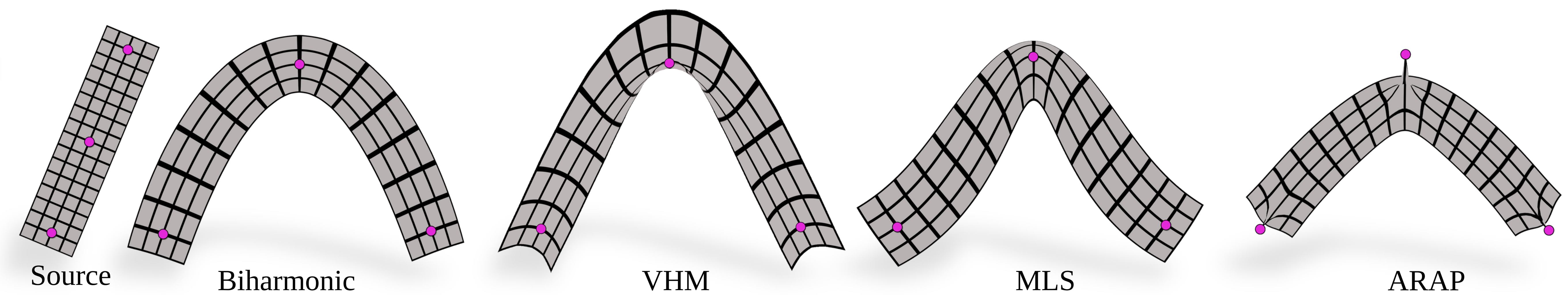
Biharmonic Coordinates for Shape Deformation

Roi Poranne[1], Ofir Weber[2] and Craig Gotsman[1]
[1]Computer Science Department, Technion, Haifa, Israel
[2]Courant Institute of Mathematical Sciences, New York University, USA



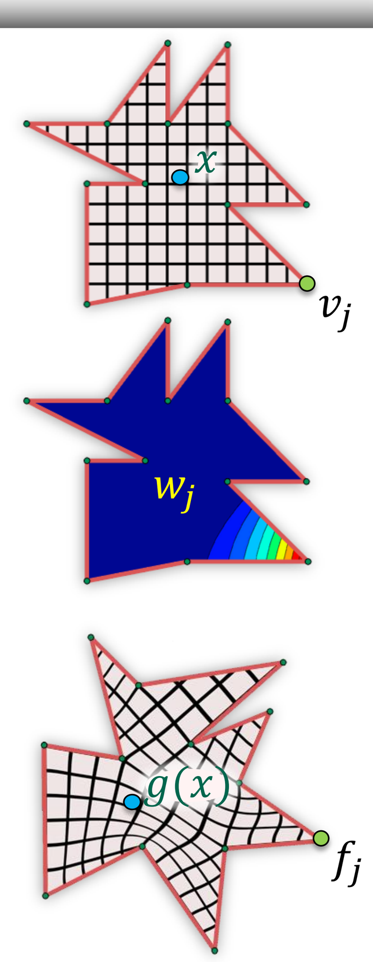
Introduction

Barycentric coordinates provide a convenient way to deform shapes or to pose characters. **Harmonic coordinates** in particular, have a long list of desirable properties. We derive a natural generalization called **biharmonic coordinates**, with the additional ability to interpolate boundary derivative data.

Deformation with barycentric coordinates

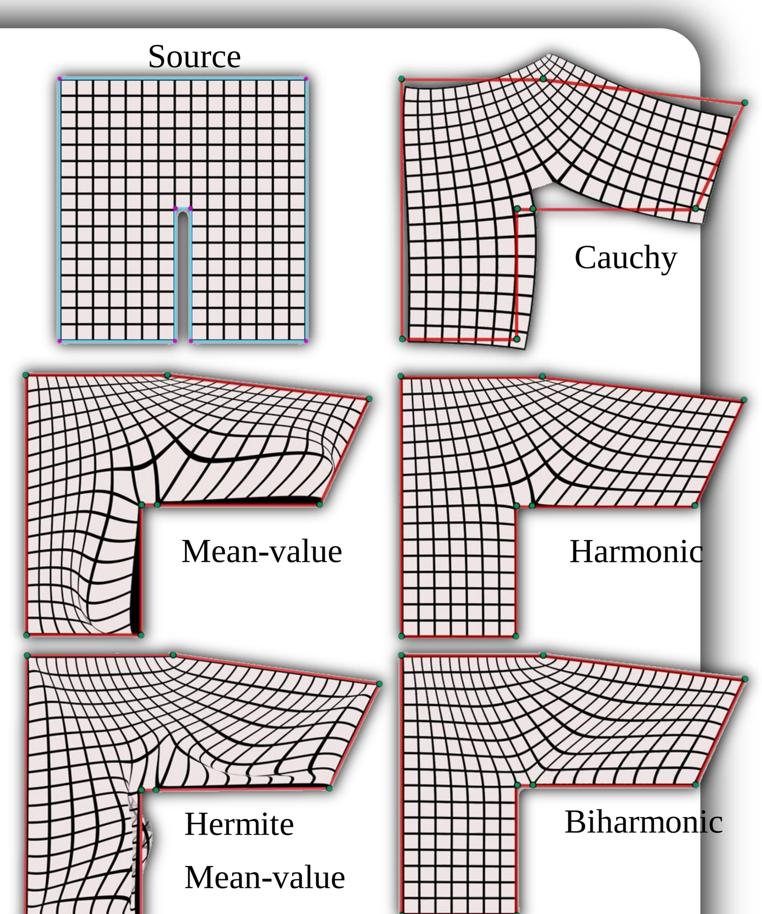
- Enclose a region in space by a polygon (the "cage")
- For each cage vertex v_j, define a basis function $w_j(x)$
- Manipulates the cage vertices v_j to new positions f_j
- The deformation function is computed as a linear combinations of the basis functions with the new positions of the cage vertices:

$$g(x) = \sum_{j=1}^{n} w_j(x)$$



Second order barycentric coordinates

Different coordinates have different attributes, and sometimes an important trait comes at the expense of the other. One of our goals was to devise **smooth** and **positive** coordinates that **interpolate** the boundary and generates maps that are **conformal** on the boundary



The boundary integrals

Green's third identity for the bi-Laplacian

$$f(x) = \int_{\partial\Omega} f(x')\frac{\partial G^H}{\partial n} - G^H\frac{\partial f(x')}{\partial n}ds + \int_{\partial\Omega} \Delta f(x')\frac{\partial G^{BH}}{\partial n} - G^{BH}\frac{\partial \Delta f(x')}{\partial n}ds$$

$$\Delta f(x) = \int_{\partial\Omega} \Delta f(x')\frac{\partial G^H}{\partial n} - G^H\frac{\partial \Delta f(x')}{\partial n}\,ds$$

The discretization

We set f and Δf to be linear, and $\partial\Delta f/\partial n$ and $\partial f/\partial n$ to be constant on each edge. Then Green's identity can be integrated and terms combined to get the BEM counterpart:

$$f(x) = \sum_{j=1}^{m} \phi_j^H(x)f_j + \psi^H(x)d_j + \phi_j^{BH}(x)l_j + \psi_j^{BH}(x)k_j$$

$$\Delta f(x) = \sum_{j=1}^{m} \phi_j^H(x)l_j + \psi_j^H(x)k_j$$

where $f_j = f(v_j)$, $l_j = \Delta f(v_j)$ and $d_j = \partial f(x')/\partial n$, $k_j = \partial\Delta f(x')/\partial n$ for $x' \in e_j$.

The coordinates

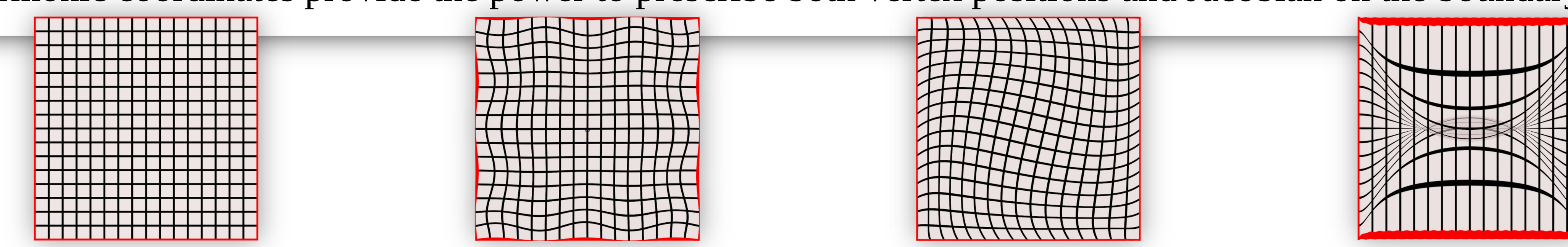
We write the BEM equation in matrix form and eliminate k and l to get **second order barycentric coordinates**:

$$f(x) = \alpha(x)f + \beta(x)d$$

Where f and d are n-vectors of vertex values and normal derivatives on each edge.

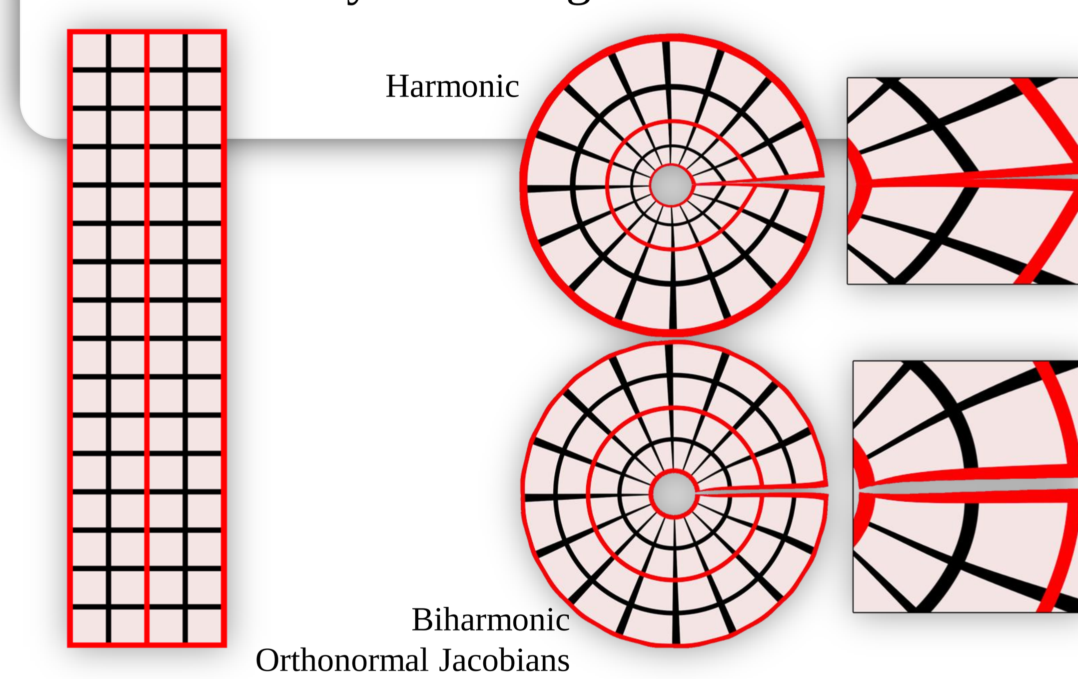
Shape Deformation

In contrast to traditional barycentric coordinates, which allow only the prescription of boundary values, biharmonic coordinates provide the power to prescribe both vertex positions and Jacobian on the boundary.



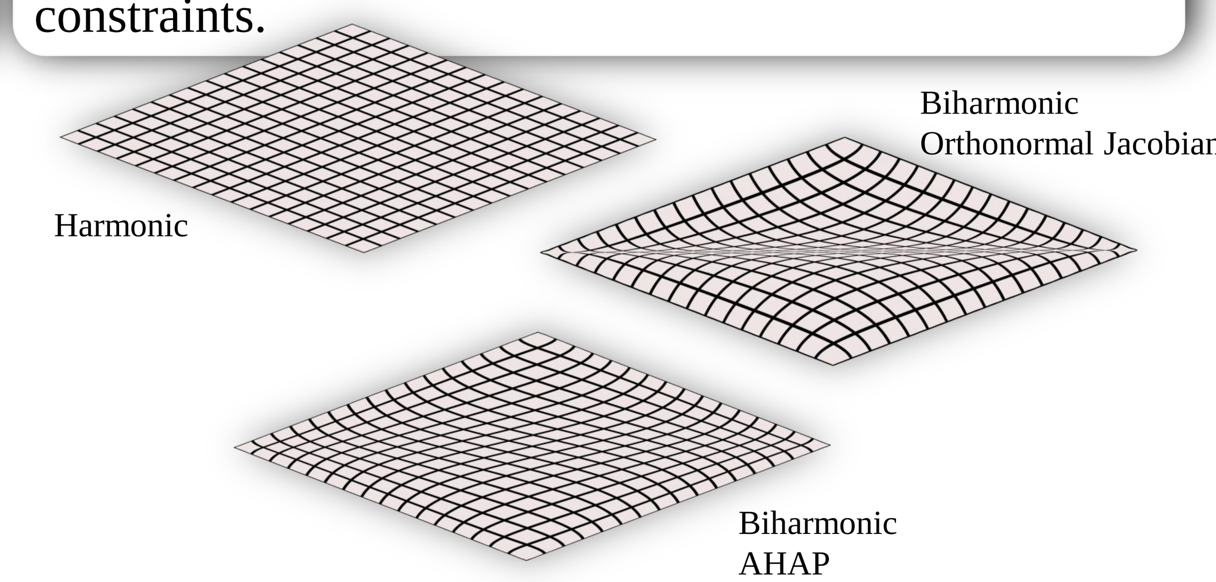
Orthogonal boundary Jacobians

A natural choice is to set the normal derivative vector to be perpendicular to the target edge. The Jacobians on the boundary are orthogonal but not orthonormal.



As-harmonic-as-possible

Since we consider harmonic coordinates to be the preferred choice for deformation among all real coordinates, we search for the **As-Harmonic-As-Possible (AHAP)** map that satisfies the user constraints.

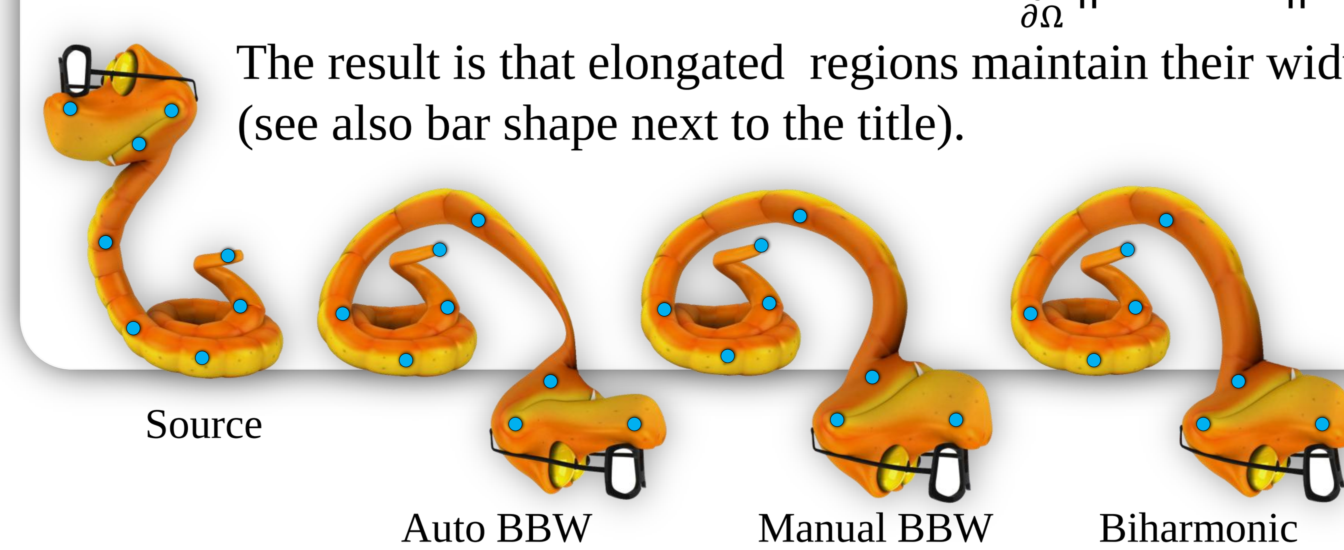


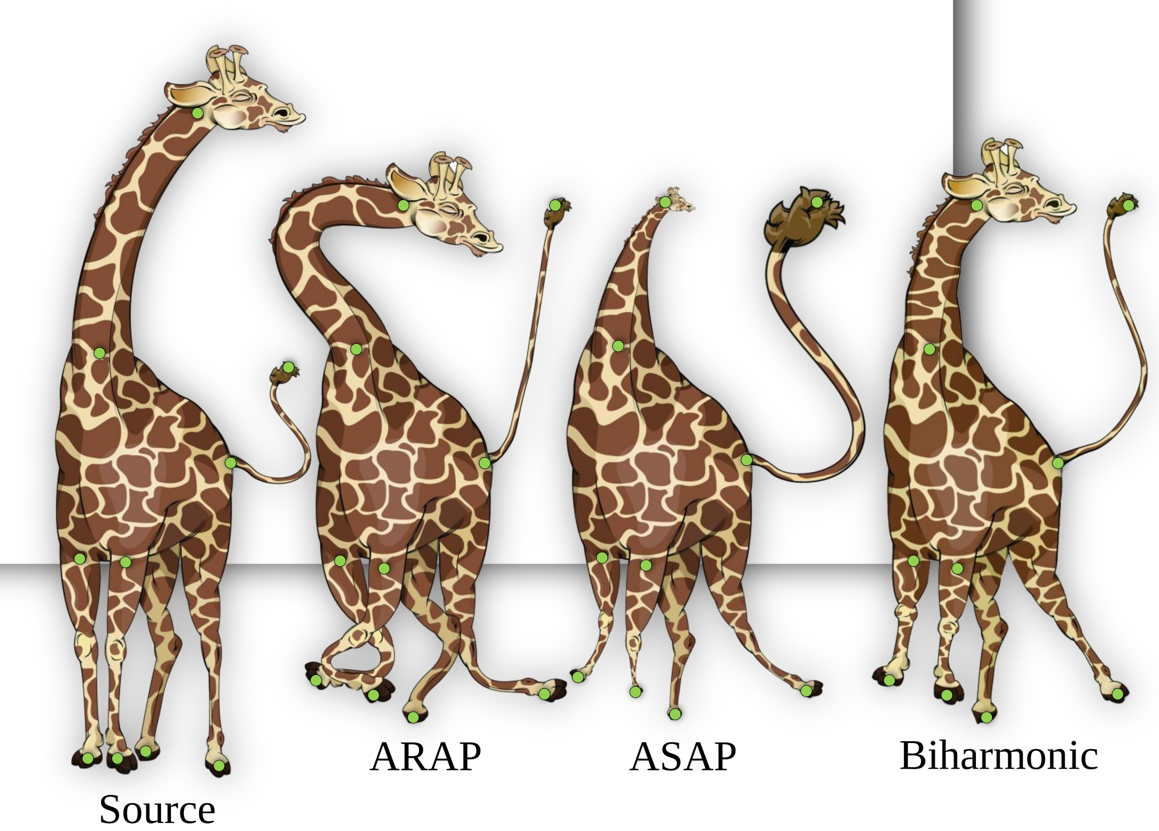
Thickness preservation

Cage-based deformation can be tedious. To simplify interaction we use the **point-to-point** deformation metaphor. The user selects a small number of **control points** and repositions them. Behind the scenes, an invisible cage that satisfies the positional constraints is found. To regularize the cage, we set the normal derivative vectors to be the unit normal vectors to the target edges and minimize the energy:

$$E^{thickness}(f) = \int_{\partial\Omega} \left\| \frac{\partial^2}{\partial n^2} f(x') \right\|^2 ds$$

The result is that elongated regions maintain their width (see also bar shape next to the title).





Reference Biharmonic Coordinates. Ofir Weber, Roi Poranne and Craig Gotsman. *Computer Graphics Forum 31, 6 (2012)*.

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