Connection between DEC and FEEC

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Basic Idea

- Lowest order FEEC compared with DEC
- Mass matrix in FEEC replaced by DEC Hodge star
- Experiments with scalar and vector Laplacians, and harmonic forms

Finite Element Exterior Calculus

L^2 de Rham Complex Examples

$$n=3:\quad \begin{array}{l} 0 \longrightarrow H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2 \longrightarrow 0 \\ 0 \longleftarrow L^2 \xleftarrow[-\text{div}]{} \mathring{H}(\text{div}) \xleftarrow[\text{curl}]{} \mathring{H}(\text{curl}) \xleftarrow[-\text{grad}]{} \mathring{H}^1 \longleftarrow 0 \end{array}$$

$$n=2:\quad \begin{array}{l} 0 \longrightarrow H^1 \xrightarrow{\text{grad}} H(\text{rot}) \xrightarrow{\text{rot}} L^2 \longrightarrow 0 \\ 0 \longleftarrow L^2 \xleftarrow[-\text{div}]{} \mathring{H}(\text{div}) \xleftarrow[\text{curl}]{} \mathring{H}(\text{curl}) \longleftarrow 0 \end{array}$$

Poisson's Equation

Mixed formulation: Given f, find (σ, u, p) such that:

$$\sigma = \delta u\,, \quad \mathrm{d}\,\sigma + \delta\,\mathrm{d}\,u = f - p \text{ in } \Omega\,,$$
$$\operatorname{tr} * u = 0\,, \quad \operatorname{tr} * \mathrm{d}\,u = 0\,, \text{ on } \partial\Omega$$
$$u \perp \text{harmonics}\,.$$

Weak mixed formulation:

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle \mathrm{d}\,\tau, u \rangle &= 0\,, \\ \langle \mathrm{d}\,\sigma, v \rangle + \langle \mathrm{d}\,u, \mathrm{d}\,v \rangle + \langle v, p \rangle &= 0\,, \\ \langle u, q \rangle &= 0\,, \end{aligned}$$

where q is in discrete harmonics.

FEM Matrix Form

Find (σ_c, u_c) such that:

$$\begin{bmatrix} * & -\mathrm{d}^T * \\ * \mathrm{d} & \mathrm{d}^T * \mathrm{d} \end{bmatrix} \begin{bmatrix} \sigma_c \\ u_c \end{bmatrix} = \begin{bmatrix} 0 \\ *(f_c - p_c) \end{bmatrix}$$

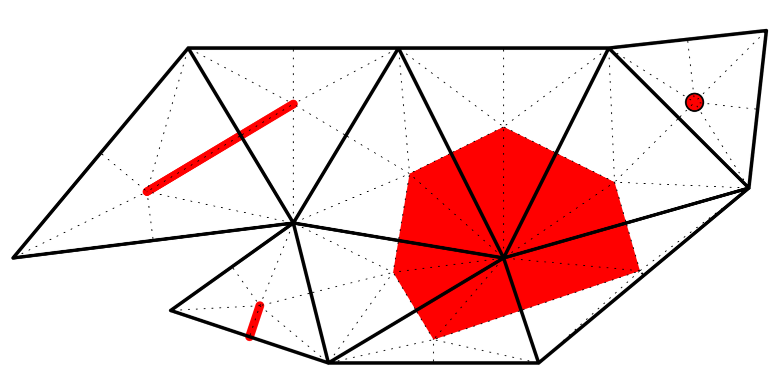
subject to: $Q^T * u_c = 0$

where p_c is the harmonic part of f and Q is a basis for discrete harmonics.

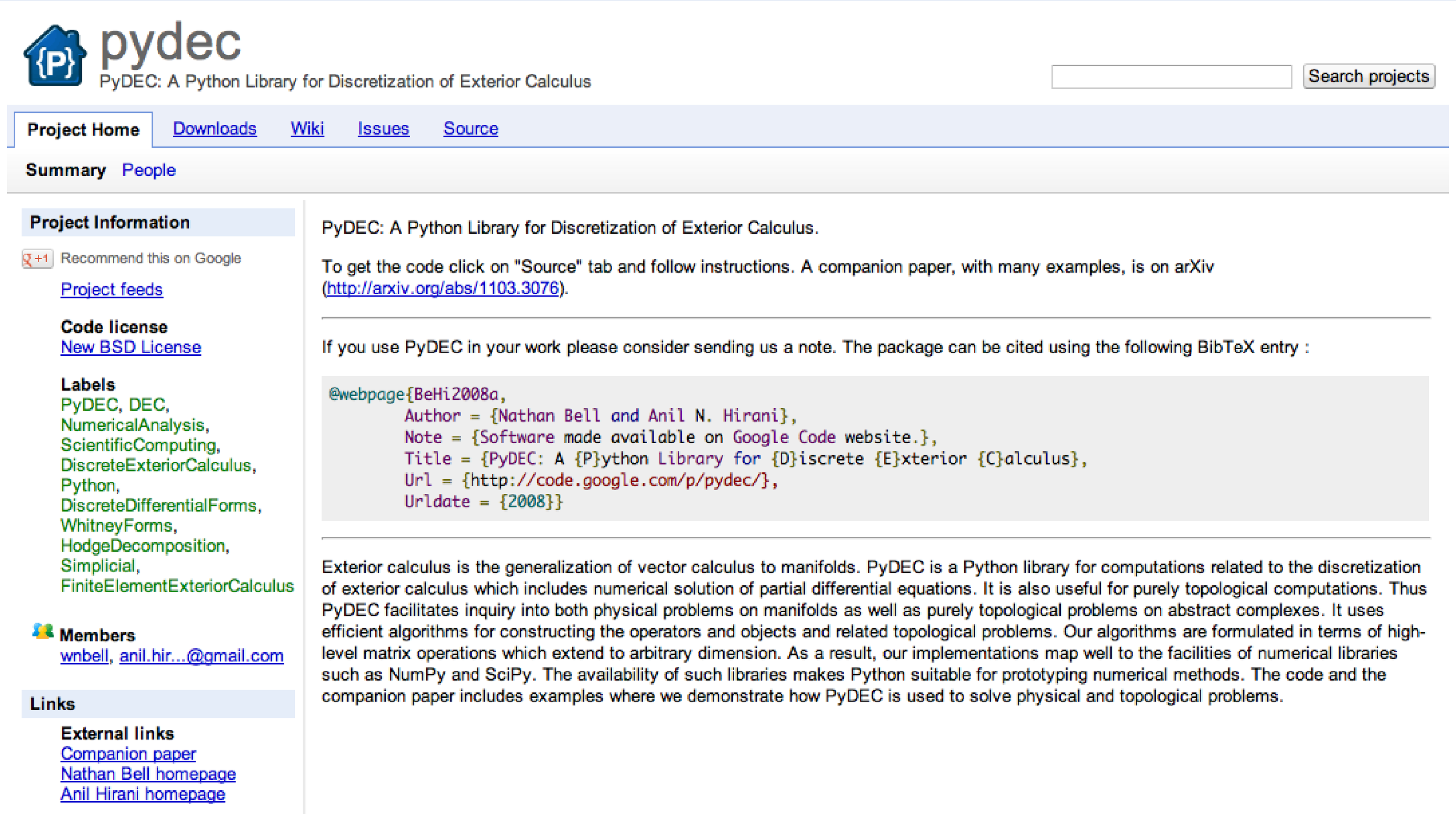
Discrete Exterior Calculus

Primal-Dual Hodge

$*_p$ is a diagonal matrix with $*_p(i,i) = \operatorname{vol} \star\sigma_i / \operatorname{vol} \sigma_i$ where σ_i is a p-simplex and $\star\sigma_i$ is its dual.



PyDEC Software


pydec
PyDEC: A Python Library for Discretization of Exterior Calculus
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Members
wnbell, anil.hir...@gmail.com
Links
External links
Companion paper
Nathan Bell homepage
Anil Hirani homepage

PyDEC: A Python Library for Discretization of Exterior Calculus.

To get the code click on "Source" tab and follow instructions. A companion paper, with many examples, is on arXiv (http://arxiv.org/abs/1103.3076).

If you use PyDEC in your work please consider sending us a note. The package can be cited using the following BibTeX entry :

```
@webpage{BeHi2008a,
        Author = {Nathan Bell and Anil N. Hirani},
        Note = {Software made available on Google Code website.},
        Title = {PyDEC: A {P}ython Library for {D}iscrete {E}xterior {C}alculus},
        Url = {http://code.google.com/p/pydec/},
        Urldate = {2008}}
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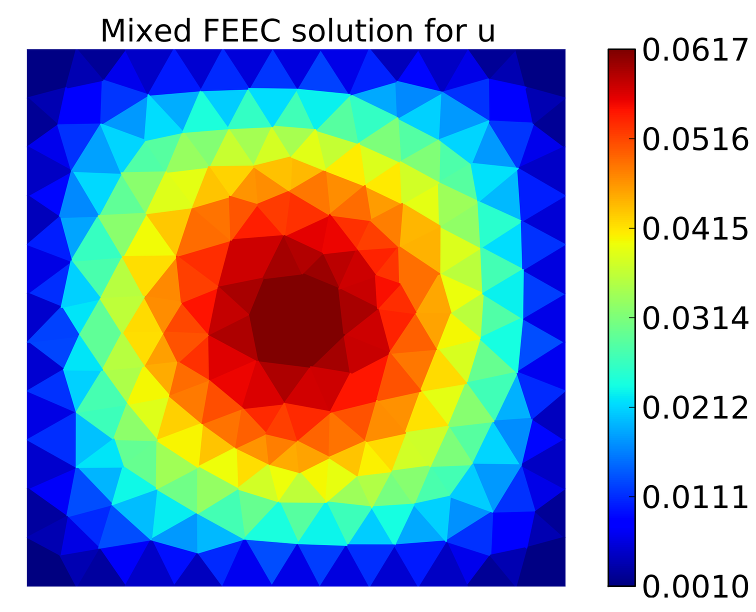
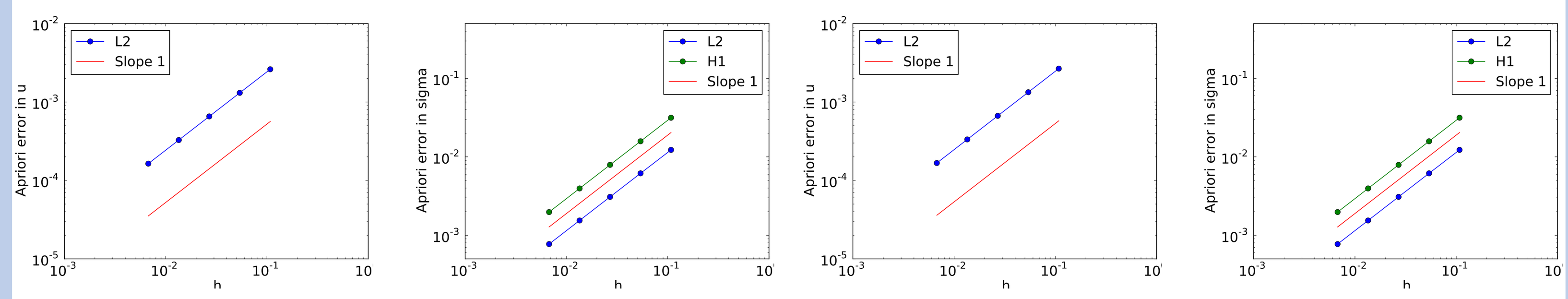
Exterior calculus is the generalization of vector calculus to manifolds. PyDEC is a Python library for computations related to the discretization of exterior calculus which includes numerical solution of partial differential equations. It is also useful for purely topological computations. Thus PyDEC facilitates inquiry into both physical problems on manifolds as well as purely topological problems on abstract complexes. It uses efficient algorithms for constructing the operators and objects and related topological problems. Our algorithms are formulated in terms of high-level matrix operations which extend to arbitrary dimension. As a result, our implementations map well to the facilities of numerical libraries such as NumPy and SciPy. The availability of such libraries makes Python suitable for prototyping numerical methods. The code and the companion paper includes examples where we demonstrate how PyDEC is used to solve physical and topological problems.

Nathan Bell and Anil N. Hirani. PyDEC: Algorithms and software for Discretization of Exterior Calculus. *ACM Transactions on Mathematical Software*, (To appear), 2012.

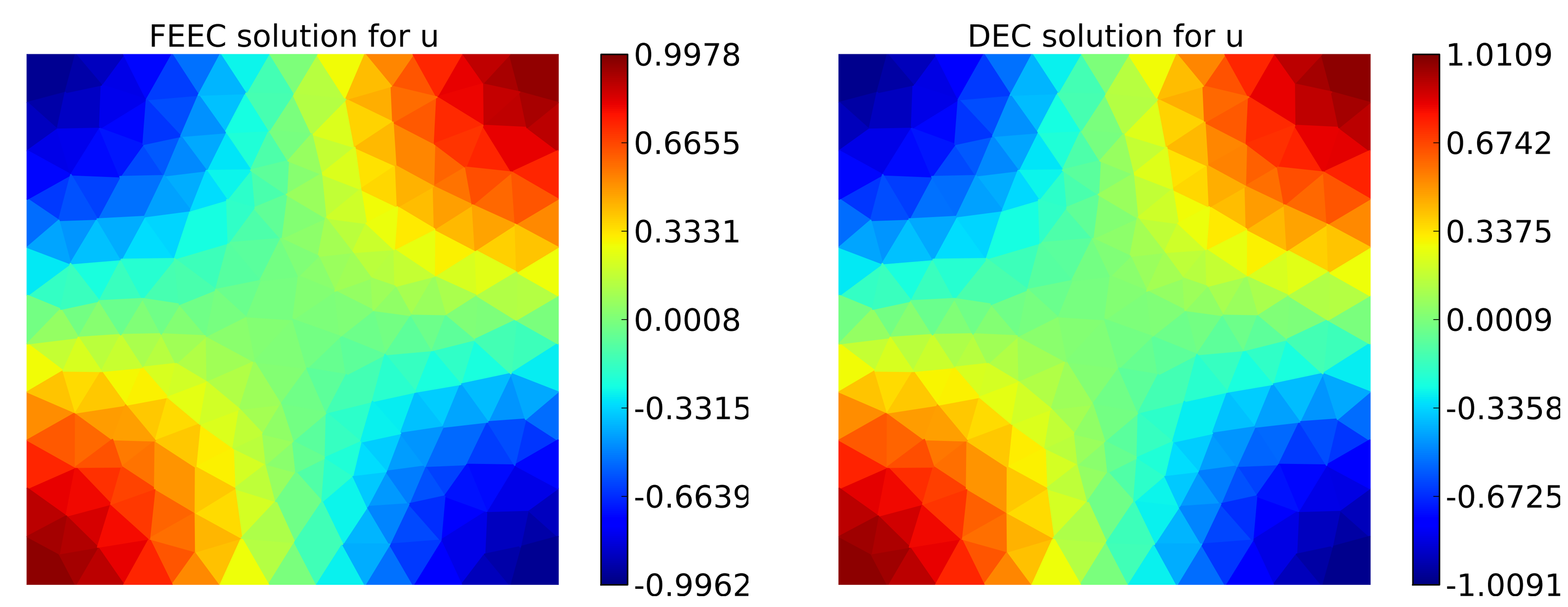
Experiments

Scalar Laplacian with Dirichlet Boundary Condition

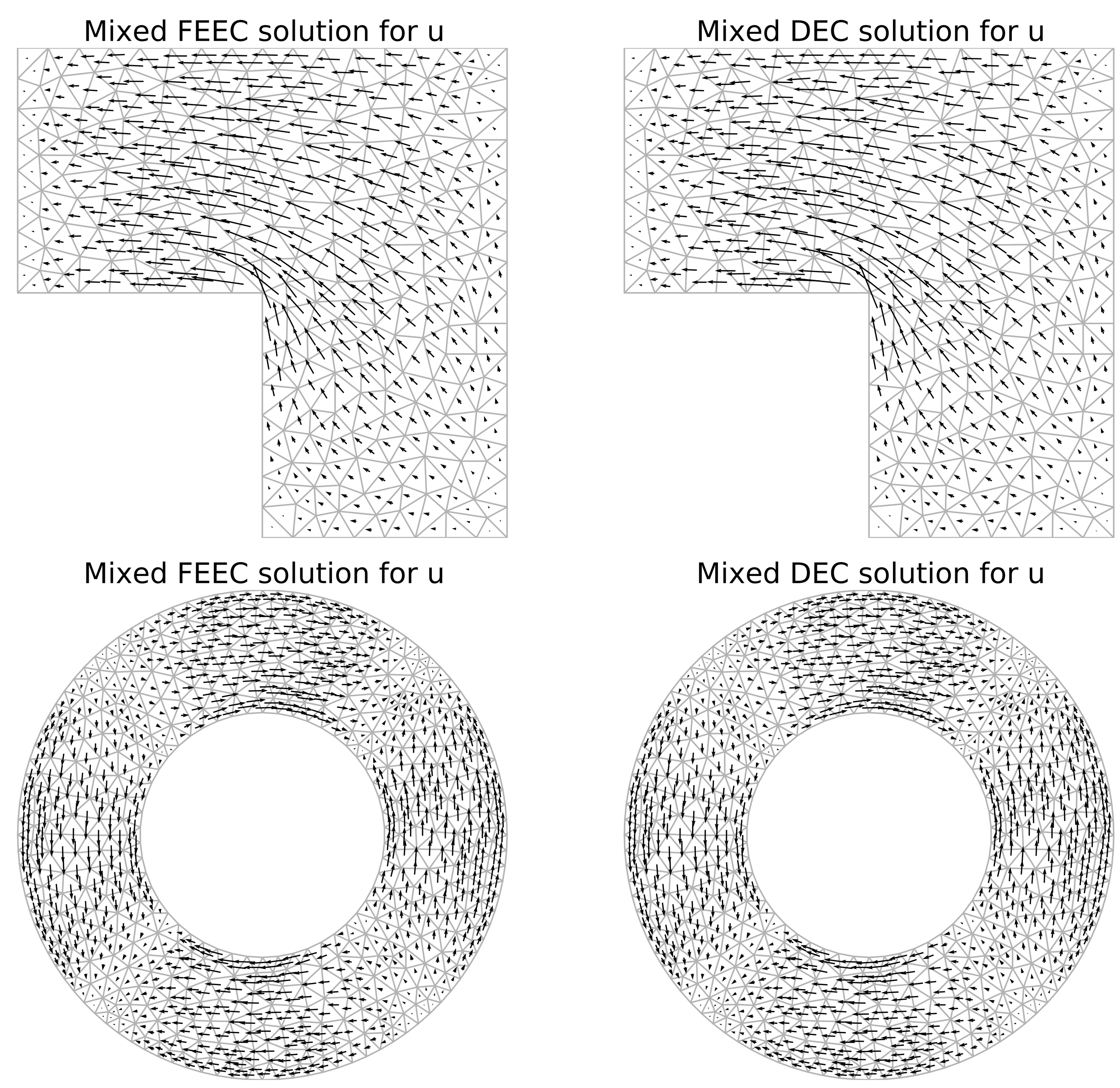




Scalar Laplacian with Neumann Boundary Condition



Vector Laplacian with Dirichlet and Neumann B.C.



Harmonic Forms