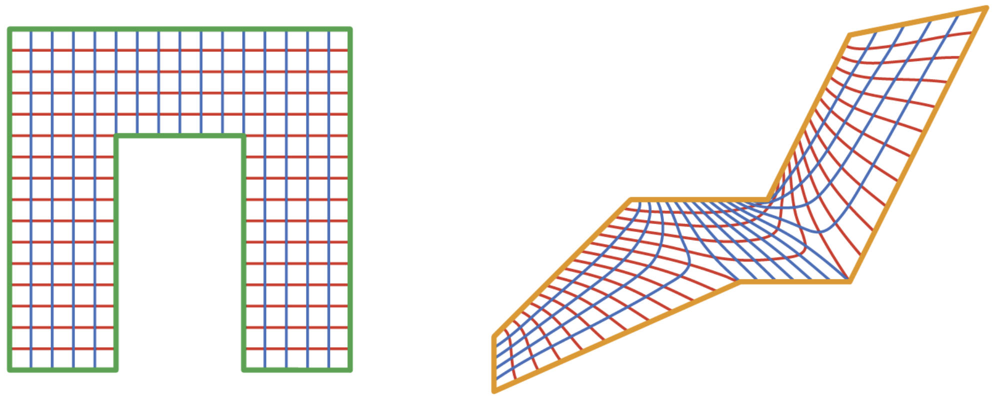
Complex Barycentric Coordinates

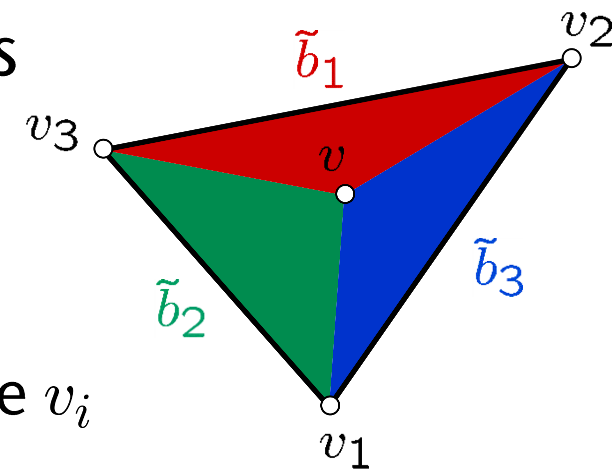


Craig Gotsman

Technion

[Möbius 1827]

- barycentric coordinates for triangles
  - ***triangle*** $= [v_1, v_2, v_3]$ in the plane
  - ***point*** v inside
  - ***area*** $\tilde{b}_i$ of triangle $[v_{i-1}, v, v_{i+1}]$ opposite v_i
  - ***barycentric coordinates*** $b_i = \dfrac{\tilde{b}_i}{\tilde{b}_1 + \tilde{b}_2 + \tilde{b}_3}$ of v w.r.t.

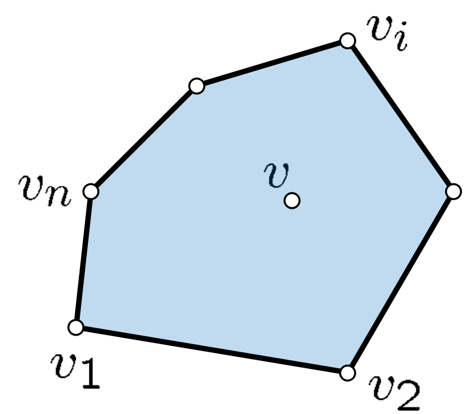


- properties
  - ***partition of unity*** $\sum_i b_i(v) = 1$
  - ***reproduction*** $\sum_i b_i(v) v_i = v$
  - ***Lagrange property*** $b_i(v_j) = \delta_{ij}$

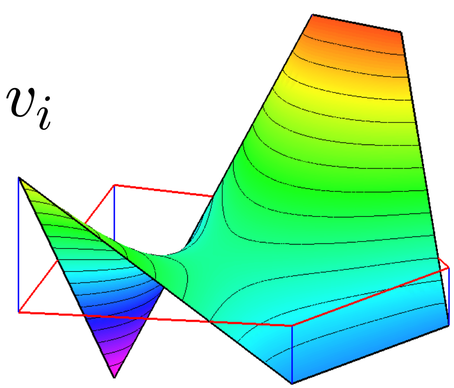
linear interpolation

$$f(v) = \sum_{i=1}^{3} b_i(v) f_i$$

- barycentric coordinates for polygons
  - *polygon* = [v_1, ..., v_n] in the plane
  - *point* v inside
  - *barycentric coordinates* b_i of v w.r.t. with
    - partition of unity $\sum_i b_i(v) = 1$
    - reproduction $\sum_i b_i(v) v_i = v$
    - Lagrange property $b_i(v_j) = \delta_{ij}$
  - *interpolation* of data f_i given at the vertices v_i



$$f(v) = \sum_{i=1}^{n} b_i(v) f_i$$

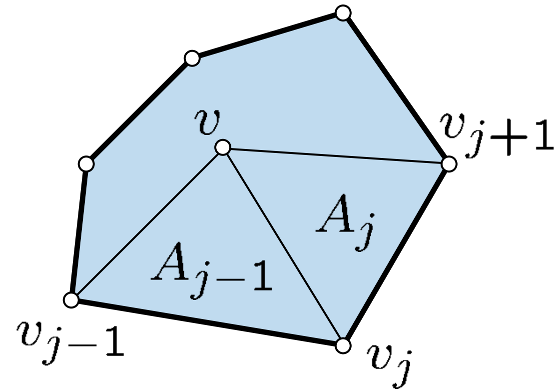


Introduction

[Floater et al. 2006]

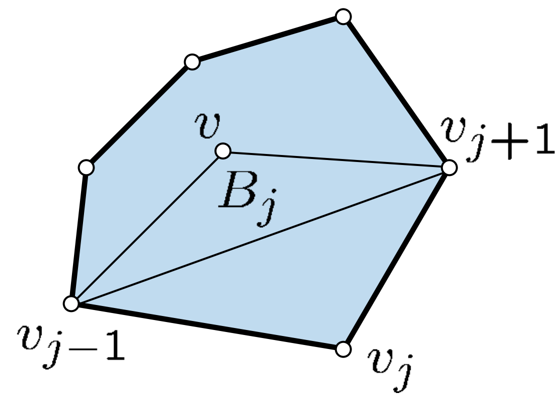
- ***normal form*** of barycentric coordinates
  - ***vertex weight*** functions $\beta_j : \to$ R
  - ***homogeneous*** coordinates

$$\tilde{b}_j = \frac{\beta_{j+1} A_{j-1} - \beta_j B_j + \beta_{j-1} A_j}{A_{j-1} A_j}$$



  - ***normalized*** coordinates

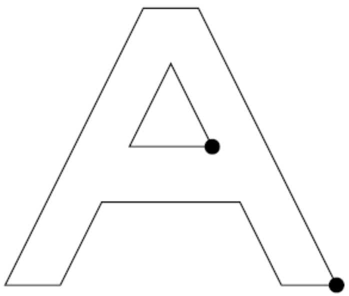
$$b_j = \frac{\tilde{b}_j}{\sum_{k=1}^{n} \tilde{b}_k}$$

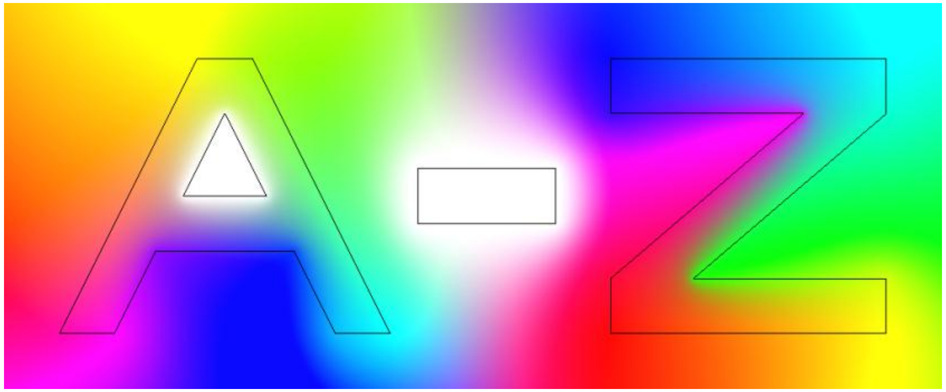


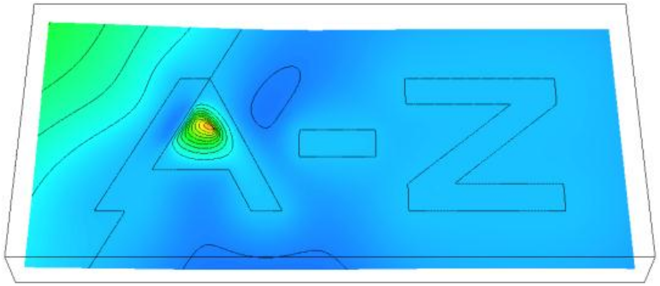
- example
  - $\beta_j = \|v_j - v\|$ gives ***mean value coordinates*** [Floater 2003]

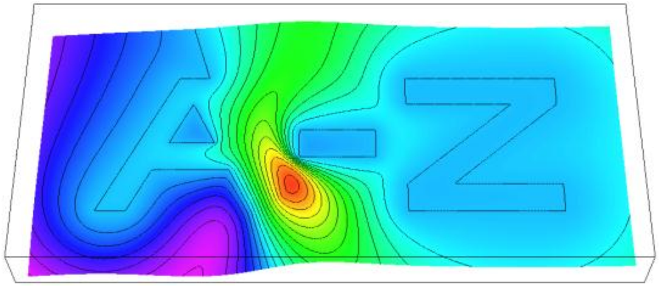
Colour interpolation

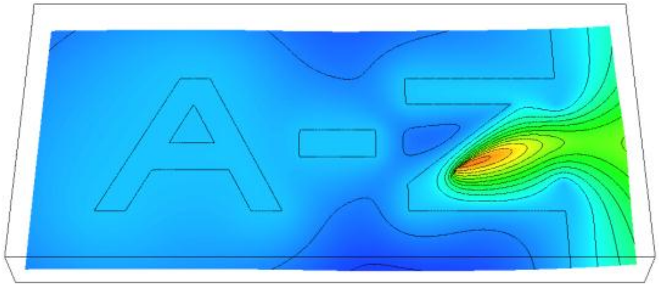
[Hormann & Floater 2006]











Smooth shading

[Hormann & Floater 2006]

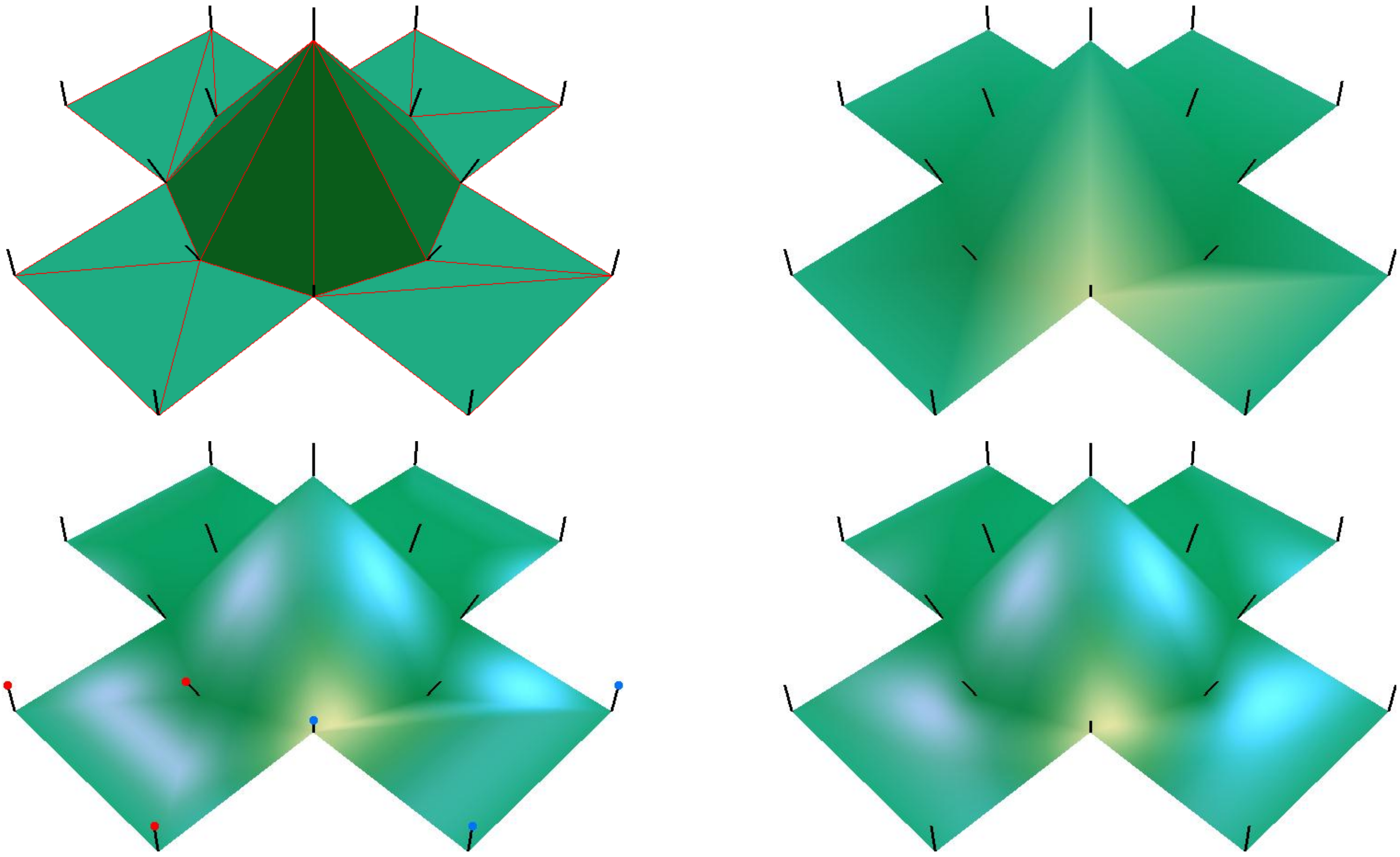
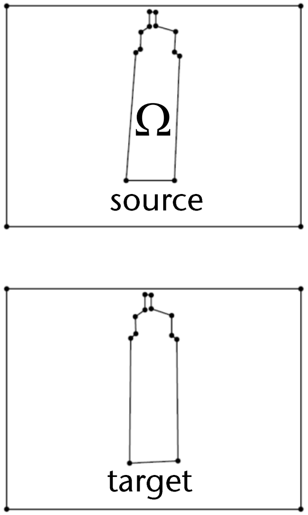


Image warping

[Hormann & Floater 2006]



original image

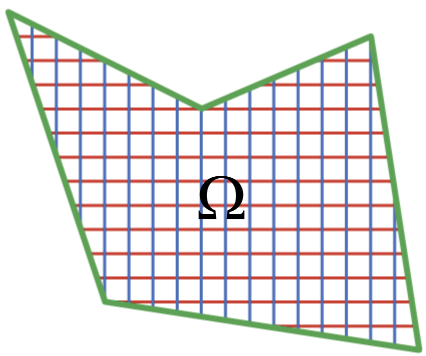
mask

warped image

$$f : \Omega \to R^2$$

Barycentric mappings

source polygon



$v = [v_1, \ldots, v_n]$

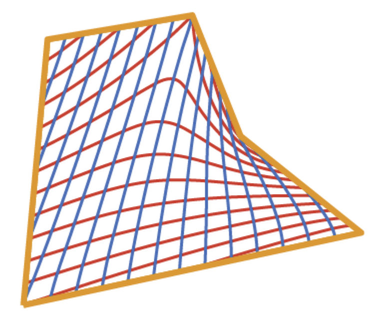
$v_j \in \mathbb{R}^2$

for $v \in \Omega$:

$$f(v) = \sum_{j=1}^{n} b_j(v)\hat{v}_j$$

reproduces affine mappings

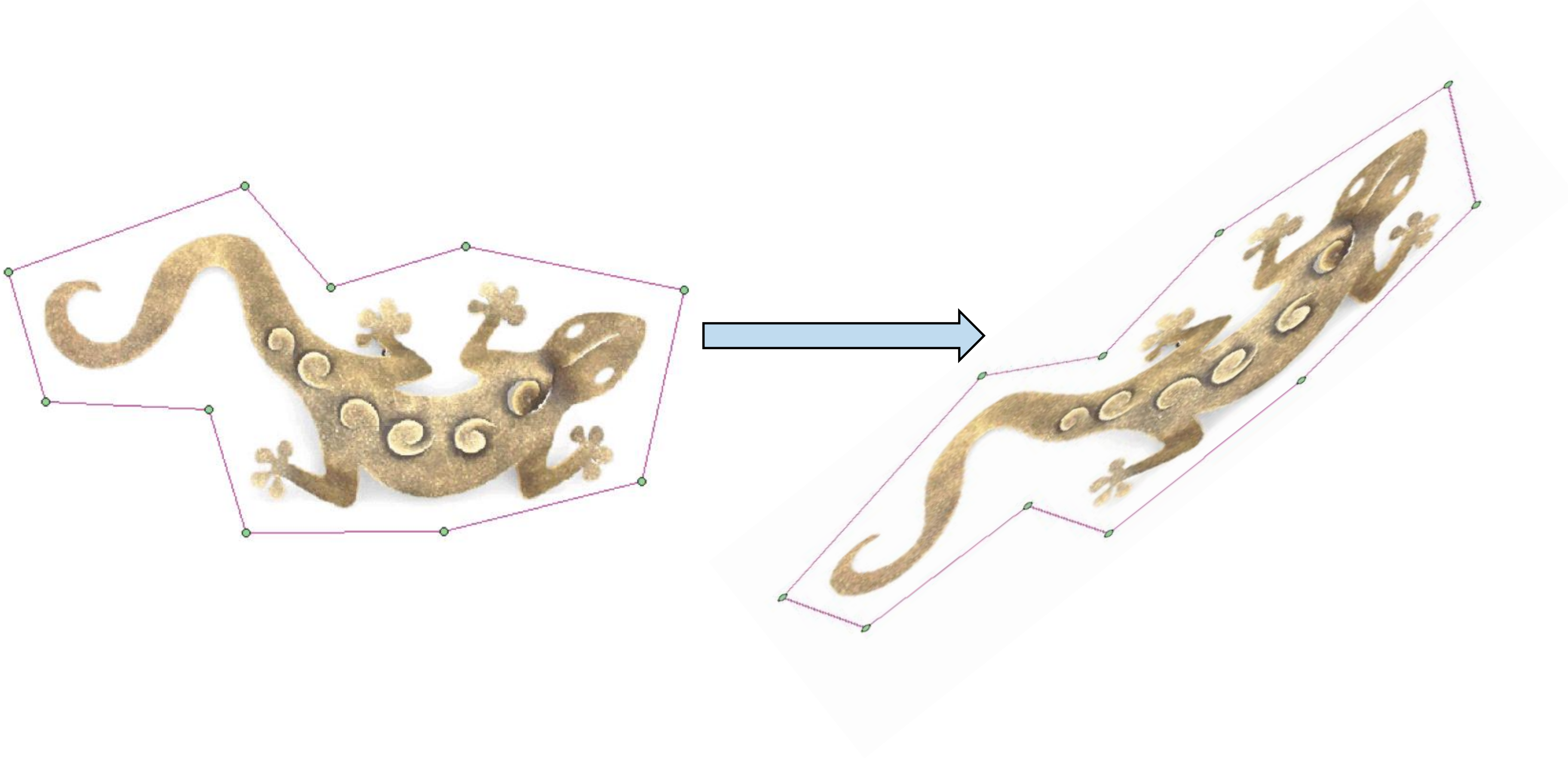
target polygon



$\hat{v} = [\hat{v}_1, \ldots, \hat{v}_n]$

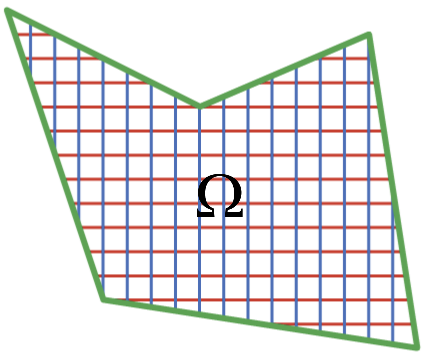
$\hat{v}_j \in \mathbb{R}^2$

Distortion !



Complex barycentric mappings

source polygon



$z = [z_1, \ldots, z_n]$

$z_j \in \mathbb{C}$

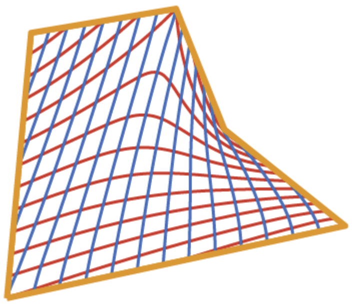
for $z \in \Omega$:

$$g(z) = \sum_{j=1}^{n} c_j(z)\hat{z}_j$$

with ***complex*** barycentric coordinates

$c_j : \Omega \to \mathbb{C}$

target polygon



$\hat{z} = [\hat{z}_1, \ldots, \hat{z}_n]$

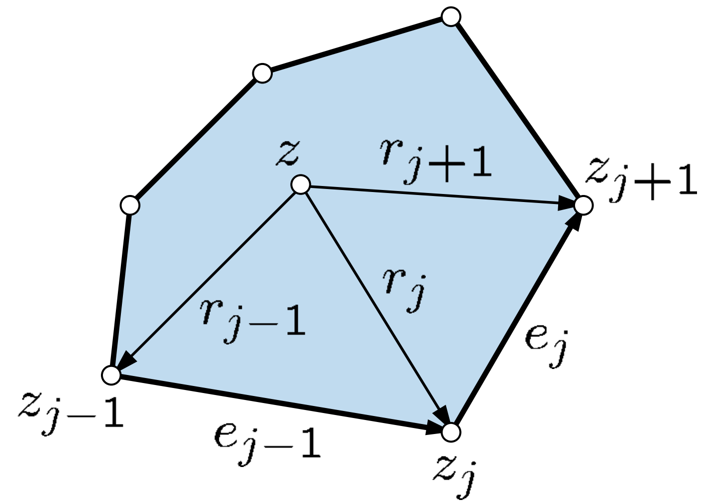
$\hat{z}_j \in \mathbb{C}$

Complex barycentric coordinates

[Weber et al. 2009]

- ***normal form*** of complex barycentric coordinates
  - ***edge weight*** functions $\gamma_j : \Omega \to \mathbb{C}$
  - ***homogeneous*** coordinates

$$\tilde{c}_j = \gamma_j \frac{r_{j+1}}{e_j} - \gamma_{j-1} \frac{r_{j-1}}{e_{j-1}}$$



  - ***normalized*** coordinates

$$c_j = \frac{\tilde{c}_j}{\sum_{k=1}^{n} \tilde{c}_k}$$

- example
  - $\gamma_j = \log(r_{j+1}/r_j)$ gives ***Cauchy–Green coordinates***

[Lipman et al. 2008]

Cauchy-Green coordinates

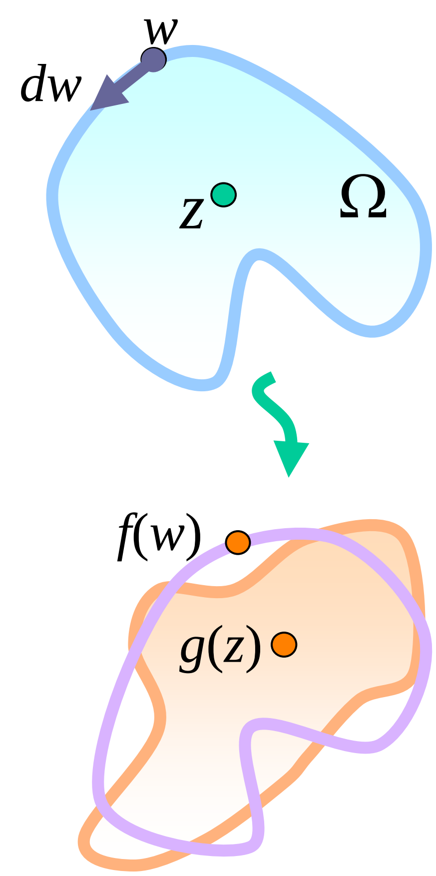
kernel: $$k(w,z) = \frac{1}{2\pi i}\frac{1}{w-z}$$

The *Cauchy Transform*:

$$g(z) = \frac{1}{2\pi i}\oint_{\partial\Omega} \frac{1}{w-z} f(w)dw$$

g is holomorphic.

Reproduces f if f is holomorphic.



$$g(z) = \oint_{\partial\Omega} k(w,z) f(w)dw$$

Discrete Cauchy-Green coordinates

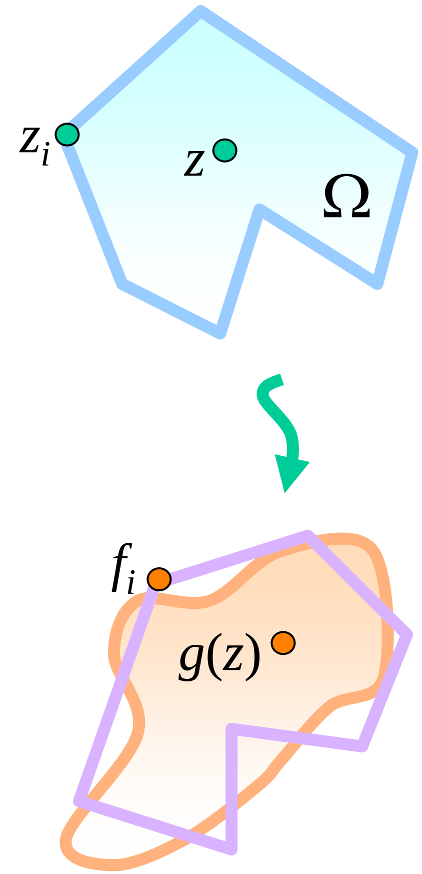
Smooth case:

$$g(z) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{1}{w-z} f(w)dw$$

Ω is polygon → integrate over edges:

$$g(z) = \frac{1}{2\pi i} \sum_{i=1}^{n} \oint_{e_i} \frac{f(w)}{w-z} dw$$

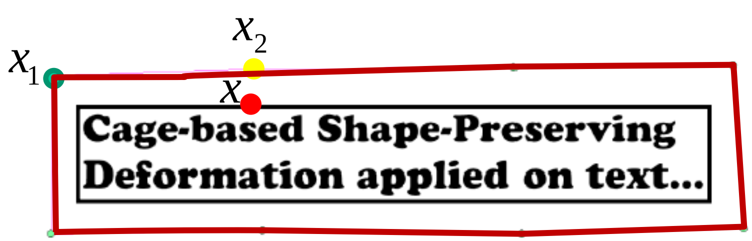
Ω is polygon → $f(w)$ is linear



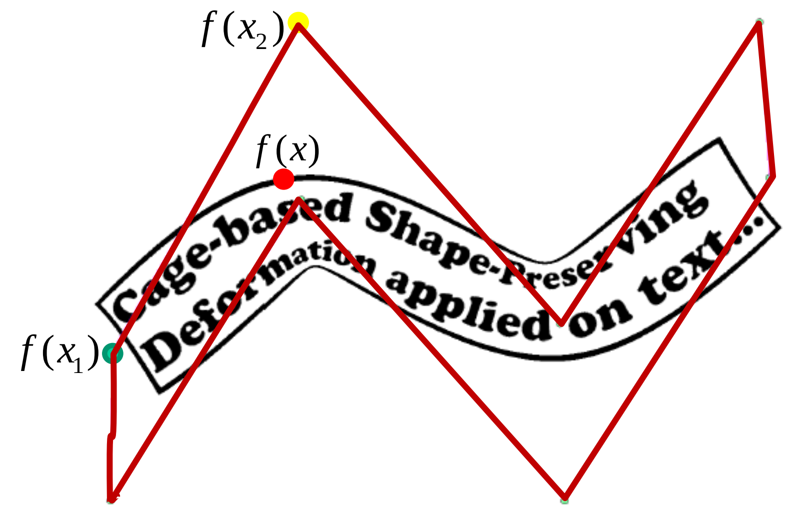
$$g(z) = \sum_{i=1}^{n} w_i(z) f_i$$

Cauchy-Green coordinates

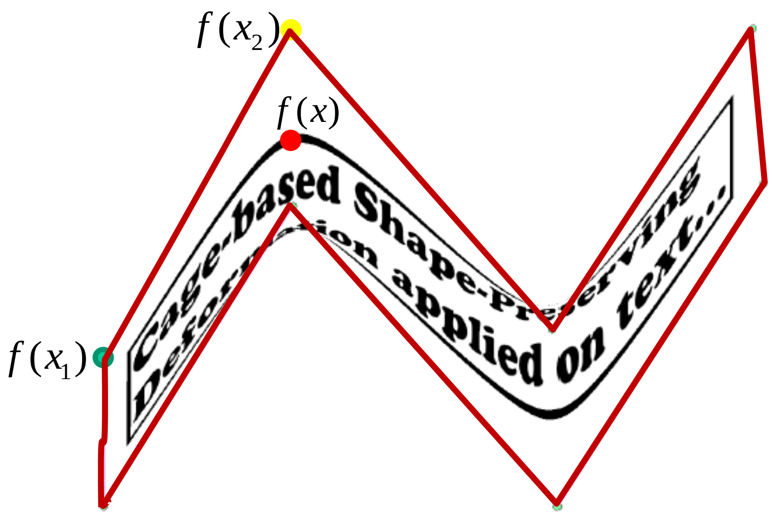
Does not reproduce affine transformations.
Does not interpolate.
Generates a conformal map of the plane.



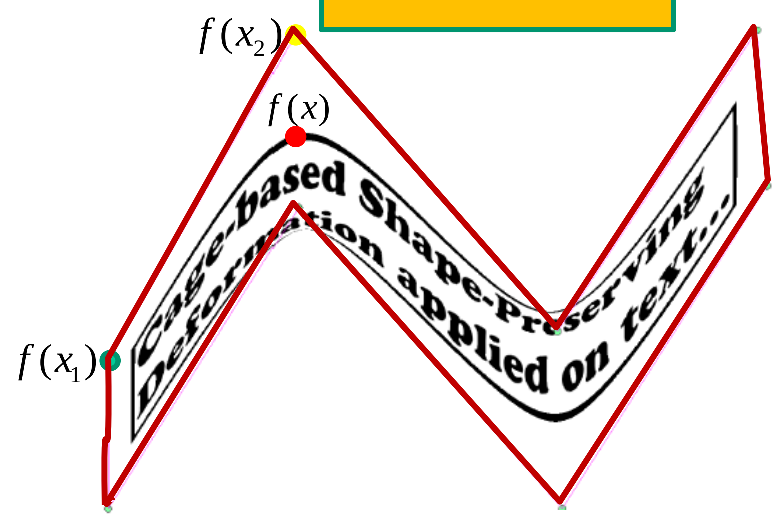
source



Cauchy-Green



mean-value

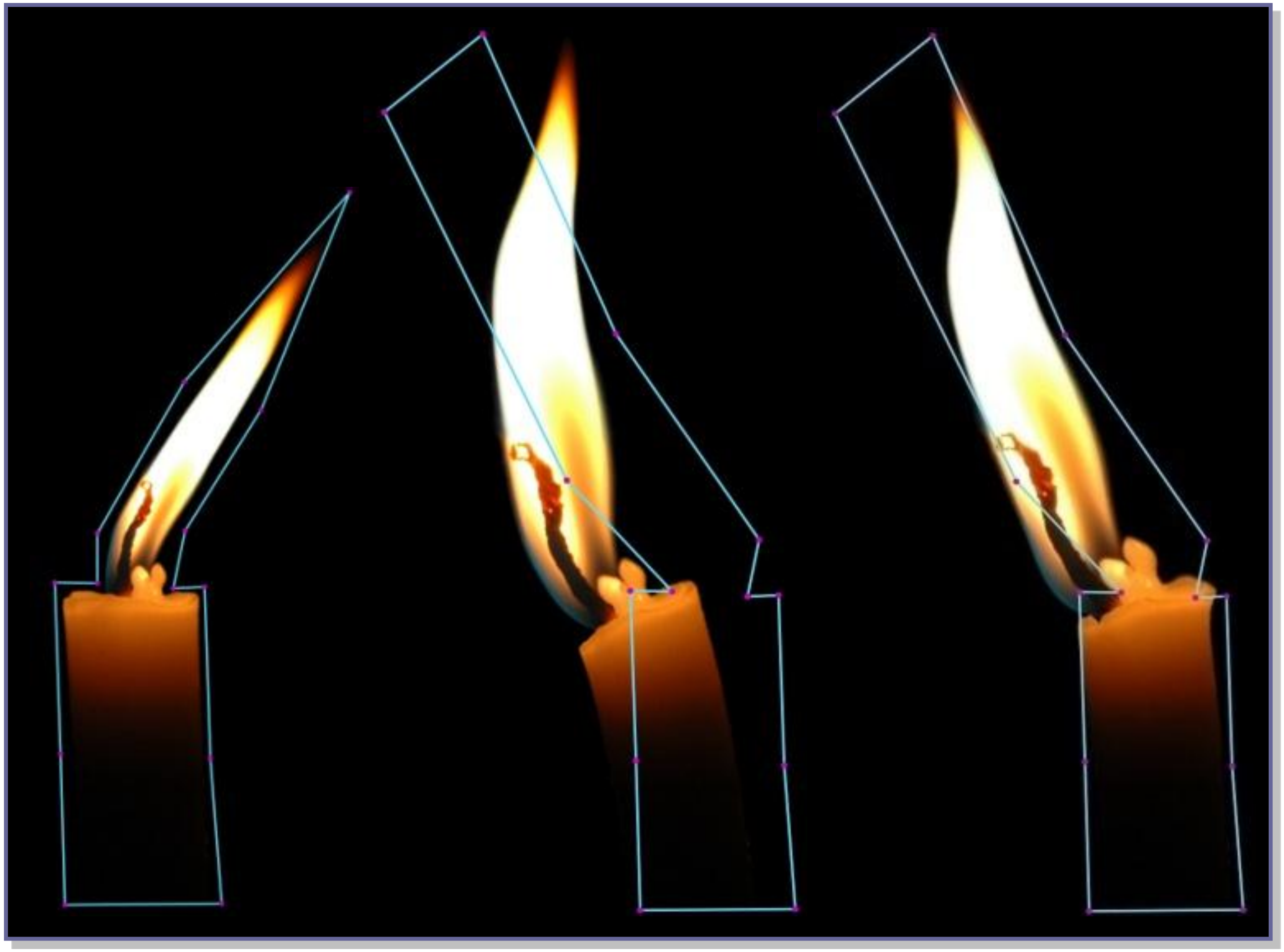


harmonic

Cauchy-Green coordinates

Szegö coordinates

[Weber et al. 2009]

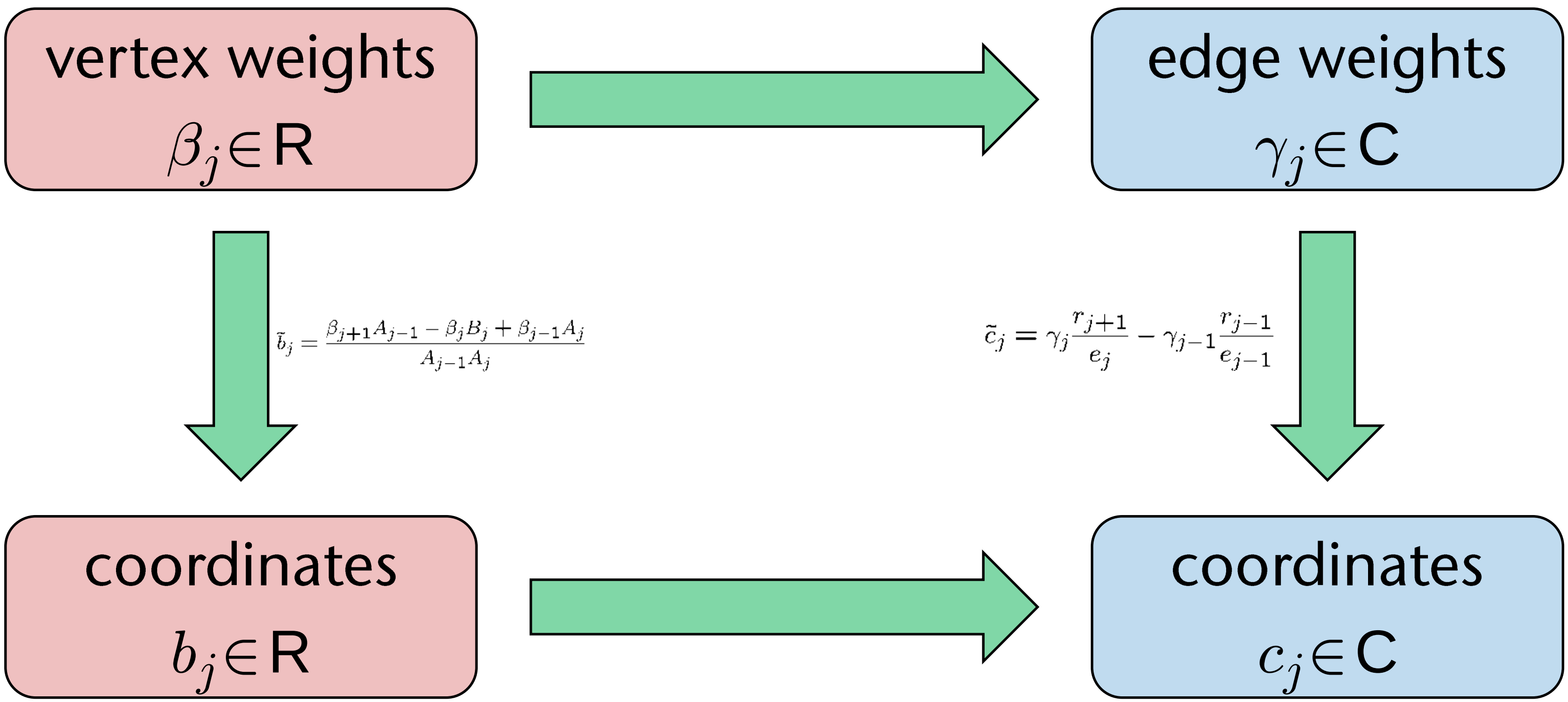


Source **Cauchy-Green** **Szegö**

Converting real-valued coordinates

[Weber et al. 2011]

$$\gamma_j = \frac{e_j}{\mathrm{Im}(\overline{r}_j r_{j+1})}\left(\frac{\beta_{j+1}}{r_{j+1}} - \frac{\beta_j}{r_j}\right)$$



$$\mathrm{Re}(c_j) = b_j,\ \mathrm{Im}(c_j) = 0$$

Some familiar cases

$$\gamma_j = \frac{e_j}{\mathrm{Im}(\overline{r}_j r_{j+1})} \left(\frac{\beta_{j+1}}{r_{j+1}} - \frac{\beta_j}{r_j} \right)$$

$p = 0$: Wachspress

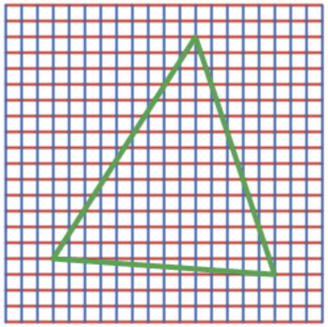
$\beta_j = |r_j|^p$ $\quad$ $p = 1$: mean value $\quad$ $k(w,z) = \dfrac{1}{(w-z)|w-z|}$

$p = 2$: discrete harmonic (cotangent)

Cauchy-Green: $\gamma_j = \log\left(\dfrac{r_{j+1}}{r_j}\right)$ $\quad$ $k(w,z) = \dfrac{1}{2\pi i}\dfrac{1}{w-z}$

Why bother?

- complex barycentric mappings are ***more general***
  - include real-valued mappings as a special case
- complex barycentric mappings are ***more powerful***
  - coordinates are not treated separately



source

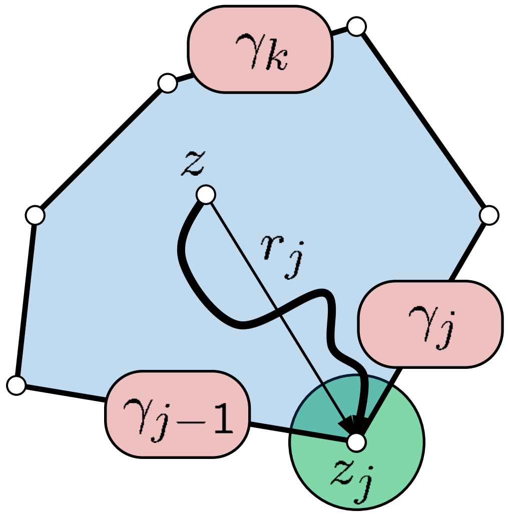
any real coords

Cauchy–Green

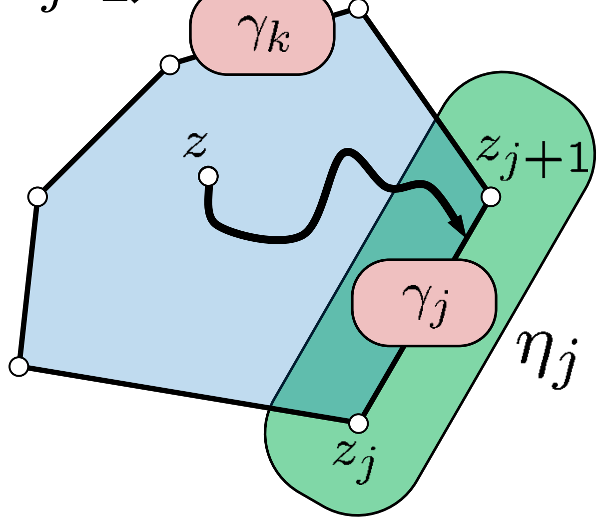
new

$$\gamma_j = \frac{r_{j+1}}{r_j} - \frac{r_j}{r_{j+1}}$$

- interpolation of ***target vertex*** $\hat{z}_j$
  - $\lim_{z\to z_j} |\gamma_{j-1}(z) + \gamma_j(z)| = \infty$
  - $\lim_{z\to z_j} |\gamma_{j-1}(z) r_j(z)| < \infty$
  - $\lim_{z\to z_j} |\gamma_j(z) r_j(z)| < \infty$
  - $\lim_{z\to z_j} |\gamma_k(z)| < \infty$



- interpolation of ***target edge*** $\hat{\eta}_j = (\hat{z}_j, \hat{z}_{j+1})$
  - $\lim_{z\to \eta_j} |\gamma_j(z)| = \infty$
  - $\lim_{z\to \eta_j} |\gamma_k(z)| < \infty$



- use conditions to design ***new*** coordinates & mappings

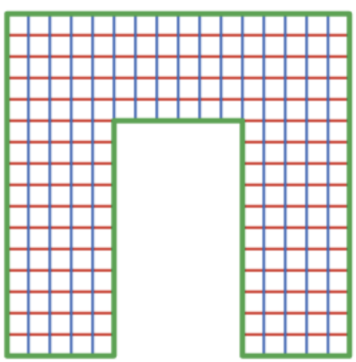
[Weber et al. 2011]

- ***M***ade-to-order ***A***ngle ***G***uided ***I***nterpolating ***C***oord's

edge scaling *edge interpolation*

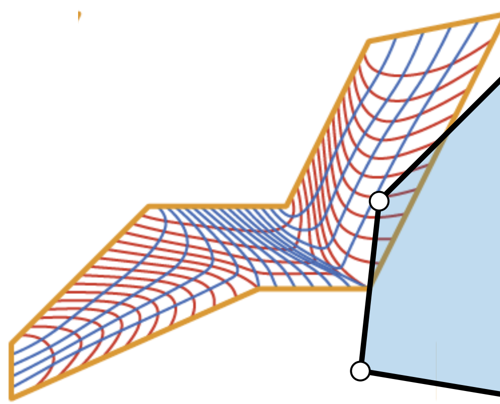
$$\gamma_j = \left|\frac{e_j}{r_j r_{j+1}}\right| \frac{1}{\pi - \underbrace{\mathrm{Im}\big(\log(r_{j+1}/r_j)\big)}_{=\,\theta_j}}$$

vertex interpolation

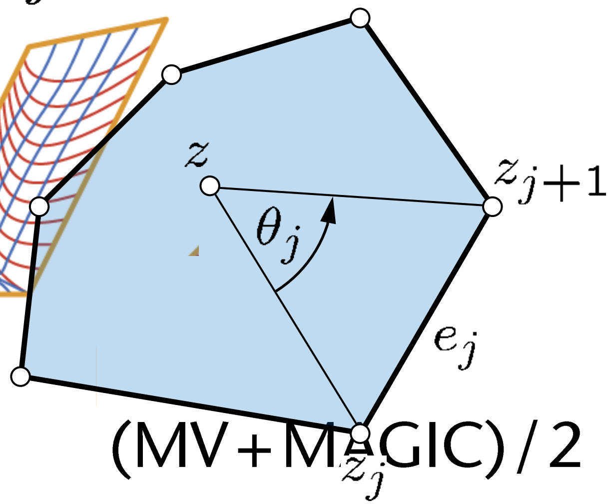


source

mean value (MV)



MAGIC



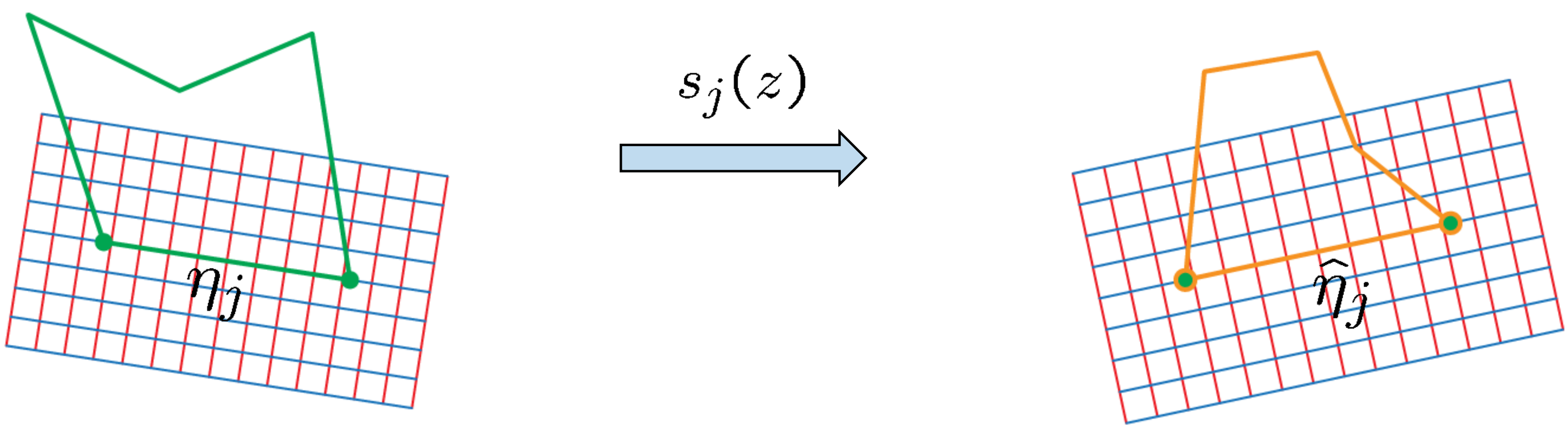
(MV+MAGIC) / 2

A new perspective

[Weber et al. 2011]

$$g(z) = \sum_{j=1}^{n} c_j(z)\hat{z}_j = \frac{\sum_j \tilde{c}_j(z)\hat{z}_j}{\sum_j \tilde{c}_j(z)} = \frac{\sum_j \gamma_j(z)s_j(z)}{\sum_j \gamma_j(z)}$$

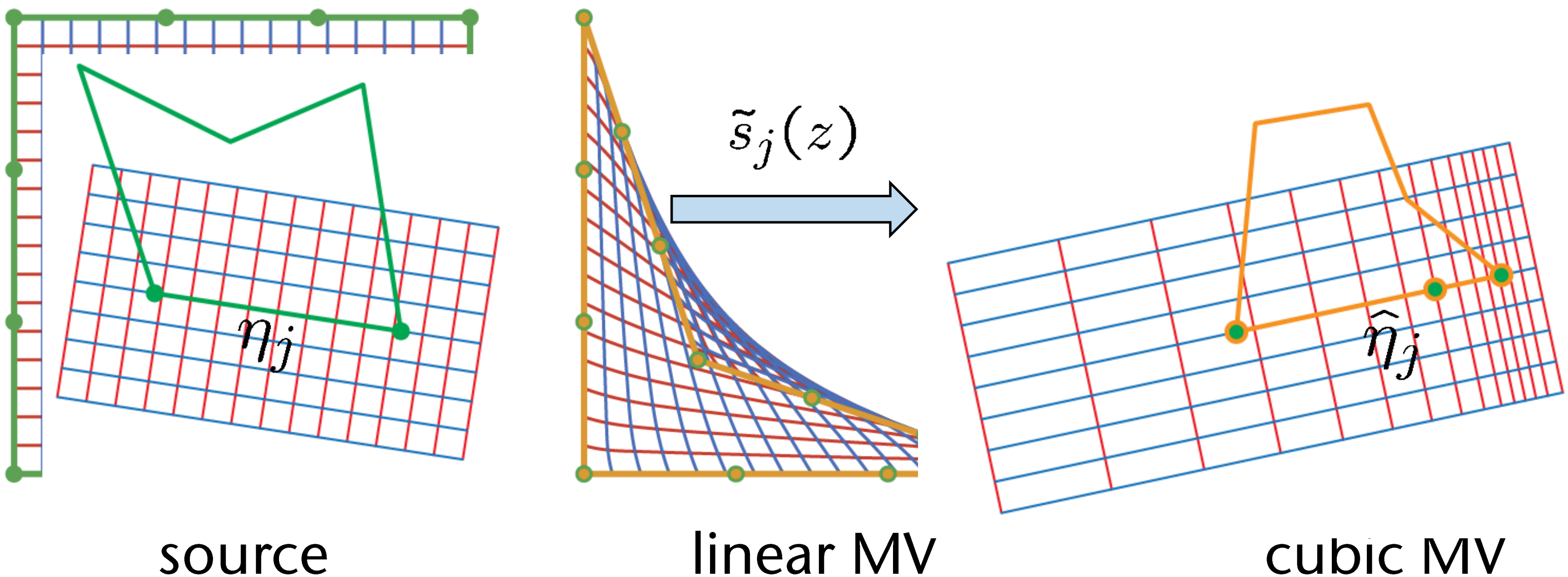
- blend of ***linear edge-to-edge*** similarity ***transforms***



Beyond barycentric mappings

$$\tilde{g}(z) = \frac{\sum_j \gamma_j(z)\tilde{s}_j(z)}{\sum_j \gamma_j(z)}$$

- blend of ***polynomial edge-to-edge transforms***



Conclusions

- work in the ***complex setting***
  - more general, more powerful
  - includes real setting as special case
- work in the ***normal form*** (edge weights γ_j)
  - simple criteria for guaranteeing interpolation
  - easy to design new coordinates/mappings
- work with ***edge-to-edge transforms*** (s_j)
  - greater flexibility by changing the speed along edges
  - potential to achieve global bijectivity

Collaborators

- Mirela Ben-Chen
- Kai Hormann
- Ofir Weber
- Roi Poranne