Error Estimates for Generalized Barycentric Coordinate Finite Element Methods

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What are *a priori* FEM error estimates?

Poisson's equation in 2D: Given a domain $\mathcal{D} \subset \mathbb{R}^2$ and $f : \mathcal{D} \to \mathbb{R}$, find u such that

strong form	$-\Delta u$	$=$	f	$u \in H^2(\mathcal{D})$
weak form	$\int_{\mathcal{D}} \nabla u \cdot \nabla \phi$	$=$	$\int_{\mathcal{D}} f\,\phi$	$\forall \phi \in H^1(\mathcal{D})$
discrete form	$\int_{\mathcal{D}} \nabla u_h \cdot \nabla \phi_h$	$=$	$\int_{\mathcal{D}} f\,\phi_h$	$\forall \phi_h \in V_h \subset H^1(\mathcal{D})$

Typical **Lagrange method:**

$\to$ Mesh $\mathcal{D}$ by polygons $\{\Omega\}$ with vertices $\{\mathbf{v}_i\}$.

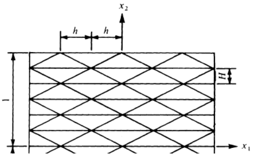
$\to$ Let $\boxed{\lambda_i := \text{barycentric coordinate}}$ defined piecewise on those Ω incident to $\mathbf{v}_i$.

$\to$ Let $\boxed{u_h := \mathcal{I}_\ell u := \sum_i u(\mathbf{v}_i)\lambda_i}$, reduce discrete form to a linear system & solve.

In this case, the associated **optimal *a priori* error estimate** is linear and has the form

$$\underbrace{\left|\left| u - \sum_{i=1}^{n} u(\mathbf{v}_i)\lambda_i \right|\right|_{H^1(\Omega)}}_{\text{approximation error}} \leq \underbrace{C\,\text{diam}(\Omega)\,|u|_{H^2(\Omega)}}_{\text{optimal error bound}}, \quad \forall u \in H^2(\Omega).$$

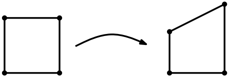
Error estimates depend on mesh geometry

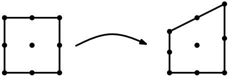


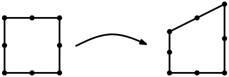
→ Triangular meshes require a maximum angle condition for the estimate to hold.

BABUŠKA, AZIZ *On the angle condition in the finite element method*, SIAM J. Num. An., 1976.

→ *A priori* error estimates are based on **affine** maps from a reference element.

Lagrange  $||u - \mathcal{I}_\ell u||_{H^1(\Omega)} \leq c\, h\, |u|_{H^2(\Omega)}, \qquad \forall u \in H^2(\Omega)$

quadratic  $||u - \mathcal{I}_q u||_{H^1(\Omega)} \leq c\, h^2\, |u|_{H^3(\Omega)}, \qquad \forall u \in H^3(\Omega)$

serendipity  $||u - \mathcal{I}_s u||_{H^1(\Omega)} \leq c\, h\, |u|_{H^3(\Omega)}, \qquad \forall u \in H^3(\Omega)$

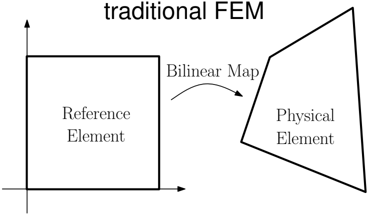
→ The non-affinely mapped serendipity element converges at a linear rate.

ARNOLD, BOFFI, FALK *Approximation by Quadrilateral Finite Elements*, Math. Comp., 2002

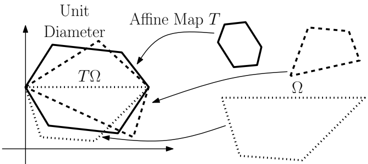
The generalized barycentric coordinate approach

Use **G**eneralized **B**arycentric **C**oordinates **(GBCs)** to create polygonal finite elements.

traditional FEM

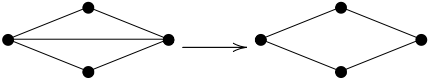


family of **GBC** reference elements

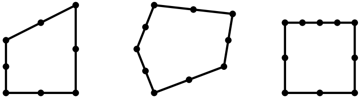


Advantages of the approach

- Builds on rich theory of **GBCs** (as exhibited at this workshop!)
- Some **GBC** elements are not sensitive to large angles, allowing remeshing:



- Serendipity elements with quadratic *a priori* error estimates can be constructed:



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4. Future Directions

Definition

Let Ω be a convex polygon in $\mathbb{R}^2$ with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Functions $\lambda_i : \Omega \to \mathbb{R}$, $i = 1, \ldots, n$ are called **barycentric coordinates** on Ω if they satisfy two properties:

1. **Non-negative**: $\lambda_i \geq 0$ on Ω.
2. **Linear Completeness**: For any linear function $L : \Omega \to \mathbb{R}$, $L = \sum_{i=1}^{n} L(\mathbf{v}_i)\lambda_i$.

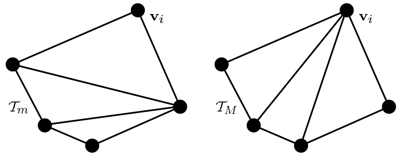
Any set of barycentric coordinates under this definition also satisfies:

3. **Partition of unity:** $\sum_{i=1}^{n} \lambda_i \equiv 1$.
4. **Linear precision:** $\sum_{i=1}^{n} \mathbf{v}_i \lambda_i(\mathbf{x}) = \mathbf{x}$.
5. **Interpolation:** $\lambda_i(\mathbf{v}_j) = \delta_{ij}$.

Theorem [Warren, 2003]

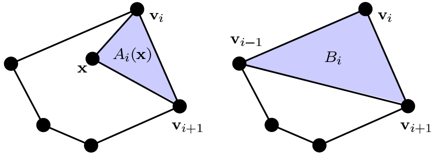
If the λ_i are rational functions of degree $n-2$, then they are unique.

Many generalizations to choose from . . .

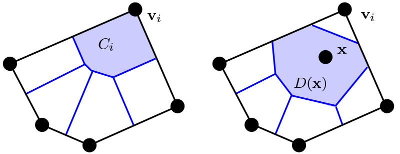


- Triangulation
  $\Rightarrow$ FLOATER, HORMANN, KÓS, *A general construction of barycentric coordinates over convex polygons*, 2006

$$0 \leq \lambda_i^{\mathcal{T}_m}(\mathbf{x}) \leq \lambda_i(\mathbf{x}) \leq \lambda_i^{\mathcal{T}_M}(\mathbf{x}) \leq 1$$

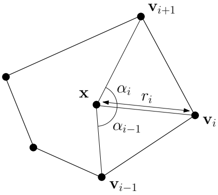


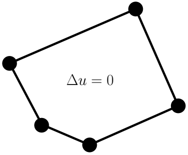
- Wachspress
  $\Rightarrow$ WACHSPRESS, *A Rational Finite Element Basis*, 1975.



- Sibson / Laplace
  $\Rightarrow$ SIBSON, *A vector identity for the Dirichlet tessellation*, 1980.
  $\Rightarrow$ HIYOSHI, SUGIHARA, *Voronoi-based interpolation with higher continuity*, 2000.

Many generalizations to choose from . . .





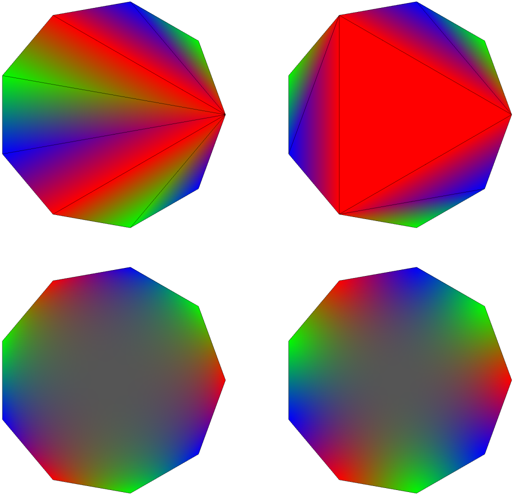
- Mean value
  ⇒ FLOATER, *Mean value coordinates*, 2003.
  ⇒ FLOATER, KÓS, REIMERS, *Mean value coordinates in 3D*, 2005.

- Harmonic
  ⇒ WARREN, *Barycentric coordinates for convex polytopes*, 1996.
  ⇒ WARREN, SCHAEFER, HIRANI, DESBRUN, *Barycentric coordinates for convex sets*, 2007.
  ⇒ CHRISTIANSEN, *A construction of spaces of compatible differential forms on cellular complexes*, 2008.

Many more papers could be cited (maximum entropy coordinates, moving least squares coordinates, etc...)

Comparison via ‘eyeball’ norm

Triangulated | Triangulated



Wachspress | Mean Value

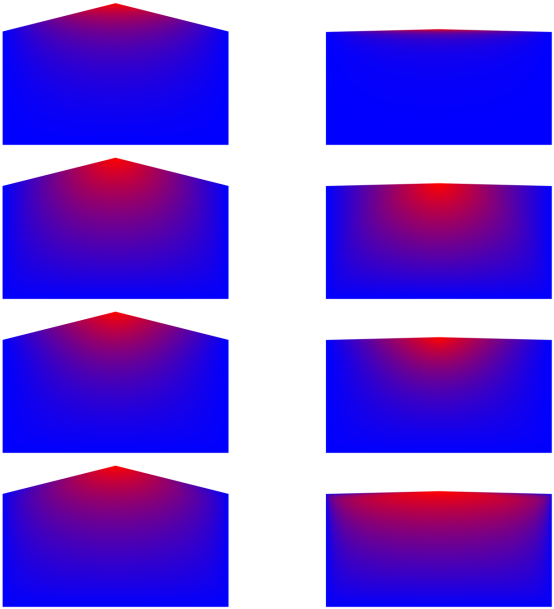
Comparison via ‘eyeball’ norm

Wachspress

Sibson

Mean Value

Discrete Harmonic



Outline

Optimal Convergence Estimates on Polygons

Let Ω be a convex polygon with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

For linear elements, an **optimal convergence estimate** has the form

$$\underbrace{\left|\left| u - \sum_{i=1}^{n} u(\mathbf{v}_i)\lambda_i \right|\right|_{H^1(\Omega)}}_{\text{approximation error}} \leq \underbrace{C\,\text{diam}(\Omega)\,|u|_{H^2(\Omega)}}_{\text{optimal error bound}}, \quad \forall u \in H^2(\Omega). \tag{1}$$

The **Bramble-Hilbert lemma** in this context says that any $u \in H^2(\Omega)$ is close to a first order polynomial in H^1 norm.

VERFÜRTH, *A note on polynomial approximation in Sobolev spaces*, Math. Mod. Num. An., 2008.
DEKEL, LEVIATAN, *The Bramble-Hilbert lemma for convex domains*, SIAM J. Math. An., 2004.

For (1), it suffices to prove an **H^1-interpolant estimate** over domains of diameter one:

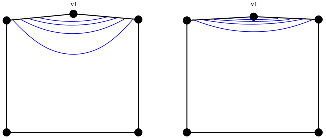
$$\left|\left| \sum_{i=1}^{n} u(\mathbf{v}_i)\lambda_i \right|\right|_{H^1(\Omega)} \leq C_I\,||u||_{H^2(\Omega)}, \quad \forall u \in H^1(\Omega). \tag{2}$$

For (2), it suffices to **bound the gradients** of the $\{\lambda_i\}$, i.e. prove $\exists\ C_\lambda \in \mathbb{R}$ such that

$$||\nabla \lambda_i||_{L^2(\Omega)} \leq C_\lambda. \tag{3}$$

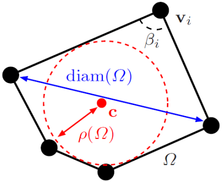
Geometric Hypotheses for Convergence Estimates

To bound the gradients of the coordinates, we need control of the element geometry.



Let $\rho(\Omega)$ denote the radius of the largest inscribed circle. The **aspect ratio** γ is defined by

$$\gamma = \frac{\mathrm{diam}(\Omega)}{\rho(\Omega)} \in (2, \infty)$$



Three possible geometric conditions on a polygonal mesh:

G1.	BOUNDED ASPECT RATIO:	$\exists\ \gamma^* < \infty$	such that	$\gamma < \gamma^*$
G2.	MINIMUM EDGE LENGTH:	$\exists\ d_* > 0$	such that	$\lvert \mathbf{v}_i - \mathbf{v}_{i-1} \rvert > d_*$
G3.	MAXIMUM INTERIOR ANGLE:	$\exists\ \beta^* < \pi$	such that	$\beta_i < \beta^*$

A key geometric proposition

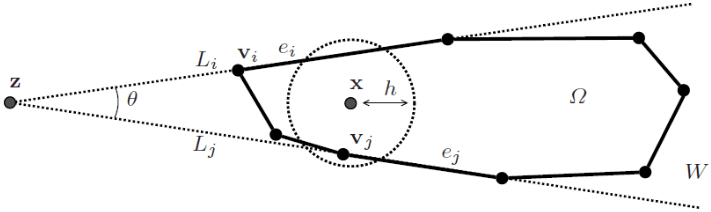
Proposition

Suppose **G1** (max aspect ratio γ^*) and **G2** (min edge length d_*) hold. Define

$$h_* := \frac{d_*}{2\gamma^*(1+d_*)} > 0$$

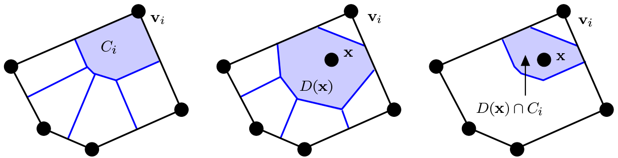
Then for all $\mathbf{x} \in \Omega$, $B(\mathbf{x}, h_*)$ does not intersect any two non-adjacent edges of Ω.

Proof: Suppose $B(\mathbf{x}, h)$ intersects two non-adjacent edges for some $h > 0$. The wedge W formed by these edges can be used to show that either **G1** fails (contradiction) or $h > h_*$.



Sibson (Natural Neighbor) coordinates

Let P denote the set of vertices $\{\mathbf{v}_i\}$ and define $P' = P \cup \{\mathbf{x}\}$.



$$
\begin{aligned}
C_i &:= |V_P(\mathbf{v}_i)| = |\{\mathbf{y} \in \Omega : |\mathbf{y} - \mathbf{v}_i| < |\mathbf{y} - \mathbf{v}_j| \, , \forall j \neq i\}| \\
&= \text{area of cell for } \mathbf{v}_i \text{ in Voronoi diagram on the points of } P,
\end{aligned}
$$

$$
\begin{aligned}
D(\mathbf{x}) &:= |V_{P'}(\mathbf{x})| = |\{\mathbf{y} \in \Omega : |\mathbf{y} - \mathbf{x}| < |\mathbf{y} - \mathbf{v}_i| \, , \forall i\}| \\
&= \text{area of cell for } \mathbf{x} \text{ in Voronoi diagram on the points of } P'.
\end{aligned}
$$

By a slight abuse of notation, we also define

$$D(\mathbf{x}) \cap C_i := |V_{P'}(\mathbf{x}) \cap V_P(\mathbf{v}_i)|.$$

The **Sibson coordinates** are defined to be

$$\lambda_i(\mathbf{x}) := \frac{D(\mathbf{x}) \cap C_i}{D(\mathbf{x})} \qquad \text{or, equivalently,} \qquad \lambda_i(\mathbf{x}) = \frac{D(\mathbf{x}) \cap C_i}{\sum_{j=1}^{n} D_j(\mathbf{x}) \cap C_j}.$$

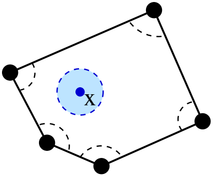
Gradient bound for Sibson coordinates

Theorem

Suppose **G1** (max aspect ratio γ^*) and **G2** (min edge length d_*) hold. Let λ_i be the **Sibson** coordinates. Then there exists $C_\lambda > 0$ such that

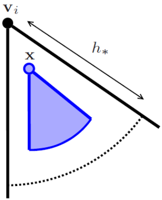
$$||\lambda_i||_{H^1(\Omega)} \leq C_\lambda$$

Proof: Recall $\lambda_i(\mathbf{x}) := \dfrac{|V_{P'}(\mathbf{x}) \cap V_P(\mathbf{v}_i)|}{|V_{P'}(\mathbf{x})|}$; we need a lower bound on the denominator. Either



$\mathbf{x}$ is away from corners and a minimum size ball $\subset V_{P'}(\mathbf{x})$

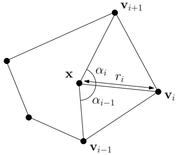
or



$\mathbf{x}$ is near a corner and a minimum size wedge $\subset V_{P'}(\mathbf{x})$

Hence $V_{P'}(\mathbf{x})$ is bounded uniformly below and there exists C_λ as desired.

Gradient bound for mean value coordinates



The **mean value coordinates** are defined by

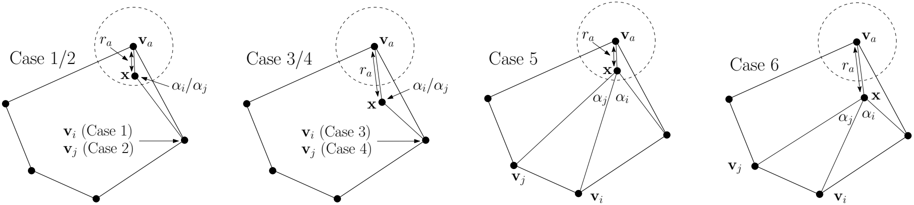
$$\lambda_i^{\mathrm{MV}}(\mathbf{x}) := \frac{w_i(\mathbf{x})}{\sum_{j=1}^{n} w_j(\mathbf{x})} \qquad w_i(\mathbf{x}) := \frac{\tan\left(\frac{\alpha_i(\mathbf{x})}{2}\right) + \tan\left(\frac{\alpha_{i-1}(\mathbf{x})}{2}\right)}{||\mathbf{v}_i - \mathbf{x}||}$$

Theorem

Suppose **G1** (max aspect ratio γ^*) and **G2** (min edge length d_*) hold. Let λ_i be the **mean value** coordinates. Then there exists $C_\lambda > 0$ such that

$$||\lambda_i||_{H^1(\Omega)} \leq C_\lambda$$

Proof: Divide analysis into six cases based on proximity to $\mathbf{v}_a$ and size of α_i and α_j



Polygonal Finite Element Optimal Convergence

Theorem

In the table, any necessary geometric criteria to achieve the ***a priori* linear error estimate** are denoted by N. The set of geometric criteria denoted by S in each row ***taken together*** are sufficient to guarantee the estimate.

		G1 (aspect ratio)	**G2** (min edge length)	**G3** (max interior angle)
Triangulated	λ^{Tri}	-	-	S,N
Wachspress	λ^{Wach}	S	S	S,N
Sibson	λ^{Sibs}	S	S	-
Mean Value	λ^{MV}	S	S	-
Harmonic	λ^{Har}	S	-	-

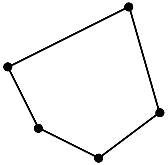
GILLETTE, RAND, BAJAJ *Error Estimates for Generalized Barycentric Interpolation* Advances in Computational Mathematics, to appear, 2011

RAND, GILLETTE, BAJAJ *Interpolation Error Estimates for Mean Value Coordinates*, submitted, 2011.

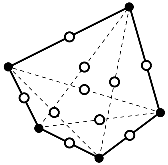
Outline

From linear to quadratic elements

A naïve quadratic element is formed by products of linear element basis functions:



$$\{\lambda_i\} \xrightarrow[\text{products}]{\text{pairwise}} \{\lambda_a\lambda_b\}$$



Why is this naïve?

- For an n-gon, this construction gives $n + \binom{n}{2}$ basis functions $\lambda_a\lambda_b$
- The space of quadratic polynomials is only dimension 6: $\{1, x, y, xy, x^2, y^2\}$
- Conforming to a linear function on the boundary requires 2 degrees of freedom per edge $\Rightarrow$ *only* $2n$ *functions needed!*

Problem Statement

Construct $2n$ basis functions associated to the vertices and edge midpoints of an arbitrary n-gon such that a quadratic convergence estimate is obtained.

Polygonal Quadratic Serendipity Elements

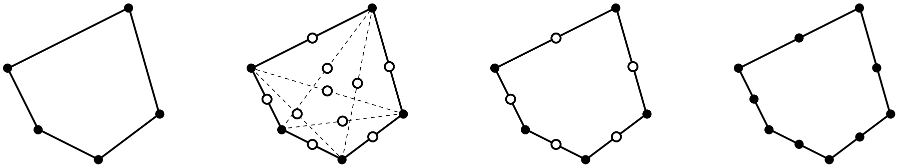
We define matrices $\mathbb{A}$ and $\mathbb{B}$ to reduce the naïve quadratic basis.

filled dot $=$ Lagrangian domain point

$=$ all functions in the set evaluate to 0 except the associated function which evaluates to 1

open dot $=$ non-Lagrangian domain point

$=$ partition of unity satisfied, but not Lagrange property



$$\{\lambda_i\} \xrightarrow[\text{products}]{\text{pairwise}} \{\lambda_a\lambda_b\} \xrightarrow{\mathbb{A}} \{\xi_{ij}\} \xrightarrow{\mathbb{B}} \{\psi_{ij}\}$$

Linear	Quadratic	Serendipity	Lagrange
n	$n + \binom{n}{2}$	$2n$	$2n$

From quadratic to serendipity

The bases are ordered as follows:

ξ_{ii}	and	$\lambda_a\lambda_a$	= basis functions associated with vertices
$\xi_{i(i+1)}$	and	$\lambda_a\lambda_{a+1}$	= basis functions associated with edge midpoints
		$\lambda_a\lambda_b$	= basis functions associated with interior diagonals, i.e. $b \notin \{a-1, a, a+1\}$

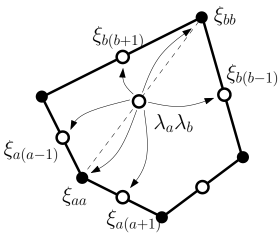
Serendipity basis functions ξ_{ij} are a linear combination of pairwise products $\lambda_a\lambda_b$:

$$\begin{bmatrix} \xi_{ii} \\ \vdots \\ \xi_{i(i+1)} \end{bmatrix} = \mathbb{A} \begin{bmatrix} \lambda_a\lambda_a \\ \vdots \\ \lambda_a\lambda_{a+1} \\ \vdots \\ \lambda_a\lambda_b \end{bmatrix} = \begin{bmatrix} c_{11}^{11} & \cdots & c_{ab}^{11} & \cdots & c_{(n-2)n}^{11} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{11}^{ij} & \cdots & c_{ab}^{ij} & \cdots & c_{(n-2)n}^{ij} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{11}^{n(n+1)} & \cdots & c_{ab}^{n(n+1)} & \cdots & c_{(n-2)n}^{n(n+1)} \end{bmatrix} \begin{bmatrix} \lambda_a\lambda_a \\ \vdots \\ \lambda_a\lambda_{a+1} \\ \vdots \\ \lambda_a\lambda_b \end{bmatrix}$$

From quadratic to serendipity

We ***require*** the serendipity basis to have quadratic approximation power:

Constant precision:	1	$=$	$\sum_i \xi_{ii} + 2\xi_{i(i+1)}$
Linear precision:	$\mathbf{x}$	$=$	$\sum_i \mathbf{v}_i \xi_{ii} + 2\mathbf{v}_{i(i+1)} \xi_{i(i+1)}$
Quadratic precision:	$\mathbf{x}\mathbf{x}^T$	$=$	$\sum_i \mathbf{v}_i \mathbf{v}_i^T \xi_{ii} + (\mathbf{v}_i \mathbf{v}_{i+1}^T + \mathbf{v}_{i+1} \mathbf{v}_i^T) \xi_{i(i+1)}$



Theorem

Constants $\{c_{ij}^{ab}\}$ exist for **any** convex polygon such that the resulting basis $\{\xi_{ij}\}$ satisfies constant, linear, and quadratic precision requirements.

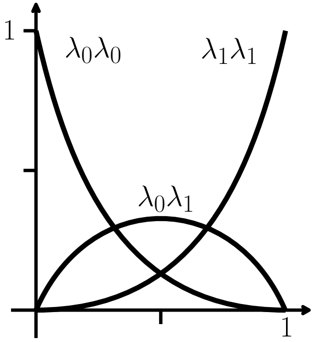
Proof: We produce a coefficient matrix $\mathbb{A}$ with the structure

$$\mathbb{A} := \left[\ \mathbb{I} \ \middle| \ \mathbb{A}' \ \right]$$

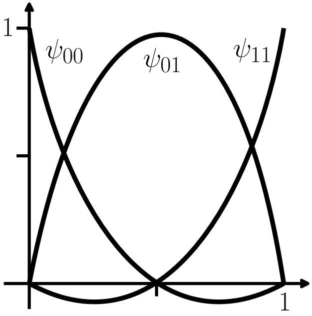
where $\mathbb{A}'$ has only six non-zero entries per column and show that the resulting functions satisfy the six precision equations.

Pairwise products vs. Lagrange basis

Even in 1D, pairwise products of barycentric functions do not form a Lagrange basis at interior degrees of freedom:



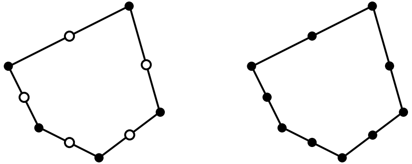
Pairwise products



Lagrange basis

Translation between these two bases is straightforward and generalizes to the higher dimensional case...

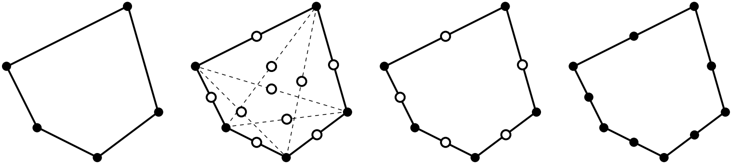
From serendipity to Lagrange



$$\{\xi_{ij}\} \xrightarrow{\quad\quad\mathbb{B}\quad\quad} \{\psi_{ij}\}$$

$$[\psi_{ij}] = \begin{bmatrix} \psi_{11} \\ \psi_{22} \\ \vdots \\ \psi_{nn} \\ \psi_{12} \\ \psi_{23} \\ \vdots \\ \psi_{n1} \end{bmatrix} = \left[\begin{array}{ccccc|ccccc} 1 & & & & & -1 & & \cdots & & -1 \\ & 1 & & & & -1 & -1 & \cdots & & \\ & & \ddots & & & & \ddots & \ddots & & \\ & & & \ddots & & & & \ddots & \ddots & \\ & & & & 1 & & & & -1 & -1 \\ \hline & & & & & 4 & & & & \\ & & & & & & 4 & & & \\ & & 0 & & & & & \ddots & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 4 \end{array}\right] \begin{bmatrix} \xi_{11} \\ \xi_{22} \\ \vdots \\ \xi_{nn} \\ \xi_{12} \\ \xi_{23} \\ \vdots \\ \xi_{n1} \end{bmatrix} = \mathbb{B}[\xi_{ij}].$$

Serendipity Theorem



$$\{\lambda_i\} \xrightarrow[\text{products}]{\text{pairwise}} \{\lambda_a\lambda_b\} \xrightarrow{\mathbb{A}} \{\xi_{ij}\} \xrightarrow{\mathbb{B}} \{\psi_{ij}\}$$

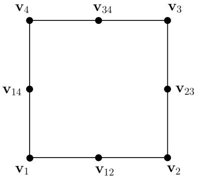
Theorem

Given bounds on polygon aspect ratio (**G1**), minimum edge length (**G2**), and maximum interior angles (**G3**):

- $||\mathbb{A}||$ is uniformly bounded,
- $||\mathbb{B}||$ is uniformly bounded, and
- $\text{span}\{\psi_{ij}\} \supset \mathcal{P}_2(\mathbb{R}^2)$ = quadratic polynomials in x and y

RAND, GILLETTE, BAJAJ *Quadratic Serendipity Finite Element on Polygons Using Generalized Barycentric Coordinates*, Submitted, 2011

Special case of a square



Bilinear functions are barycentric coordinates:

$$\begin{aligned}\lambda_1 &= (1-x)(1-y)\\ \lambda_2 &= x(1-y)\\ \lambda_3 &= xy\\ \lambda_4 &= (1-x)y\end{aligned}$$

Compute $[\xi_{ij}] := \left[\ \mathbb{I}\ \middle|\ \mathbb{A}'\ \right][\lambda_a\lambda_b]$

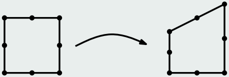
$$\begin{bmatrix}\xi_{11}\\ \xi_{22}\\ \xi_{33}\\ \xi_{44}\\ \xi_{12}\\ \xi_{23}\\ \xi_{34}\\ \xi_{14}\end{bmatrix} = \begin{bmatrix}1 & & 0 & -1 & 0\\ 0 & & 0 & 0 & -1\\ 0 & & 0 & -1 & 0\\ 0 & \cdots & 0 & 0 & -1\\ 0 & \cdots & 0 & 1/2 & 1/2\\ 0 & & 0 & 1/2 & 1/2\\ 0 & & 0 & 1/2 & 1/2\\ 0 & & 1 & 1/2 & 1/2\end{bmatrix}\begin{bmatrix}\lambda_1\lambda_1\\ \lambda_2\lambda_2\\ \lambda_3\lambda_3\\ \lambda_4\lambda_4\\ \lambda_1\lambda_2\\ \lambda_2\lambda_3\\ \lambda_3\lambda_4\\ \lambda_1\lambda_4\end{bmatrix} = \begin{bmatrix}(1-x)(1-y)(1-x-y)\\ x(1-y)(x-y)\\ xy(-1+x+y)\\ (1-x)y(y-x)\\ (1-x)x(1-y)\\ x(1-y)y\\ (1-x)xy\\ (1-x)(1-y)y\end{bmatrix}$$

$$\text{span}\left\{\xi_{ii},\xi_{i(i+1)}\right\} = \text{span}\left\{1,x,y,x^2,y^2,xy,x^2y,xy^2\right\} =: \mathcal{S}_2(I^2)$$

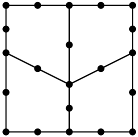
Hence, this provides a computational basis for the serendipity space $\mathcal{S}_2(I^2)$ defined in

ARNOLD, AWANOU *The serendipity family of finite elements*, Found. Comp. Math, 2011.

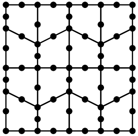
Numerical evidence for non-affine image of a square

Instead of mapping ,
use quadratic serendipity **GBC** interpolation with mean value coordinates:

$$u_h = I_q u := \sum_{i=1}^{n} u(\mathbf{v}_i)\psi_{ii} + u\left(\frac{\mathbf{v}_i + \mathbf{v}_{i+1}}{2}\right)\psi_{i(i+1)}$$



$n = 2$



$n = 4$

Non-affine bilinear mapping

	$\|\|u - u_h\|\|_{L^2}$		$\|\|\nabla(u - u_h)\|\|_{L^2}$	
n	error	rate	error	rate
2	5.0e-2		6.2e-1	
4	6.7e-3	2.9	1.8e-1	1.8
8	9.7e-4	2.8	5.9e-2	1.6
16	1.6e-4	2.6	2.3e-2	1.4
32	3.3e-5	2.3	1.0e-2	1.2
64	7.4e-6	2.1	4.96e-3	1.1

ARNOLD, BOFFI, FALK, Math. Comp., 2002

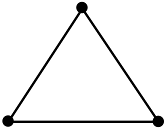
Quadratic serendipity **GBC** method

	$\|\|u - u_h\|\|_{L^2}$		$\|\|\nabla(u - u_h)\|\|_{L^2}$	
n	error	rate	error	rate
2	2.34e-3		2.22e-2	
4	3.03e-4	2.95	6.10e-3	1.87
8	3.87e-5	2.97	1.59e-3	1.94
16	4.88e-6	2.99	4.04e-4	1.97
32	6.13e-7	3.00	1.02e-4	1.99
64	7.67e-8	3.00	2.56e-5	1.99
128	9.59e-9	3.00	6.40e-6	2.00
256	1.20e-9	3.00	1.64e-6	1.96

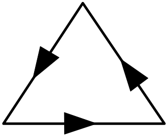
Outline

From scalar to vector elements

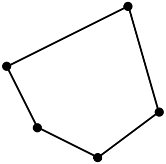
Barycentric functions are used to define $H(\text{curl})$ vector elements on triangles:



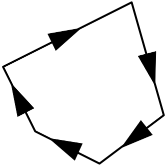
$$\{\lambda_i\} \xrightarrow[\text{construction}]{\text{Whitney}} \{\lambda_a\nabla\lambda_b - \lambda_b\nabla\lambda_a\}$$



Generalized barycentric functions provide $H(\text{curl})$ elements on polygons:



$$\{\lambda_i\} \xrightarrow[\text{construction}]{\text{Whitney}} \{\lambda_a\nabla\lambda_b - \lambda_b\nabla\lambda_a\}$$

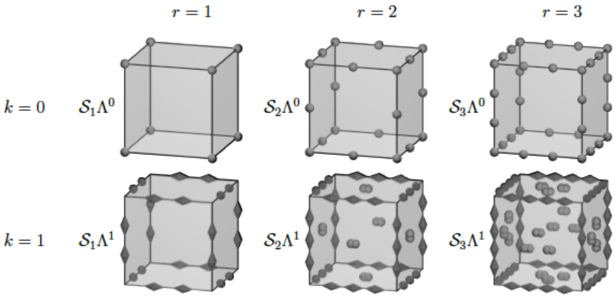


This idea fits naturally into the framework of **Discrete Exterior Calculus** and suggests a wide range of applications. . . . . . *work in progress*

GILLETTE, BAJAJ *Dual Formulations of Mixed Finite Element Methods with Applications*
Computer-Aided Design 43:10, pages 1213-1221, 2011.

Basis functions for serendipity spaces

Recent work characterized serendipity spaces in *n* dimensions for scalar fields, vector fields, and their generalization to differential *k*-forms:



ARNOLD, AWANOU *Finite Element Differential Forms on Cubical Meshes*
arXiv:1204.2595, 2012.

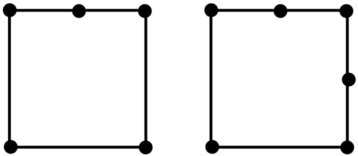
The results of that paper coupled with the techniques used here suggest a variety of new research directions. . . . . .*work in progress*

GBCs for adaptive algorithms

Recall from the table of geometric dependencies for gradient bounds:

		G1 (aspect ratio)	**G2** (min edge length)	**G3** (max interior angle)
Mean Value	λ^{MV}	S	S	-

Thus, the quadratic serendipity construction with mean value coordinates can still allow quadratic convergence for elements with interior angles of π, including typical elements from adaptive methods:



References

- Thank to the organizers for the invitation and for organizing!
- Slides and pre-prints are available at:

http://ccom.ucsd.edu/∼agillette