Barycentric coordinates and transfinite interpolation

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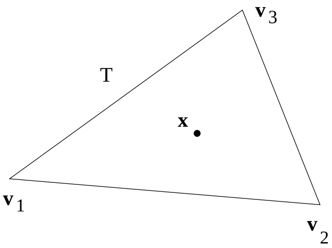
In this talk:

1. Barycentric coordinates
2. Generalization to polygons: Wachspress, mean value, etc.
3. Transfinite interpolation

Barycentric coordinates on a triangle



Given $\mathbf{x} \in T$, want $\lambda_1, \lambda_2, \lambda_3 \geq 0$ such that

$$\lambda_1 + \lambda_2 + \lambda_3 = 1,$$

and

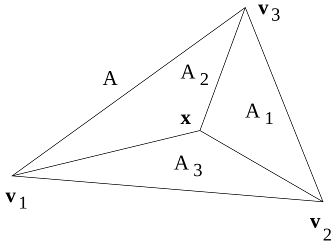
$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \lambda_3 \mathbf{v}_3 = \mathbf{x}.$$

Linear system of three equations,

$$\begin{pmatrix} 1 & 1 & 1 \\ v_1^1 & v_2^1 & v_3^1 \\ v_1^2 & v_2^2 & v_3^2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 \\ x^1 \\ x^2 \end{pmatrix}.$$

By Cramer's rule, unique solution is

$$\lambda_1 = \frac{A_1}{A}, \quad \lambda_2 = \frac{A_2}{A}, \quad \lambda_3 = \frac{A_3}{A}.$$



Properties

- Lagrange property: $\lambda_i(\mathbf{v}_j) = \delta_{ij}$.
- Interpolation: if

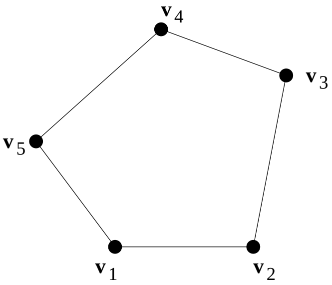
$$g(\mathbf{x}) = \sum_{i=1}^{3} \lambda_i(\mathbf{x}) f(\mathbf{v}_i),$$

  then $g(\mathbf{v}_i) = f(\mathbf{v}_i)$.
- Linear precision: if f is linear then $g = f$.
- Linearity: λ_i is linear.
- Bernstein-Bezier basis for polynomials, degree d:

$$p(\mathbf{x}) = \sum_{i+j+k=d} \frac{n!}{i!j!k!} \lambda_1^i(\mathbf{x}) \lambda_2^j(\mathbf{x}) \lambda_3^k(\mathbf{x}) c_{ijk}.$$

Barycentric coordinates on polygons

Let Ω be a convex polygon.



Functions $\lambda_i : \Omega \to \mathbb{R}$, $i = 1, \ldots, n$, are *barycentric coordinates* if for $\mathbf{x} \in \Omega$, $\lambda_i(\mathbf{x}) \geq 0$, $i = 1, \ldots, n$, and

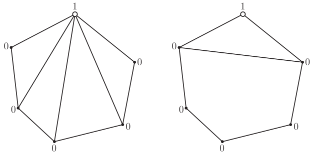
$$\sum_{i=1}^{n} \lambda_i(\mathbf{x}) = 1, \qquad \sum_{i=1}^{n} \lambda_i(\mathbf{x})\mathbf{v}_i = \mathbf{x}.$$

Properties

- Lagrange property: $\lambda_i(\mathbf{v}_j) = \delta_{ij}$.
- Interpolation: if

$$g(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i(\mathbf{x}) f(\mathbf{v}_i),$$

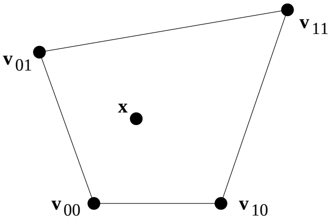
  then $g(\mathbf{v}_i) = f(\mathbf{v}_i)$.
- Linear precision: if f is linear then $g = f$.
- λ_i lies between the two functions:



- Generalized Bernstein-Bezier:

$$p(\mathbf{x}) = \sum_{|\mathbf{i}|=d} \binom{d}{\mathbf{i}} \lambda_1^{i_1}(\mathbf{x}) \cdots \lambda_n^{i_n}(\mathbf{x}) c_{\mathbf{i}}.$$

Coordinates for quadrilaterals



Can find $\lambda, \mu \in [0, 1]$ s.t.

$$\mathbf{x} = (1-\lambda)(1-\mu)\mathbf{v}_{00} + \lambda(1-\mu)\mathbf{v}_{10} + (1-\lambda)\mu\mathbf{v}_{01} + \lambda\mu\mathbf{v}_{11},$$

giving an interpolant

$$g(\mathbf{x}) = (1-\lambda)(1-\mu)f_{00} + \lambda(1-\mu)f_{10} + (1-\lambda)\mu f_{01} + \lambda\mu f_{11}.$$

To find λ and μ: let

$\mathbf{a} = \mathbf{v}_{00} - \mathbf{x}$, $\mathbf{b} = \mathbf{v}_{10} - \mathbf{v}_{00}$, $\mathbf{c} = \mathbf{v}_{01} - \mathbf{v}_{00}$, $\mathbf{d} = \mathbf{v}_{00} - \mathbf{v}_{10} - \mathbf{v}_{01} + \mathbf{v}_{11}$.

Then need to solve

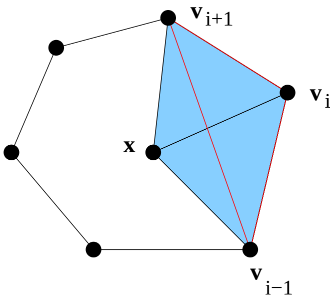
$$\mathbf{a} + \mathbf{b}\lambda + \mathbf{c}\mu + \mathbf{d}\lambda\mu = 0.$$

Eliminating λ gives a quadratic in μ:

$$(\mathbf{a} + \mathbf{c}\mu) \times (\mathbf{b} + \mathbf{d}\mu) = 0.$$

Similar for λ. The special cases $\mathbf{c} \times \mathbf{d} = 0$ and $\mathbf{b} \times \mathbf{d} = 0$ need to be treated separately.

Wachspress coordinates: Wachspress 1973, Warren 1996



For a convex polygon, n vertices, let

$$\lambda_i(\mathbf{x}) = \frac{w_i(\mathbf{x})}{\sum_{j=1}^{n} w_j(\mathbf{x})}, \qquad w_i(\mathbf{x}) = \frac{A(\mathbf{v}_{i-1}, \mathbf{v}_i, \mathbf{v}_{i+1})}{A(\mathbf{x}, \mathbf{v}_{i-1}, \mathbf{v}_i) A(\mathbf{x}, \mathbf{v}_i, \mathbf{v}_{i+1})}.$$

Then $\lambda_1, \ldots, \lambda_n$ are barycentric coordinates.

Proof: Meyer et al. 2002

Let

$$A_i = A(\mathbf{x}, \mathbf{v}_i, \mathbf{v}_{i+1}) \qquad \text{and} \qquad B_i = A(\mathbf{v}_{i-1}, \mathbf{v}_i, \mathbf{v}_{i+1}).$$

Then express $\mathbf{x}$ as

$$\mathbf{x} = \frac{A_i}{B_i}\mathbf{v}_{i-1} + \frac{(B_i - A_{i-1} - A_i)}{B_i}\mathbf{v}_i + \frac{A_{i-1}}{B_i}\mathbf{v}_{i+1},$$

and rearrange:

$$\frac{B_i}{A_{i-1}A_i}(\mathbf{v}_i - \mathbf{x}) = \frac{1}{A_{i-1}}(\mathbf{v}_i - \mathbf{v}_{i-1}) - \frac{1}{A_i}(\mathbf{v}_{i+1} - \mathbf{v}_i).$$

Summing both sides over i gives

$$\sum_i \frac{B_i}{A_{i-1}A_i}(\mathbf{v}_i - \mathbf{x}) = 0$$

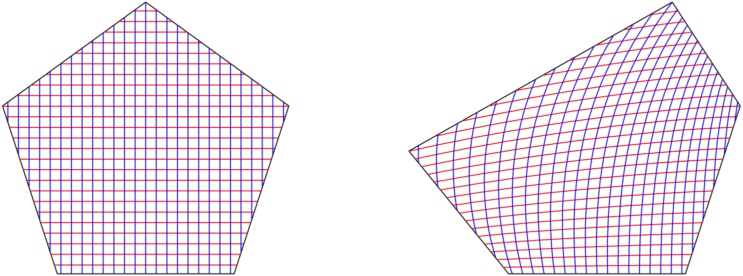
or

$$\sum_i w_i(\mathbf{x})(\mathbf{v}_i - \mathbf{x}) = 0.$$

Wachspress coordinates are rational

$$\lambda_i(\mathbf{x}) = \frac{B_i \prod_{j \neq i-1,i} A_j(\mathbf{x})}{\sum_k B_k \prod_{j \neq k-1,k} A_j(\mathbf{x})} = \frac{\text{degree } (n-2)}{\text{degree } (n-3)}.$$

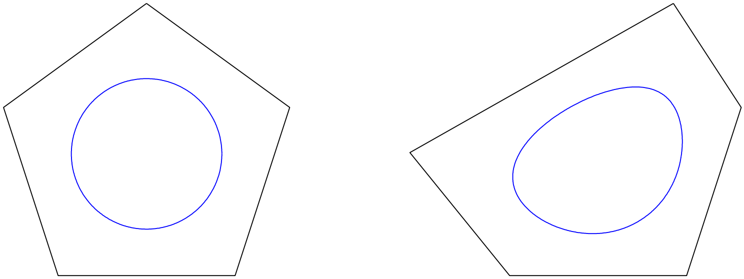
Application: barycentric mappings



Recipe: given $\mathbf{x} \in \Omega$,

1. express $\mathbf{x}$ in Wachspress form $\mathbf{x} = \sum_i \lambda_i(\mathbf{x})\mathbf{v}_i$,
2. set $\mathbf{x}' = \sum_i \lambda_i(\mathbf{x})\mathbf{v}'_i$.

Application: curve deformation



Theorem (F. & Kosinka). **Wachspress mappings are injective**

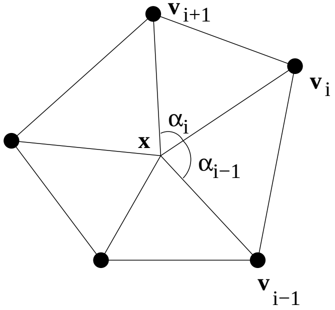
Idea of proof: express Jacobian as

$$J = 2 \sum_{1 \le i < j < k \le n} \begin{vmatrix} \lambda_i & \lambda_j & \lambda_k \\ \partial_1 \lambda_i & \partial_1 \lambda_j & \partial_1 \lambda_k \\ \partial_2 \lambda_i & \partial_2 \lambda_j & \partial_2 \lambda_k \end{vmatrix} A(\mathbf{v}_i', \mathbf{v}_j', \mathbf{v}_k').$$

Can show that

$$\begin{vmatrix} \lambda_i & \lambda_j & \lambda_k \\ \partial_1 \lambda_i & \partial_1 \lambda_j & \partial_1 \lambda_k \\ \partial_2 \lambda_i & \partial_2 \lambda_j & \partial_2 \lambda_k \end{vmatrix} > 0.$$

Mean value coordinates



For a convex polygon, n vertices, let $\lambda_i(\mathbf{x}) = w_i(\mathbf{x}) / \sum_{j=1}^{n} w_j(\mathbf{x})$, where

$$w_i(\mathbf{x}) = \frac{1}{\|\mathbf{v}_i - \mathbf{x}\|} \left(\tan\left(\frac{\alpha_{i-1}(\mathbf{x})}{2} \right) + \tan\left(\frac{\alpha_i(\mathbf{x})}{2} \right) \right).$$

Then $\lambda_1, \ldots, \lambda_n$ are barycentric coordinates.

Proof:

With $\mathbf{e}_i = (\mathbf{v}_i - \mathbf{x})/\|\mathbf{v}_i - \mathbf{x}\|$, need to show that

$$\sum_{i=1}^{n} \big(\tan(\alpha_{i-1}/2) + \tan(\alpha_i/2) \big) \mathbf{e}_i = 0,$$

or

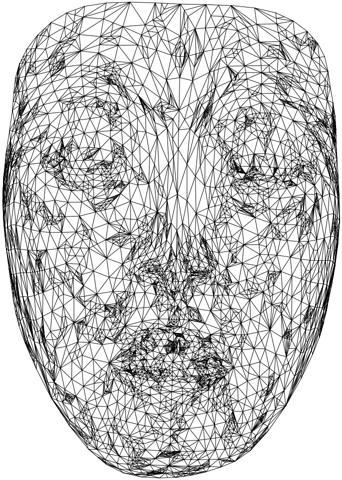
$$\sum_{i=1}^{n} \tan(\alpha_i/2)(\mathbf{e}_i + \mathbf{e}_{i+1}) = 0. \tag{1}$$

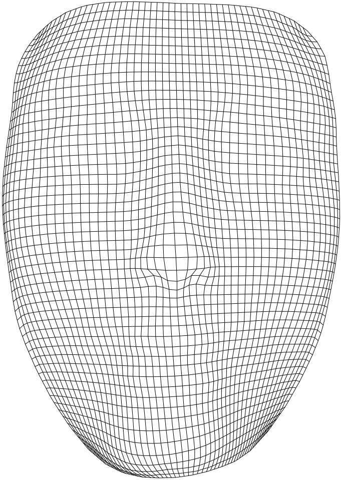
To show this, let $\mathbf{e}_i = (\cos\theta_i, \sin\theta_i)$. Then $\alpha_i = \theta_{i+1} - \theta_i$, and so

$$\begin{aligned} \tan\left(\frac{\alpha_i}{2}\right)(\mathbf{e}_i + \mathbf{e}_{i+1}) &= \tan\left(\frac{\alpha_i}{2}\right)(\cos\theta_i + \cos\theta_{i+1}, \sin\theta_i + \sin\theta_{i+1}) \\ &= (\sin\theta_{i+1} - \sin\theta_i, \cos\theta_i - \cos\theta_{i+1}). \end{aligned}$$

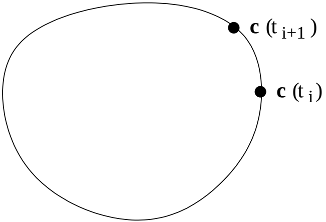
Summing this over $i = 1, \ldots, n$ gives (1).

Application: mesh parameterization





Transfinite interpolation, Warren et al 2004

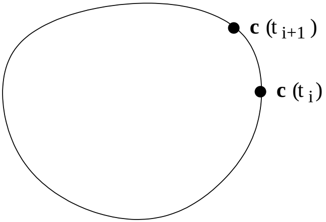


For convex curve $\mathbf{c} : [a, b] \to \mathbb{R}^2$ sample points $\mathbf{v}_i = \mathbf{c}(t_i)$ and take limit of Wachspress interpolants as $\max_i(t_{i+1} - t_i) \to 0$. Gives a 'transfinite interpolant'

$$g(\mathbf{x}) = \int_a^b w(\mathbf{x}, t) f(\mathbf{c}(t))\, dt \Bigg/ \int_a^b w(\mathbf{x}, t)\, dt,$$

where $w(\mathbf{x}, t) = (\mathbf{c}'(t) \times \mathbf{c}''(t))/((\mathbf{c}(t) - \mathbf{x}) \times \mathbf{c}'(t))^2$.

MV transfinite interpolation



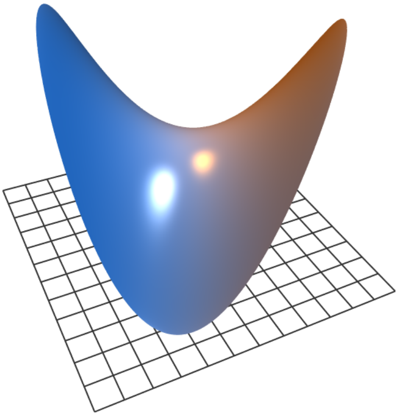
Analogously, the transfinite MV interpolant is

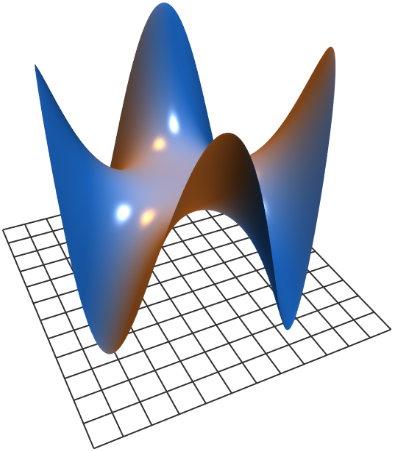
$$g(\mathbf{x}) = \int_a^b w(\mathbf{x}, t) f(\mathbf{c}(t))\, dt \Bigg/ \int_a^b w(\mathbf{x}, t)\, dt,$$

where

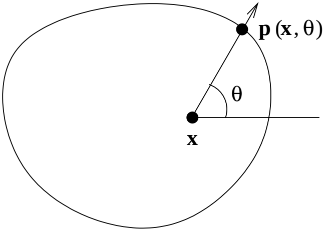
$$w(\mathbf{x}, t) = \frac{(\mathbf{c}(t) - \mathbf{x}) \times \mathbf{c}'(t)}{\|\mathbf{c}(t) - \mathbf{x}\|^3}.$$

Examples of MV interpolants





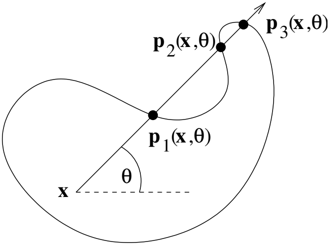
Non-convex curves? Angle formulation



For a convex curve, the MV interpolant is

$$g(\mathbf{x}) = \int_0^{2\pi} \frac{f(\mathbf{p}(\mathbf{x},\theta))}{\|\mathbf{p}(\mathbf{x},\theta) - \mathbf{x}\|}\, d\theta \Bigg/ \int_0^{2\pi} \frac{1}{\|\mathbf{p}(\mathbf{x},\theta) - \mathbf{x}\|}\, d\theta.$$

On arbitrary curves



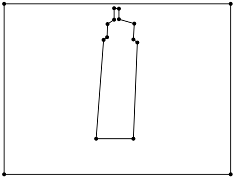
$$g(\mathbf{x}) = \int_0^{2\pi} \sum_{j=1}^{n(\mathbf{x},\theta)} \frac{(-1)^{j-1}}{\|\mathbf{p}_j(\mathbf{x},\theta) - \mathbf{x}\|} f(\mathbf{p}_j(\mathbf{x},\theta))\, d\theta \Big/ \phi(\mathbf{x}),$$

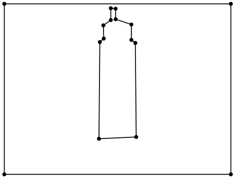
where

$$\phi(\mathbf{x}) = \int_0^{2\pi} \sum_{j=1}^{n(\mathbf{x},\theta)} \frac{(-1)^{j-1}}{\|\mathbf{p}_j(\mathbf{x},\theta) - \mathbf{x}\|}\, d\theta.$$

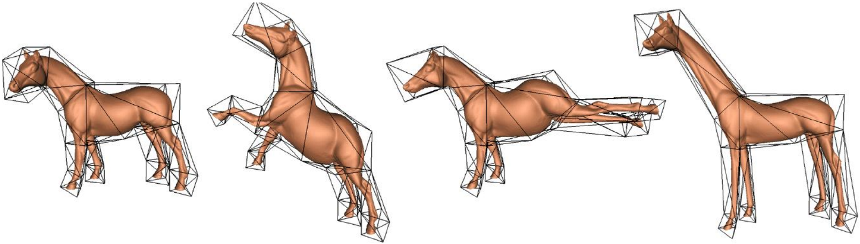
Hormann: $\phi(\mathbf{x}) > 0$.

Application: image warping, Hormann





Application: mesh deformation, Ju, Schaefer, Warren



Hermite interpolation

Two methods:

- Use MV weight function (with Chris Dyken / Solveig Bruvoll)
- Radial minimization (with Christian Schulz)

Method 1: weight function

The cubic interpolant p to the data $f(0)$, $f'(0)$, $f(1)$, $f'(1)$, can be expressed as

$$p(x) = g_0(x) + \psi(x) g_1(x),$$

where

$$\begin{aligned} g_0(x) &= (1 - x) f(0) + x f(1), \\ \psi(x) &= x(1 - x), \\ g_1(x) &= (1 - x) m_0 + x m_1, \end{aligned}$$

and

$$m_0 = f'(0) - f(1) + f(0), \qquad m_1 = -f'(1) + f(1) - f(0).$$

In $\mathbb{R}^n$ we interpolate the data f and $\frac{\partial f}{\partial \mathbf{n}}$ on $\partial\Omega$ by the function

$$p(\mathbf{x}) = g_0(\mathbf{x}) + \psi(\mathbf{x}) g_1(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$

where

$$g_0(\mathbf{x}) = \int_S \frac{f(\mathbf{p}(\mathbf{x},\mathbf{v}))}{\|\mathbf{p}(\mathbf{x},\mathbf{v}) - \mathbf{x}\|}\, d\mathbf{v} \Big/ \phi(\mathbf{x}),$$

$$\psi(\mathbf{x}) = 1 \Big/ \phi(\mathbf{x}),$$

$$g_1(\mathbf{x}) = \int_S \frac{m(\mathbf{p}(\mathbf{x},\mathbf{v}))}{\|\mathbf{p}(\mathbf{x},\mathbf{v}) - \mathbf{x}\|}\, d\mathbf{v} \Big/ \phi(\mathbf{x}),$$

$$\phi(\mathbf{x}) = \int_S \frac{1}{\|\mathbf{p}(\mathbf{x},\mathbf{v}) - \mathbf{x}\|}\, d\mathbf{v}.$$

and

$$m(\mathbf{y}) = \left(\frac{\partial f}{\partial \mathbf{n}}(\mathbf{y}) - \frac{\partial g_0}{\partial \mathbf{n}}(\mathbf{y})\right) \Big/ \frac{\partial \psi}{\partial \mathbf{n}}(\mathbf{y}), \qquad \mathbf{y} \in \partial\Omega.$$

To use this construction we need to find $\frac{\partial \psi}{\partial \mathbf{n}}(\mathbf{y})$ and $\frac{\partial g_0}{\partial \mathbf{n}}(\mathbf{y})$.

Theorem

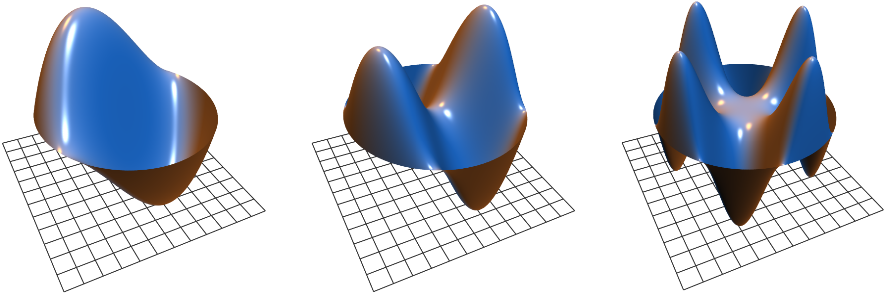
If $d(M_E, \partial\Omega) > 0$ and $d(M_I, \partial\Omega) > 0$ and $\mathbf{y} \in \partial\Omega$ then

$$\frac{\partial \psi}{\partial \mathbf{n}}(\mathbf{y}) = \frac{1}{V_{n-1}},$$

$$\frac{\partial g_0}{\partial \mathbf{n}}(\mathbf{y}) = \frac{1}{V_{n-1}} \int_{H(\mathbf{n})} \frac{f(\mathbf{p}(\mathbf{y},\mathbf{v})) - f(\mathbf{y})}{\|\mathbf{p}(\mathbf{y},\mathbf{v}) - \mathbf{y}\|} \, d\mathbf{v},$$

where V_{n-1} is the volume of the unit sphere in $\mathbb{R}^{n-1}$: $V_1 = 2$, $V_2 = \pi$, $V_3 = 4\pi/3$, $V_4 = \pi^2/2$,...

Hermite examples



Method 2: radial minimization

MV interpolation revisited: the value $g(\mathbf{x})$ is the unique minimizer $a = g(\mathbf{x})$ of the local 'energy' function

$$E_{\mathbf{x}}(a) = \int_S \int_0^{\rho(\mathbf{x},\mathbf{v})} \left(q'_{\mathbf{x},\mathbf{v}}(r)\right)^2 dr\, d\mathbf{v},$$

where

$$q_{\mathbf{x},\mathbf{v}}(r) = \frac{\rho(\mathbf{x},\mathbf{v}) - r}{\rho(\mathbf{x},\mathbf{v})} a + \frac{r}{\rho(\mathbf{x},\mathbf{v})} f(\mathbf{p}(\mathbf{x},\mathbf{v})),$$

and $\rho(\mathbf{x},\mathbf{v}) = \|\mathbf{p}(\mathbf{x},\mathbf{v}) - \mathbf{x}\|$.

Hermite interpolation

Find $a \in \mathbb{R}$ and $\mathbf{b} \in \mathbb{R}^n$ that minimize

$$E_{\mathbf{x}}(a, \mathbf{b}) = \int_S \int_0^{\rho(\mathbf{x},\mathbf{v})} \left(q''_{\mathbf{x},\mathbf{v}}(r)\right)^2 \, dr \, d\mathbf{v},$$

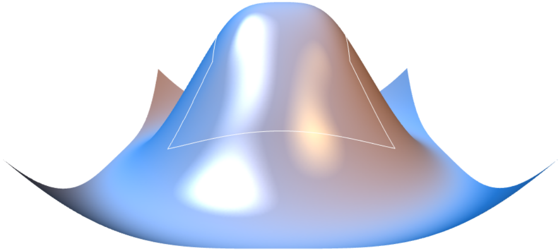
where $q_{\mathbf{x},\mathbf{v}}$ is the cubic polynomial such that

$$\begin{aligned} q_{\mathbf{x},\mathbf{v}}(0) &= a, & q_{\mathbf{x},\mathbf{v}}(\rho(\mathbf{x},\mathbf{v})) &= f(\mathbf{p}(\mathbf{x},\mathbf{v})), \\ q'_{\mathbf{x},\mathbf{v}}(0) &= \mathbf{v} \cdot \mathbf{b}, & q'_{\mathbf{x},\mathbf{v}}(\rho(\mathbf{x},\mathbf{v})) &= D_{\mathbf{v}} f(\mathbf{p}(\mathbf{x},\mathbf{v})), \end{aligned}$$

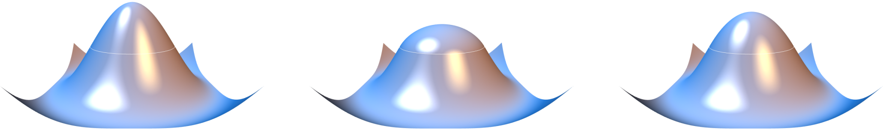
and set $g(\mathbf{x}) = a$.

Has cubic precision!

Example of radial minimization



Comparison of the two Hermite methods



Radial cosine   Method 1   Method 2