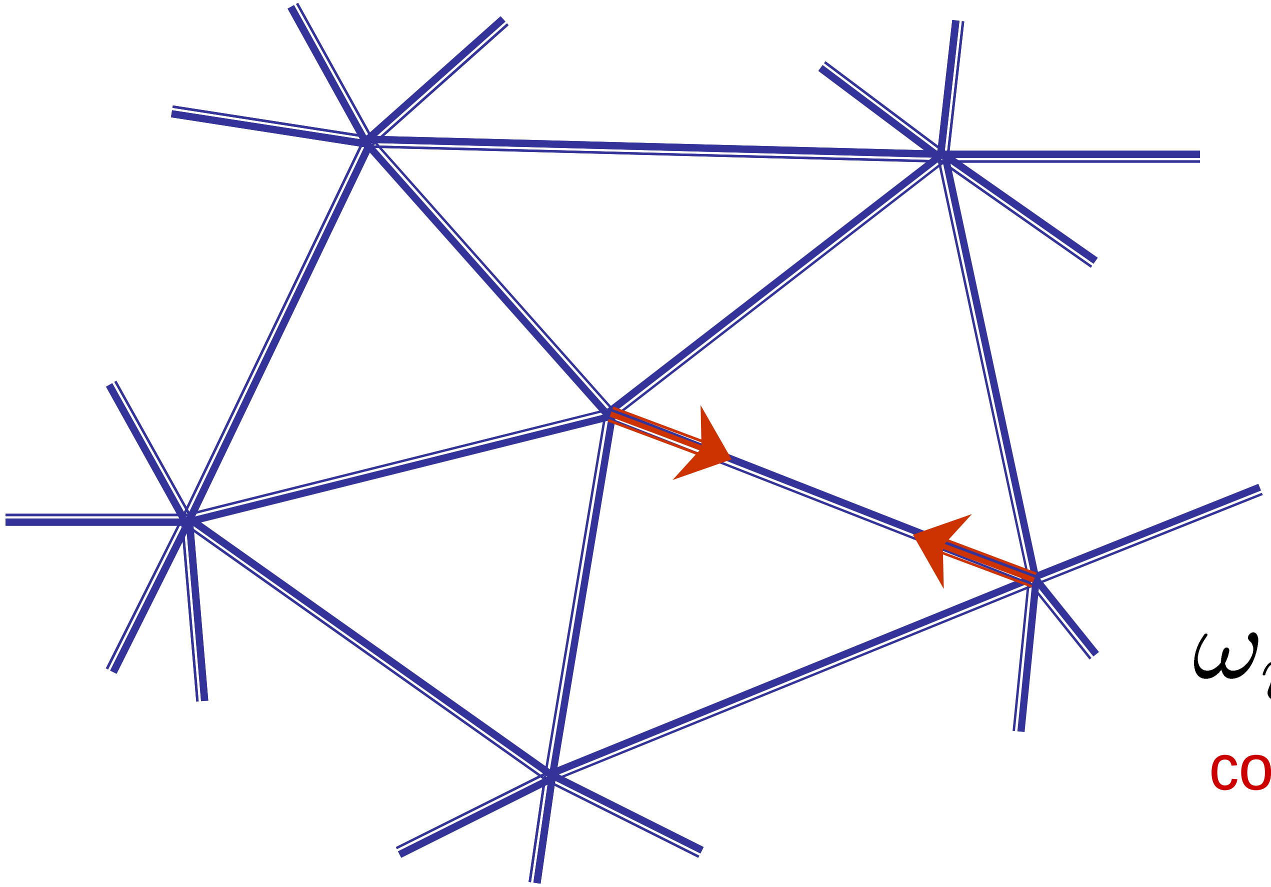
Main result:

No-free-lunch-theorem (preliminary version)

General meshes do not allow for discrete Laplacians with (Sym)+(Loc)+(Lin)+(Pos).

Proof: planar stress frameworks

1. (Sym)+(Loc) & stress frameworks in the plane

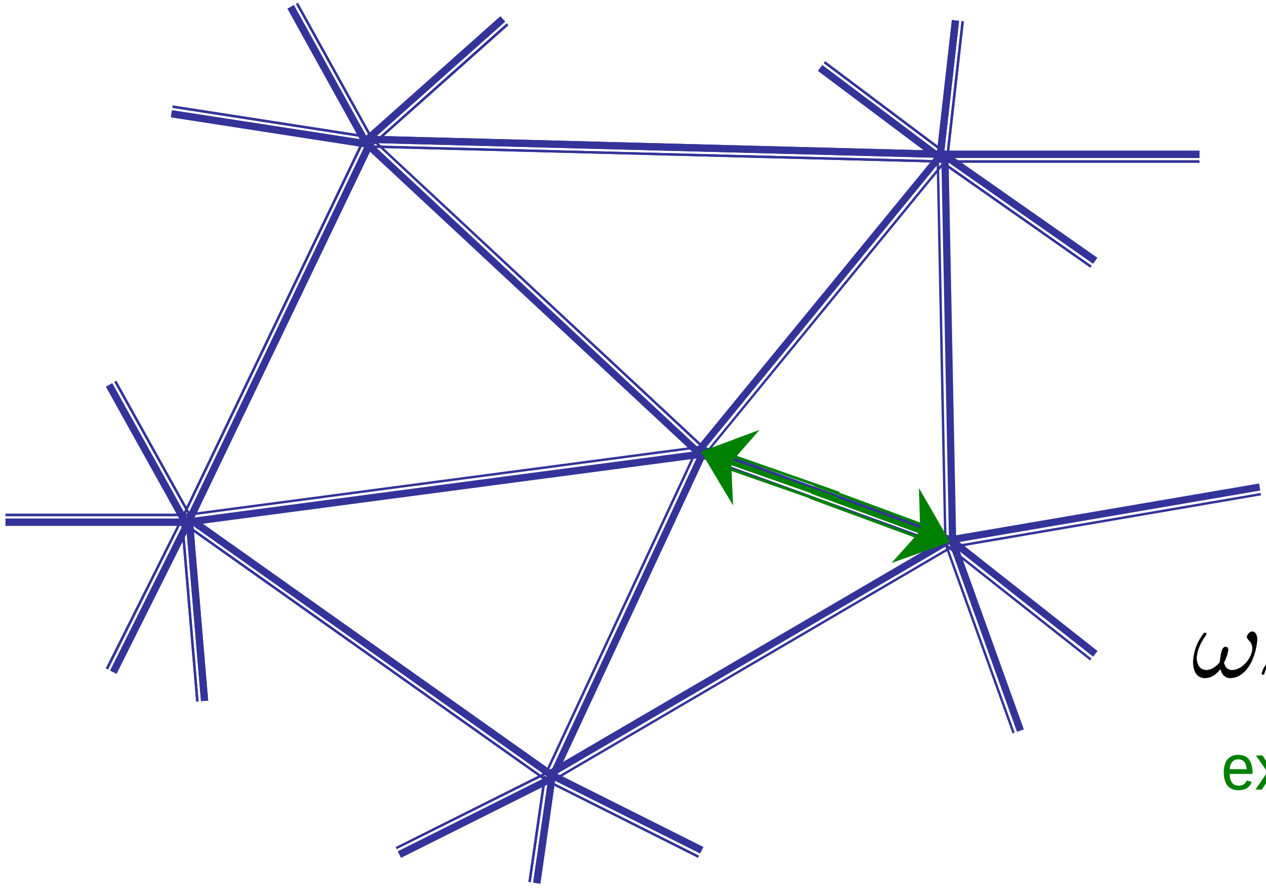


$\omega_{ij} > 0$

contracting

Proof: planar stress frameworks

1. (Sym)+(Loc) & stress frameworks in the plane



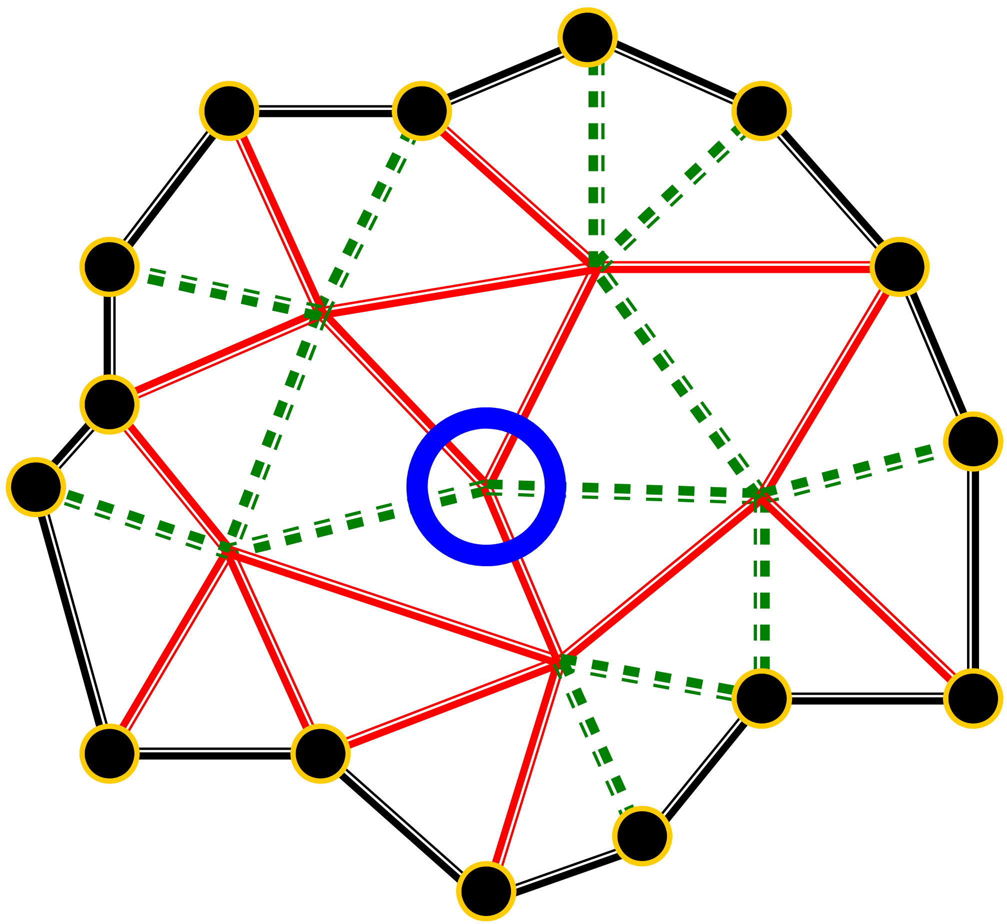
$\omega_{ij} < 0$

expanding

Proof: planar stress frameworks

2. (Sym) + (Loc) + (Lin) $\Longleftrightarrow$ inner vertices are in force balance

[e.g., use cotan weights]



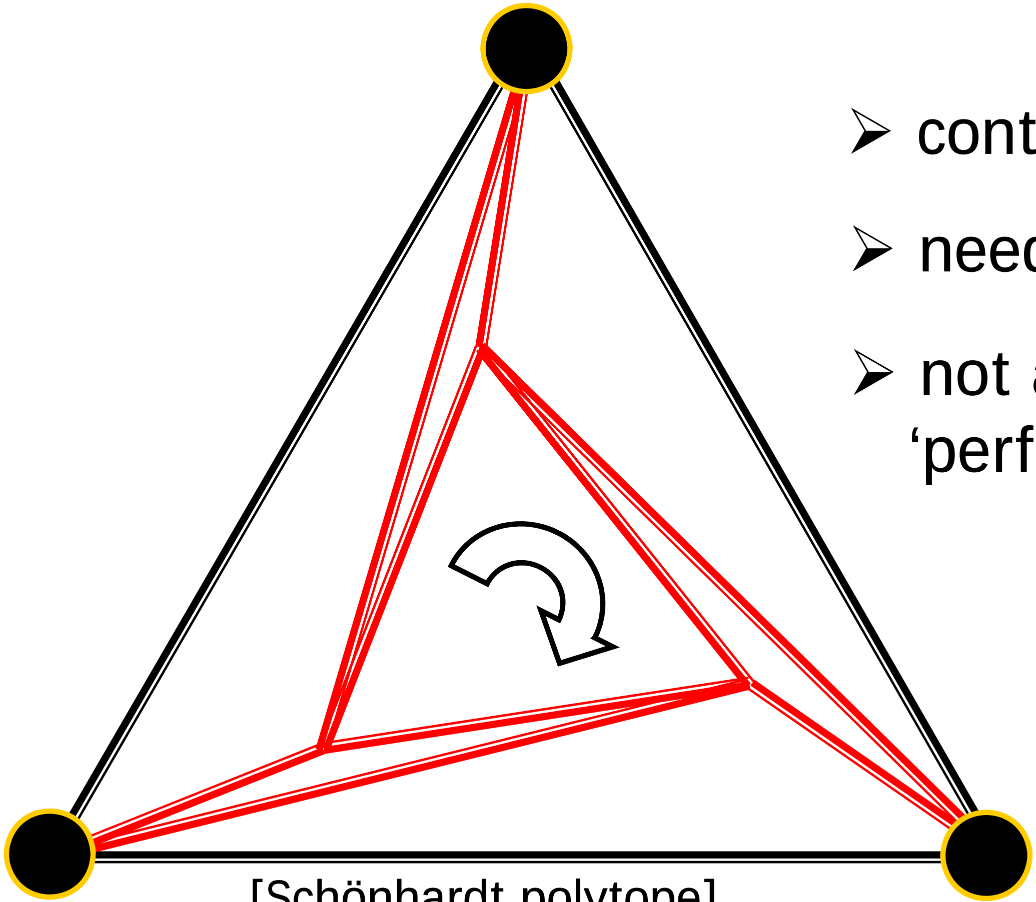
fixed boundary vertices

contracting edges

expanding edges

Proof: planar stress frameworks

3. (Sym) + (Loc) + (Lin) + (Pos)

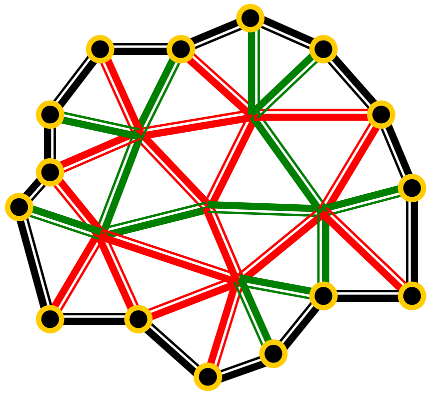


[Schönhardt polytope]

- contracting forces: net torque
- need negative weights
- not all meshes allow for 'perfect' Laplacians

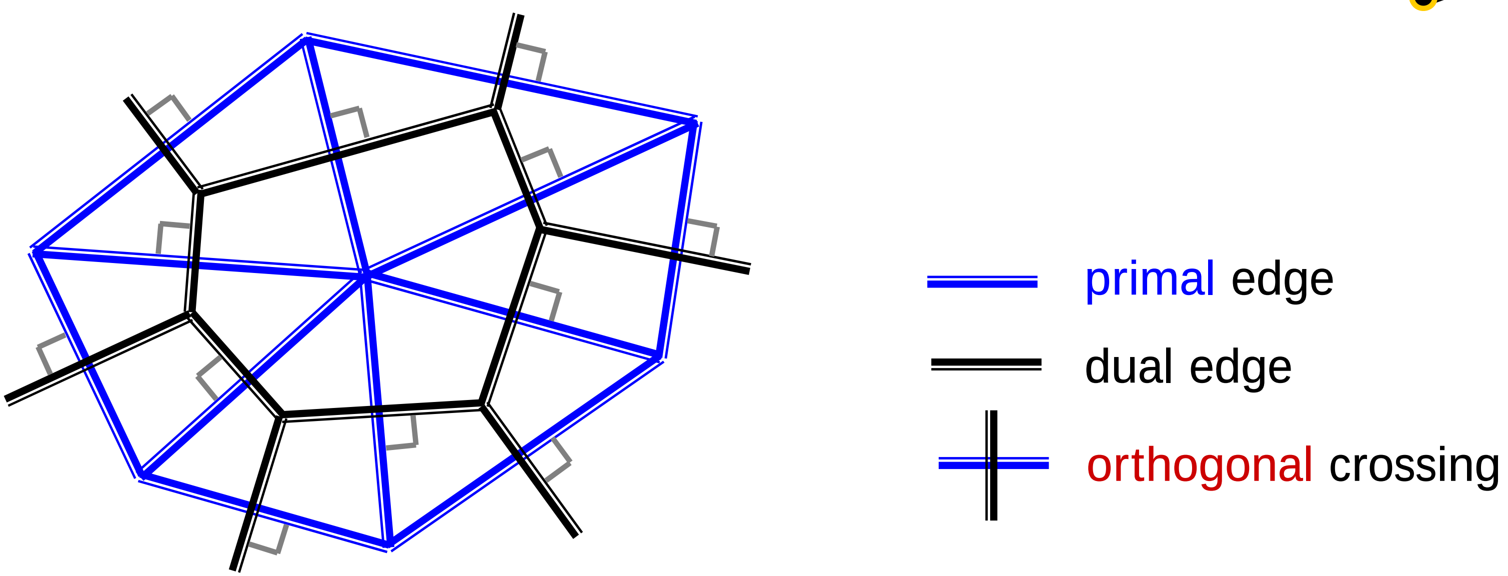
"QED"

Which meshes allow for ‘perfect’ Laplacians?



Theorem (Maxwell-Cremona 1864)

A stress framework in the plane is in force-balance iff there exists an orthogonal dual graph.



Example: Delaunay triangulation & Voronoi dual

Which meshes allow for 'perfect' Laplacians?

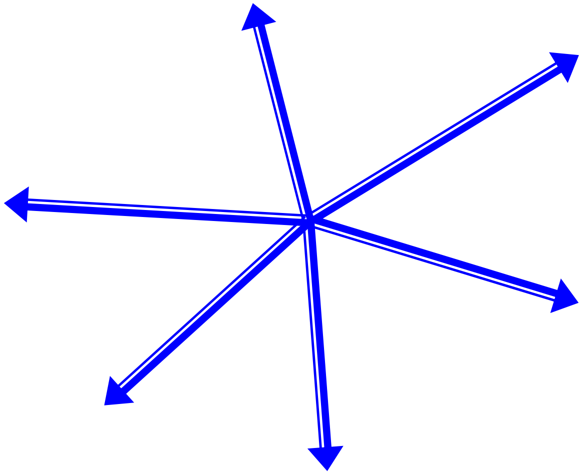
Theorem (Maxwell-Cremona 1864)

(Sym)+(Loc)+(Lin) $\Longleftrightarrow$ orthogonal duals

Proof:

1) Given (Sym)+(Loc)+(Lin), observe that

$$\sum_j \omega_{ij} \vec{e}_{ij} = \sum_j \omega_{ij} (p_j - p_i) = 0$$



Which meshes allow for ‘perfect’ Laplacians?

Theorem (Maxwell-Cremona 1864)

(Sym)+(Loc)+(Lin) $\Longleftrightarrow$ orthogonal duals

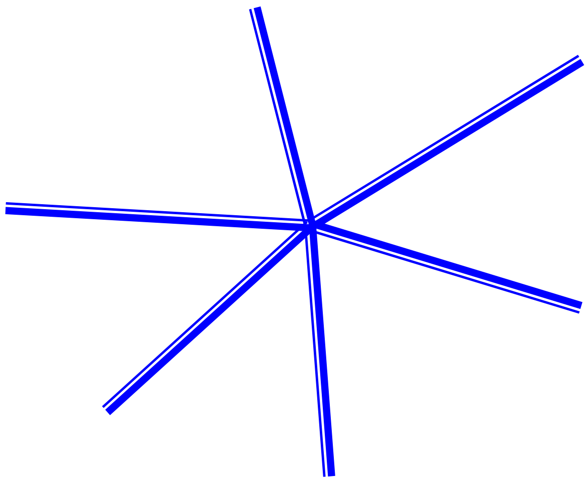
Proof:

1) Define dual edges by

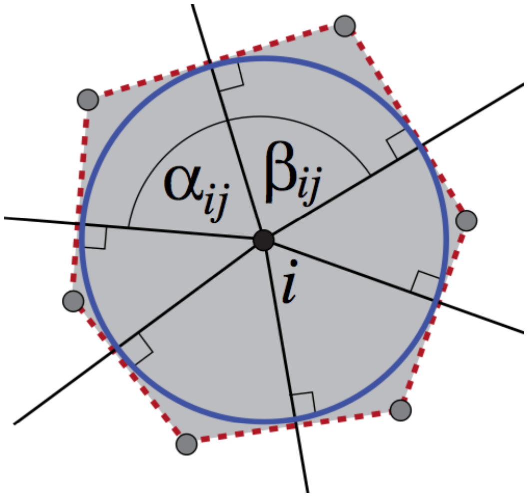
$$\star\vec{e}_{ij} = R^{90}(\omega_{ij}\vec{e}_{ij})$$

Get closed dual cycles.

$$\sum_j \star\vec{e}_{ij} = 0$$



Which meshes allow for 'perfect' Laplacians?



Theorem (Maxwell-Cremona 1864)

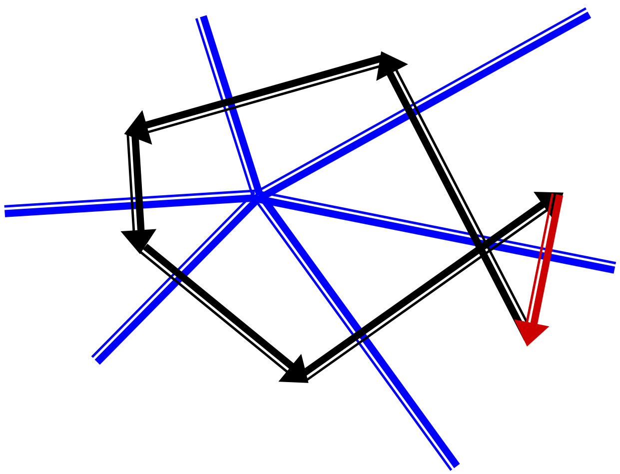
(Sym)+(Loc)+(Lin) $\Longleftrightarrow$ orthogonal duals

<u>Proof:</u>

2) Vice-versa, given orthogonal dual, define

$$\omega_{ij} = \frac{|\star e_{ij}|}{|e_{ij}|}$$

Closed dual cycles give (Lin).

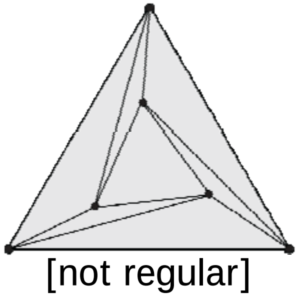


[negative dual edge]

Which meshes allow for 'perfect' Laplacians?

Theorem (Maxwell-Cremona 1864)

(Sym)+(Loc)+(Lin) $\Longleftrightarrow$ orthogonal duals



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Theorem (Aurenhammer '87)

Orthogonal duals w/ pos. weights $\Longleftrightarrow$ regular triangulations

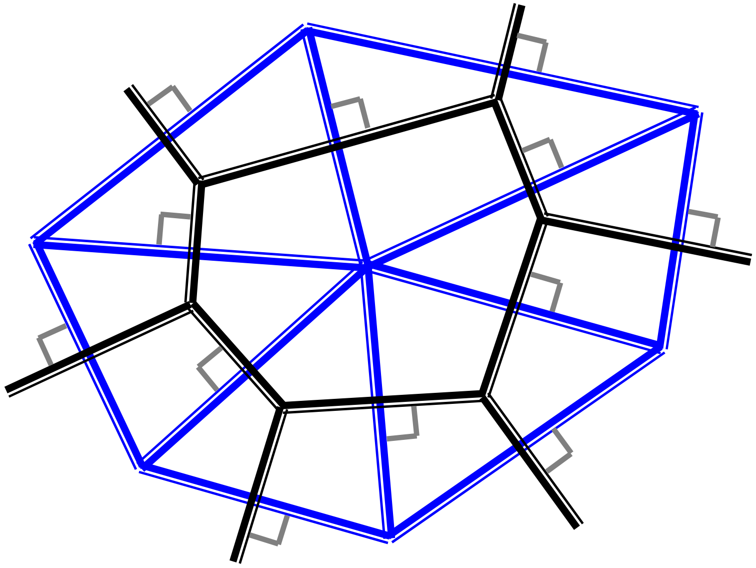
=

No-free-lunch-theorem

(Sym)+(Loc)+(Lin)+(Pos) $\Longleftrightarrow$ regular triangulations

Regular triangulations:

- Delaunay
- more generally: weighted Delaunay



Taxonomy on literature

Laplacian Zoo

- dropping (Loc): weighted Delaunay Laplacians
- dropping (Sym): barycentric coordinates
- dropping (Lin): combinatorial Laplacians
- dropping (Pos): cotan weights and generalizations

... no free lunch!