Anisotropic Voronoi diagrams from distance graphs

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Abstract

We present a new type of an anisotropic Voronoi diagram, constructed from a distance graph, which is a set of distances between given points. Our anisotropic Voronoi diagram is a generalization of the Euclidean Voronoi diagram, using an anisotropic metric, which approximates a given distance graph best in the sense of least squares.
The anisotropic metric is based on a 2-dimensional, continuous one-to-one embedding into $\mathbb{R}^m$ for $m \geq 2$. This embedding is constructed from the distance graph via a fitting procedure which is based on the Gauss-Newton algorithm.

Mathematical Background

Voronoi diagram (cf. [1])
Let $m \in \mathbb{Z}_+$ with $m \geq 2$ and let $\mathbf{P} = \{\mathbf{p}_1, \mathbf{p}_2, \ldots\}$ with $\mathbf{p}_i \in \mathbb{R}^m$. Let D be a metric on $\mathbb{R}^m$. Then we define the Voronoi cell $V_D^i(\mathbf{P})$ of the point $\mathbf{p}_i \in \mathbf{P}$ as follows

$$V_D^i(\mathbf{P}) = \{\mathbf{p} \in \mathbb{R}^m \ : \ D(\mathbf{p}, \mathbf{p}_i) < D(\mathbf{p}, \mathbf{p}_j) \text{ for all } j \neq i\}.$$

Then the **Voronoi diagram** $V_D(\mathbf{P})$ is given by

$$V_D(\mathbf{P}) = \mathbb{R}^m \setminus (\cup_i V_D^i(\mathbf{P})).$$

We denote the Voronoi diagram using the Euclidian metric by $V(\mathbf{P})$. We call a Voronoi diagram orphan-free if each Voronoi cell is connected.

Anisotropic metric framework

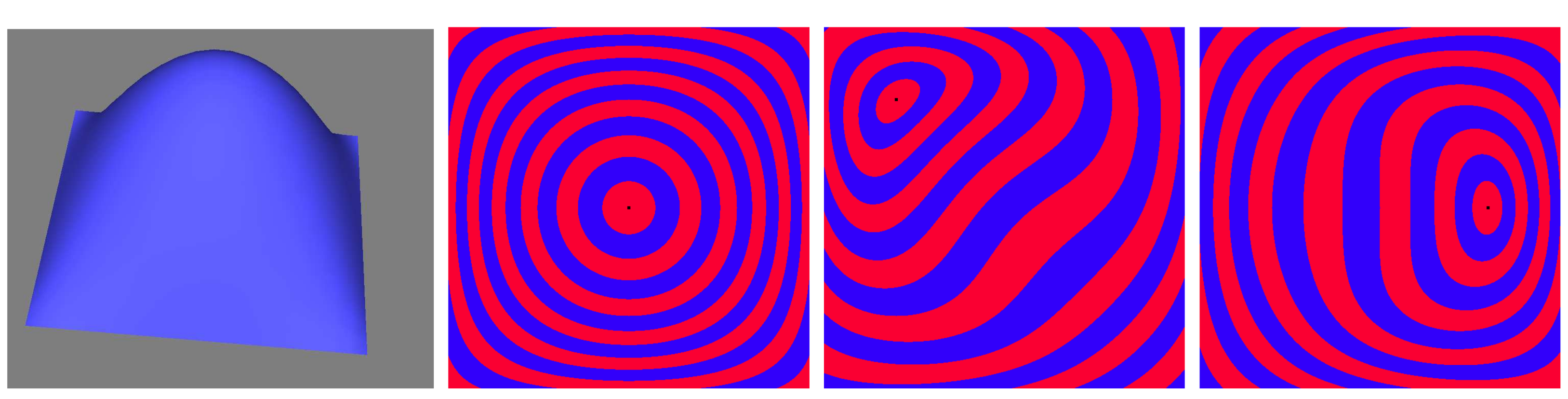
Let $m \in \mathbb{Z}_+$ with $m \geq 2$ and let $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^m$ be a continuous one-to-one embedding with $\mathbf{x}(u, v) = (x_1(u, v), \ldots, x_m(u, v))$. Let $d(r)$ for $r \geq 0$ be a scalar-valued function with the following properties

- $d(0) = 0$,
- $d'(r) \geq 0$ for $r \geq 0$,
- $d(r) > 0$ for $r > 0$,
- $\frac{d(r)}{r}$ for $r \geq 0$ is monotonic decreasing.

Then we define the distance $D : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ between two points $\mathbf{u}_1 = (u_1, v_1) \in \mathbb{R}^2$ and $\mathbf{u}_2 = (u_2, v_2) \in \mathbb{R}^2$ as follows

$$D(\mathbf{u}_1, \mathbf{u}_2) = d(||\mathbf{x}(u_1, v_1) - \mathbf{x}(u_2, v_2)||). \qquad (1)$$

Lemma
The distance D, given by (1), defines a metric on $\mathbb{R}^2$.



Example of an embedding into $\mathbb{R}^3$ and the induced anisotropic metric visualized by generalized circles.

Anisotropic graph fitting

Distance graph
Let $n \in \mathbb{Z}^+$, let $\mathbf{I} = [0, 1]^2$ and let $\mathbf{Q} = \{\mathbf{q}_1, \ldots, \mathbf{q}_n\}$ be a set of points in $\mathbf{I}$. In addition, let $\mathbf{Y}$ be a set of double indices

$$\mathbf{Y} = \{(y, z)\} \subset \{1, \ldots, n\}^2,$$

satisfying $y < z$ for all $(y, z) \in \mathbf{Y}$. Then a **distance graph** $\mathcal{G}$ is given by the points of $\mathbf{Q}$ and by the edges $\mathbf{e}_{y,z} = (\mathbf{q}_y, \mathbf{q}_z)$ for $(y, z) \in \mathbf{Y}$ with assigned lengths $l_{y,z}$. The lengths $l_{y,z}$ are not the real lengths of the corresponding edges $\mathbf{e}_{y,z}$ in $\mathbf{I}$, rather appropriate distances between the points $\mathbf{q}_y$ and $\mathbf{q}_z$, fulfilling the triangle inequality for existing triangles in the distance graph $\mathcal{G}$.

Goal: Construction of a continuous one-to-one embedding $\mathbf{x}(u, v)$, given by a B-spline surface of degree (p_1, p_2), i.e.

$$\mathbf{x}(u, v) = \sum_{j=0}^{n_1} \sum_{k=0}^{n_2} \mathbf{c}_{j,k} M_j^{p_1}(u) N_k^{p_2}(v)$$

with $\mathbf{c}_{j,k} \in \mathbb{R}^m$, which approximates a given distance graph best in the sense of least squares. For simplicity we choose $d(r) = r$.

Construction of the embedding $\mathbf{x}(u, v)$
We compute the unknown coefficients $\mathbf{c} = (\mathbf{c}_{0,0}, \ldots)$ by solving the minimization problem

$$\mathbf{c} = \arg\min \sum_{(y,z) \in \mathbf{Y}} \underbrace{||\mathbf{x}(\mathbf{q}_y) - \mathbf{x}(\mathbf{q}_z)||^2 - l_{y,z}^2}_{R_{y,z}(\mathbf{c})^2}.$$

We solve this non-linear optimization problem by using the Gauss-Newton algorithm, which minimizes in each iteration step the following objective function

$$\Big(\sum_{(y,z) \in \mathbf{Y}} (R_{y,z}(\mathbf{c}^0) + \nabla R_{y,z}(\mathbf{c}^0)(\Delta \mathbf{c} - \mathbf{c}^0))^2 \Big) + \omega ||\Delta \mathbf{c} - \mathbf{c}^0||^2$$

with respect to $\Delta \mathbf{c}$, where $\mathbf{c}^0$ denotes the solution from the last step, $\Delta \mathbf{c}$ the update, $\nabla R_{y,z}$ is the row vector given by the partial derivatives of $R_{y,z}$ with respect to the control points and $\omega > 0$ is the parameter for the Tikhonov regularization term.

Anisotropic Voronoi diagram computation

By using an anisotropic metric D, given by (1), we can construct an anisotropic Voronoi diagram $V_D(\mathbf{P})$ in $\mathbb{R}^2$.

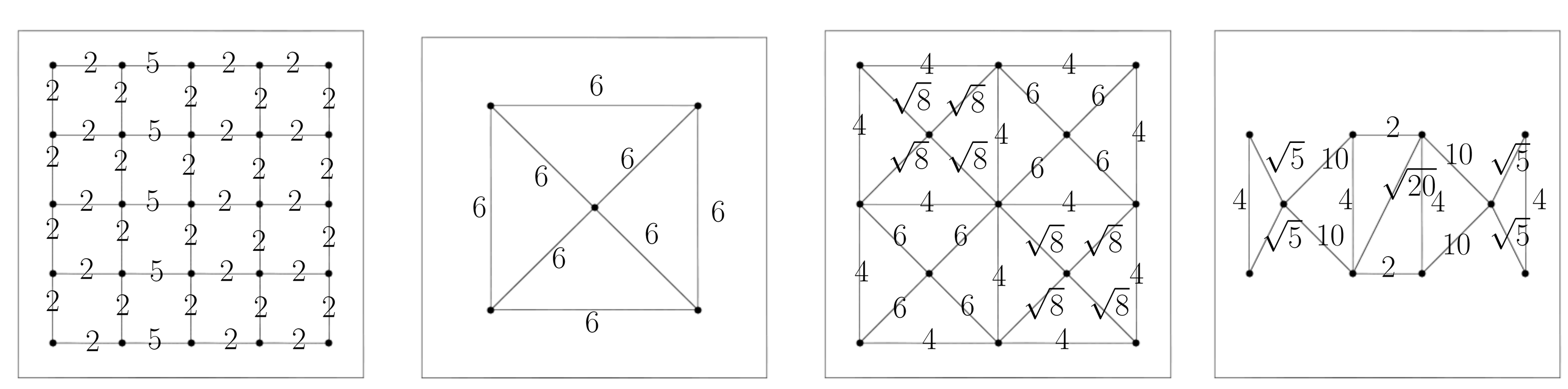
Computation of the anisotropic Voronoi diagram $V_D(\mathbf{P})$

- Given the points $\mathbf{P} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots\}$ with $\mathbf{u}_i \in \mathbb{R}^2$, we first compute the corresponding points $\mathbf{x}_i = \mathbf{x}(\mathbf{u}_i)$.
- Then we construct for the set of points $\mathbf{P}_\mathbf{x} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$ an Euclidean Voronoi diagram $V(\mathbf{P}_\mathbf{x})$ in $\mathbb{R}^m$.
- By intersecting the resulting Voronoi cells with $\mathbf{x}(u, v)$ we obtain a Voronoi diagram on $\mathbf{x}(u, v)$, which defines for the corresponding parameter values $(u, v) \in \mathbb{R}^2$ the anisotropic Voronoi diagram $V_D(\mathbf{P})$ in $\mathbb{R}^2$.

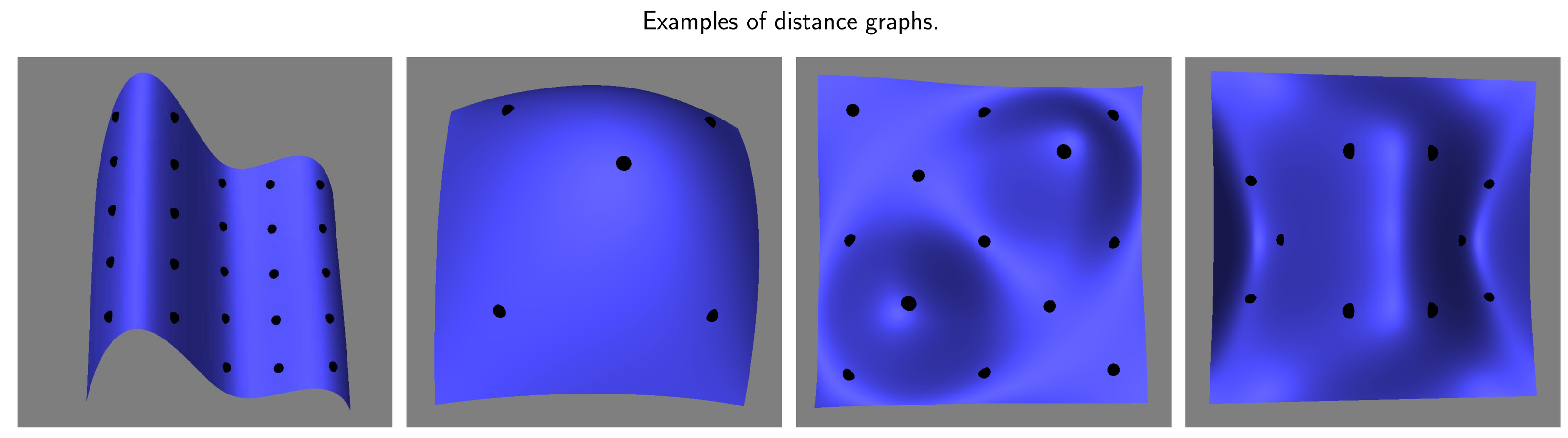
Lemma
Let $m = 2$. The anisotropic Voronoi diagram $V_D(\mathbf{P})$ is orphan-free.

Lemma
Let $m = 3$ and let $\mathbf{x}(u, v)$ be also C^2-smooth. If the set of points $\mathbf{P}_\mathbf{x} = \{\mathbf{x}(\mathbf{u}_1), \mathbf{x}(\mathbf{u}_2), \ldots\}$ is a 0.18-sample of $\mathbf{X} = \mathbf{x}(\mathbb{R}^2)$, then the resulting anisotropic Voronoi diagram $V_D(\mathbf{P})$ is orphan-free.

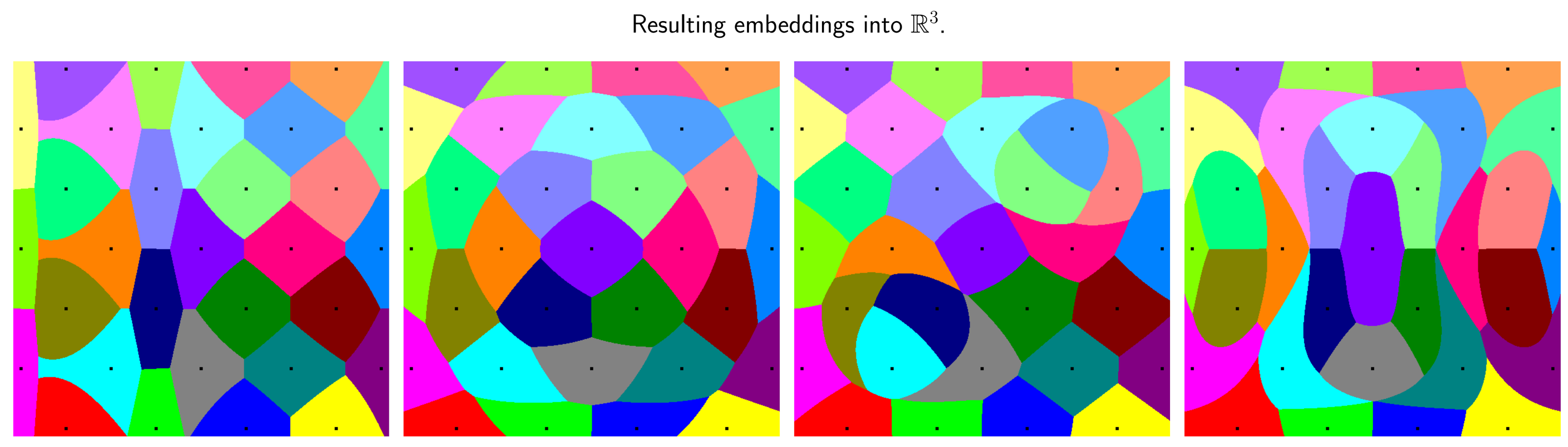
Examples



Examples of distance graphs.



Resulting embeddings into $\mathbb{R}^3$.



Resulting Voronoi diagrams for a set of points $\mathbf{P}$.

References

[1] Franz Aurenhammer and Rolf Klein. Voronoi diagrams. In *Handbook of computational geometry*, pages 201–290. North-Holland, Amsterdam, 2000.