Università della Svizzera italiana

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Finite differences with high orders of approximation for irregularly discretized domains

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Introduction

Our new method solves numerically the Poisson Equation, with the advantage of easily permitting the choice of the convergence order. Although the method is independent of the dimensionality, we develop it for bivariate functions. Moreover, the method does not require any assumption on the shape of the domain. The construction of the method is based on local Taylor expansions.

Construction

The two best known methods (*Finite Elements* and *Finite Differences*) have the same Laplace discretization pattern

$$\sum_{j\in N_i} a_{ij}(u(p_j)-u(p_i)) = \Delta u(p_i) + O(h^k)$$

To reproduce this pattern, we start with Taylor expansion of order 2 around p_i evaluated at p_j

$$u(p_j)-u(p_i) = (p_j-p_i)^T \begin{pmatrix} u_x \\ u_y \end{pmatrix} + \frac{1}{2}(p_j-p_i)^T \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix} (p_j-p_i) + O(h^3)$$

Then we multiply by a_{ij} and sum over the neighbours

$$\sum_{j\in N_i} a_{ij}(u(p_j)-u(p_i)) = \sum_{j\in N_i} a_{ij}(x_j-x_i)u_x + \sum_{j\in N_i} a_{ij}(y_j-y_i)u_y + \frac{1}{2}\sum_{j\in N_i} a_{ij}\left((x_j-x_i)^2u_{xx}+(y_j-y_i)^2u_{yy}\right) + \sum_{j\in N_i} a_{ij}(x_j-x_i)(y_j-y_i)u_{xy} + O(h^3)$$

In order to obtain the Laplacian we assemble a linear system of constraints

$$\left.\begin{array}{l} \sum_{j\in N_i} a_{ij}(x_j-x_i)=0 \\ \sum_{j\in N_i} a_{ij}(y_j-y_i)=0 \end{array}\right\} \text{cancel first derivatives}$$

$$\sum_{j\in N_i} a_{ij}\left((x_j-x_i)^2-(y_j-y_i)^2\right)=0 \quad \text{equal coefficients for second derivative}$$

$$\sum_{j\in N_i} a_{ij}(x_j-x_i)(y_j-y_i)=0 \quad \text{cancel mixed derivative}$$

$$\sum_{j\in N_i} a_{ij}=1 \quad \text{avoid trivial solution}$$

If a_{ij} satisfy the constraints then the expansion becomes

$$\sum_{j\in N_i} a_{ij}(u(p_j)-u(p_i)) = \frac{1}{4}\sum_{j\in N_i} a_{ij}||p_j-p_i||^2\Delta u(p_i) + O(h^3)$$

By rearranging the terms we obtain the Laplacian

$$\frac{4}{\sum_{j\in N_i} a_{ij}||p_j-p_i||^2}\sum_{j\in N_i} a_{ij}(u(p_j)-u(p_i)) = \Delta u(p_i) + O(h)$$

Higher convergence order

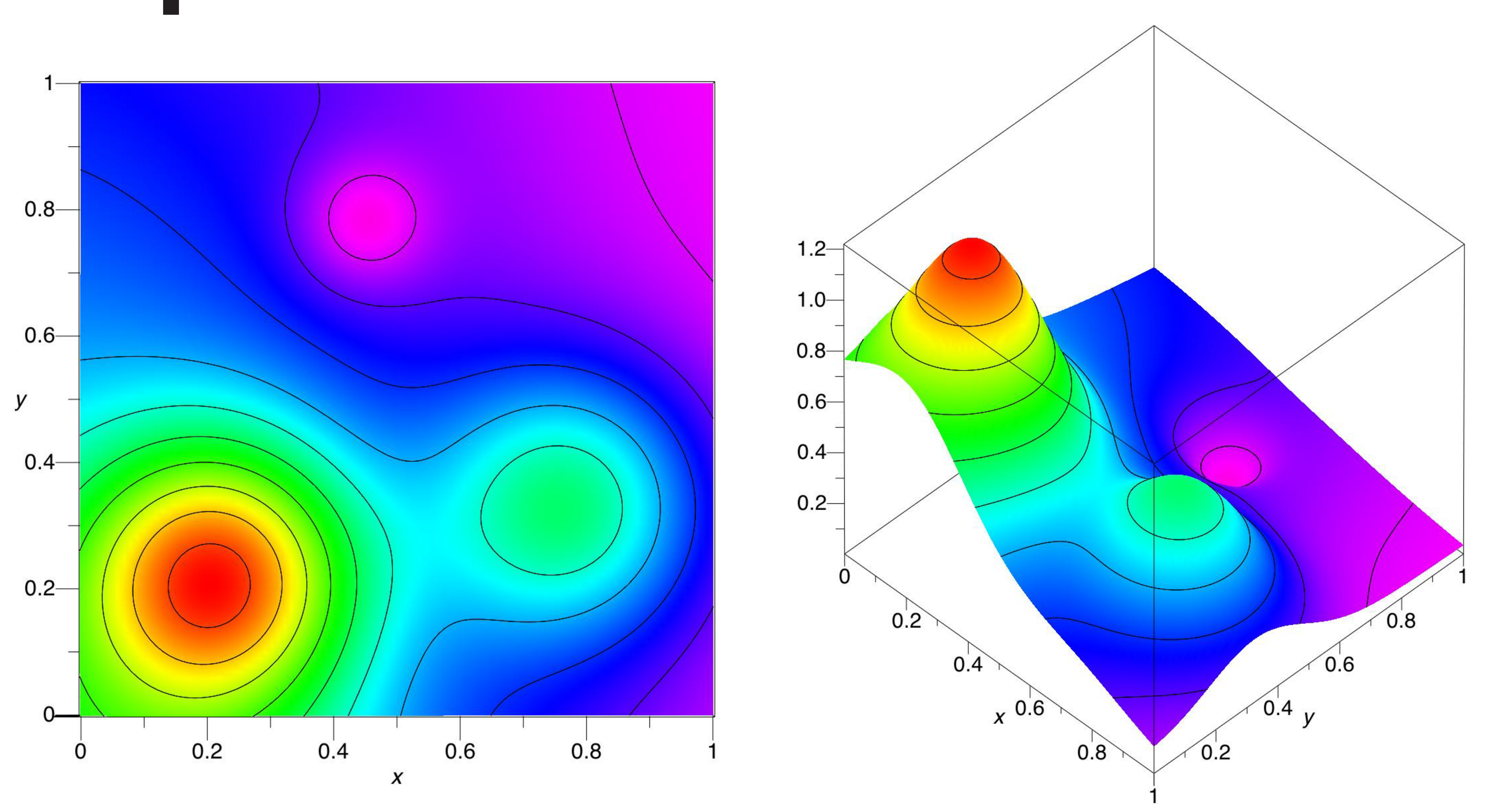
The same derivation can be done with higher Taylor expansion order, which leads to higher convergence order and more linear constraints

Expansion Order	Number of Constraints	Convergence Order
2	5	1
3	9	2
4	14	3
5	20	4

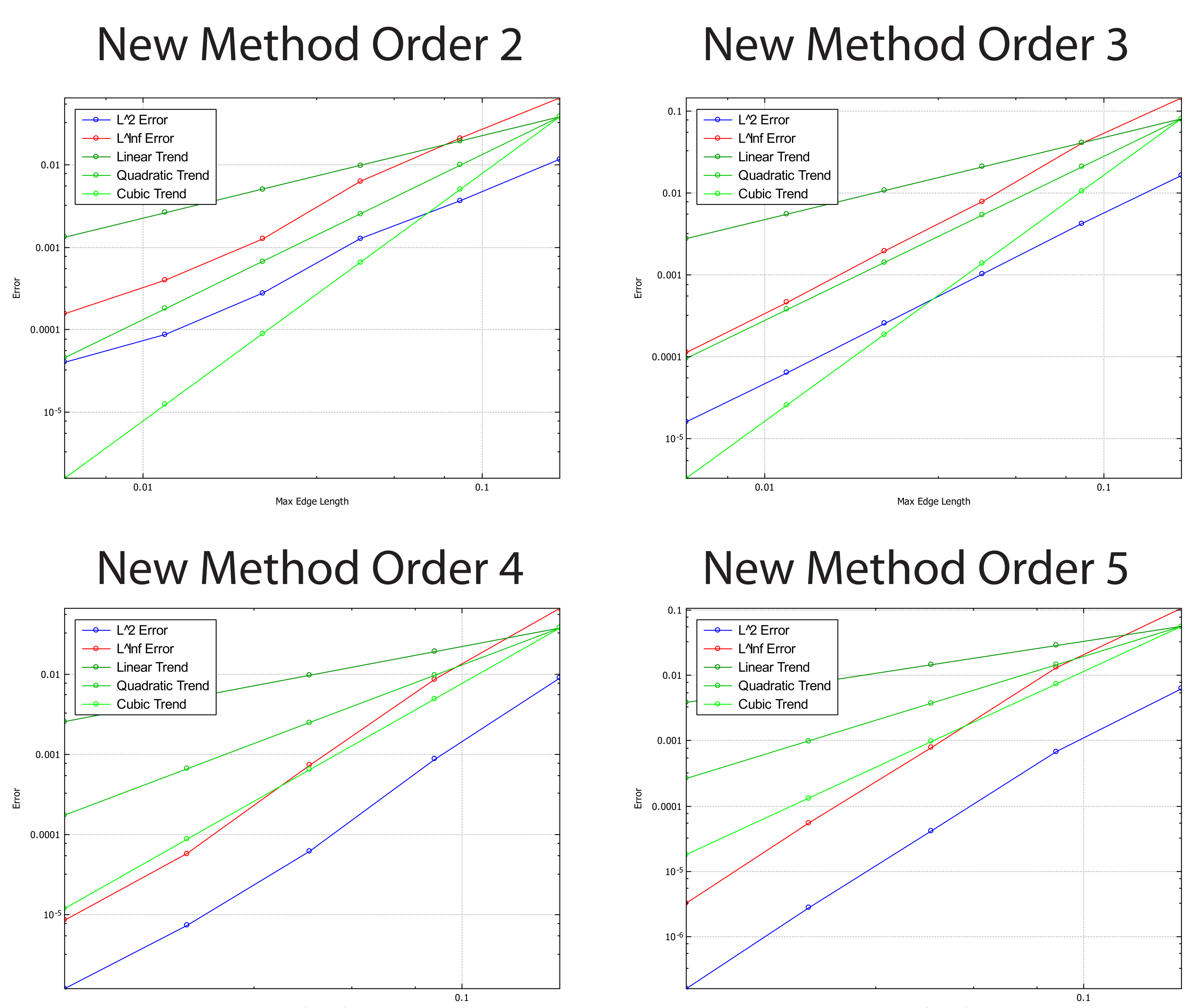
Advantages

- Simple to extend to higher order of convergence
- Easy to implement
- No need of triangulation
- Exact with functions with finite Taylor series

Empirical results



Franke Test Function



Convergence orders of the new method with different Taylor expansion orders for the Franke test function