Algebraic Multigrid for fracture modeled with XFEM

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Introduction

This work deals with developing optimal iterative solvers for linear systems arising from XFEM.

- XFEM is an attractive approach to model discontinuities e.g. fracture
- Smooth basis functions are enriched with appropriate discontinuities
- AMG is an optimal solver/preconditioner for elliptic PDEs discretized with FEM
- However, AMG loses its optimality for XFEM
- We present approaches to adapt AMG/XFEM to retain convergence properties
- 3D cracks, material nonlinearity and parallel implementation are considered.

Algebraic Multigrid

- Iterative smoothers can rapidly damp high-frequency errors but are much less effective for low-frequency "smooth" errors.
- Multigrid uses a hierarchy of discretizations to attenuate errors across a wide spectrum of frequencies
- Algebraic Multigrid builds the hierarchy and grid-transfer operators from the graph of stiffness $\boldsymbol{A}$

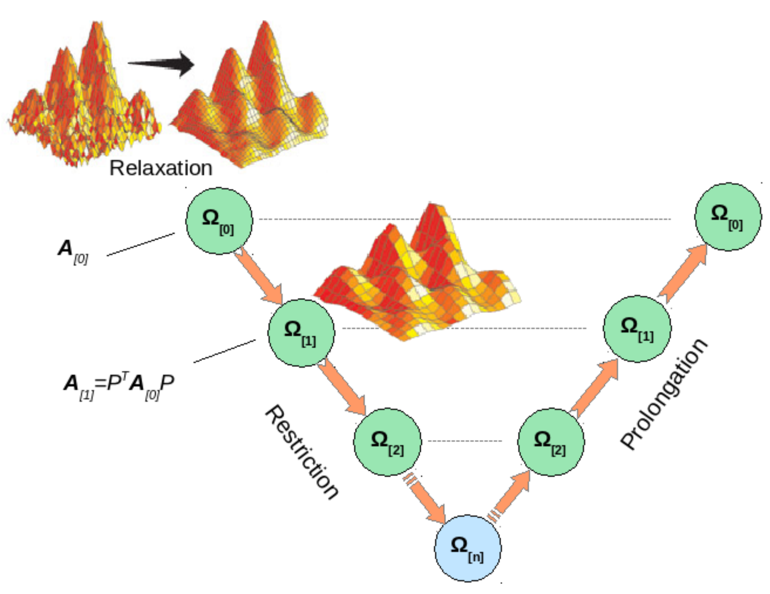


Figure: Multigrid V-cycle

Algebraic Multigrid for XFEM

Why does not "out-of-the-box" AMG work for XFEM ?

- XFEM matrix graph patterns do not follow cracks $\Longrightarrow$ non-conforming AMG aggregates.
- Coarse representation of discontinuities in AMG hierarchy not accurate.
- Additional low energy modes from fractured domains.
- Bad conditioning arising from tip singularity functions.
- XFEM block size is variable $\Longrightarrow$ non-standard AMG.

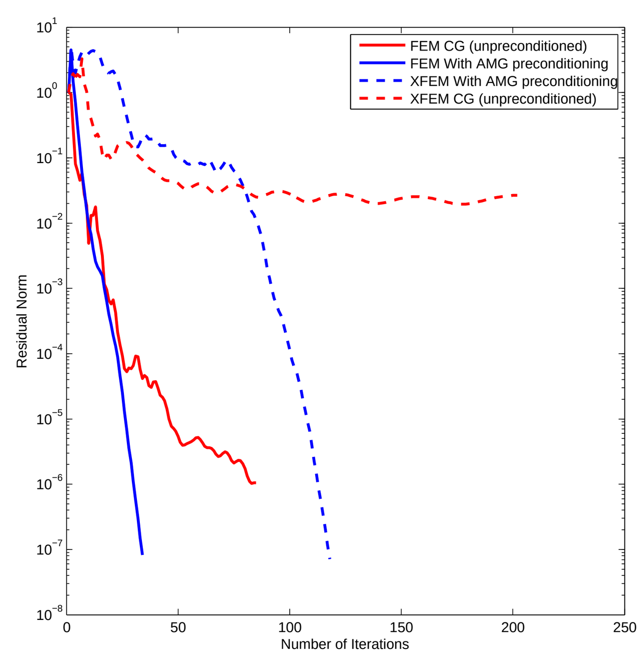


Figure: CG convergence

How to improve AMG performance for XFEM?

- *Schur complements*[1]: XFEM enrichments may condensed out.
- *Domain Decomposition*[2]: Geometrically partition domain to "cracked" and "healthy"
- *Phantom node representation*[3]: XFEM basis is transformed to decouple stiffness across discontinuities.

The last method is extended to 3D + nonlinear materials. MPI parallel implementation with *parFEAP*. Linked with Sandia's *Trilinos* framework.

Phantom node representation

Phantom node representation decouples the stiffness matrix terms across the crack resulting in conforming multigrid aggregates.

- Phantom node representation:

$$u^p = \sum N_I u_I + N_I \Psi_{I^a} u_{I^a} + N_I \Psi_{I^b} u_{I^b}$$

- Phantom basis functions:

$$\Psi^\alpha = \begin{cases} 1 \text{ in } \Omega_\alpha \\ 0 \text{ otherwise} \end{cases}$$

- Original XFEM:

$$u(\mathrm{x}) = \sum_{i=1}^{nel} N_i u_i + \sum_{i=1}^{nel} N_i \Psi a_i$$

- Modified shifted enrichment XFEM:

$$u(\mathrm{x}) = \sum_{i=1}^{nel} N_i u_i + \sum_{i=1}^{nel} N_i \frac{1}{2} \left(|\Psi - \Psi_i| \right) a_i$$

- The above enrichment form allows for an easy transformation from XFEM to phantom-node form.

Linear system transformation

A simple transformation exists for modified XFEM to phantom-node representation:

$$\left(\mathrm{G}^T \mathrm{A} \mathrm{G}\right)\left(\mathrm{G}^{-1} \boldsymbol{u}\right) = \left(\mathrm{G}^T \boldsymbol{f}\right)$$
$$\Longrightarrow \bar{\mathrm{A}} \bar{\boldsymbol{u}} = \bar{\boldsymbol{f}}$$

- The transformation matrix has a block-sparse structure - it is an identity matrix with off-diagonal "-1" terms corresponding to the Heaviside enrichments.

$$\mathrm{G} = \begin{bmatrix} 1 &&&&&&&& \\ & 1 &&&&&&& \\ && 1 &&&&&& \\ && -1 & 1 &&&&& \\ &&&& 1 &&&& \\ &&&& -1 & 1 &&& \\ &&&&&& 1 && \\ &&&&&&& 1 & \\ &&&&&&&& \ddots \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ a_3 \\ u_4 \\ a_4 \\ u_5 \\ u_6 \\ \vdots \end{Bmatrix}$$

- The original nullspace vectors also are represented in the transformed space:

$$\bar{N}_\Theta = \mathrm{G}^{-1} N_\Theta$$

- The original solution may be obtained from a simple back-transformation:

$$\boldsymbol{u} = \mathrm{G} \bar{\boldsymbol{u}}$$

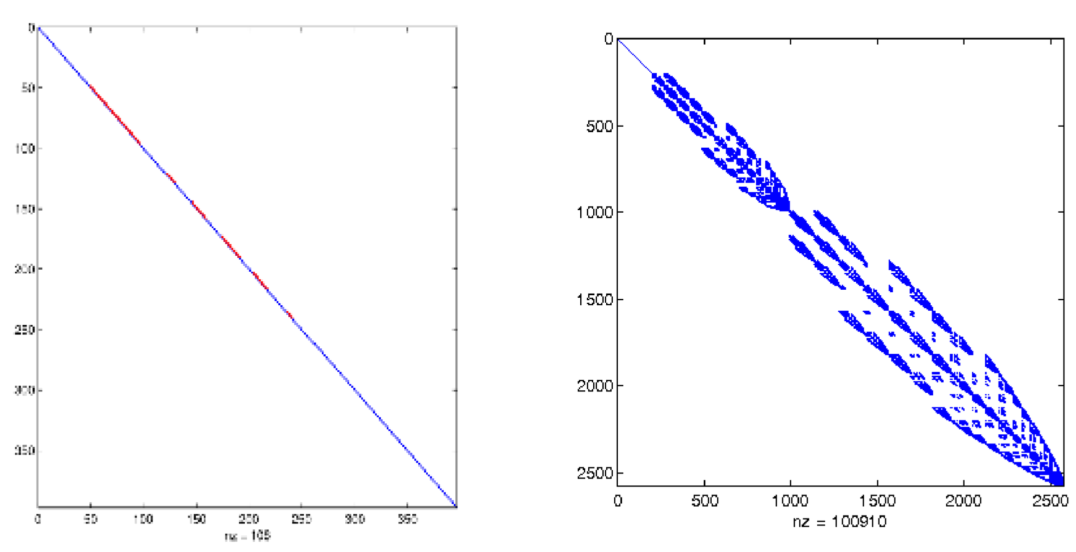
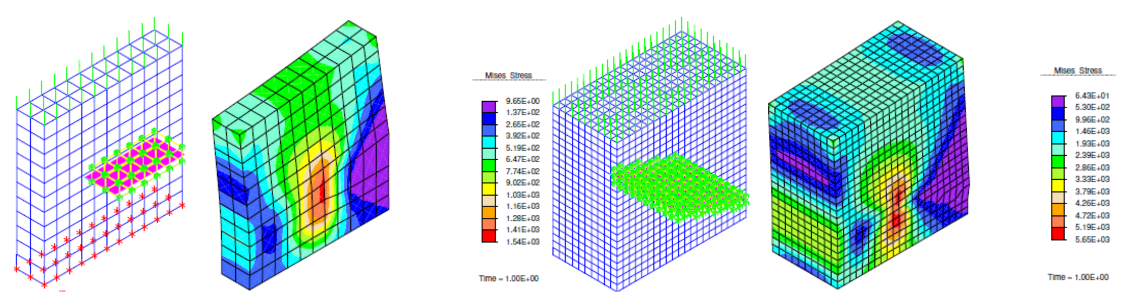


Figure: (a) G sparsity (b) $\mathrm{G}^T \boldsymbol{A} \mathrm{G}$ (c) AMG aggregates in 2D

Numerical Results

Linear Elastic Brittle Fracture:

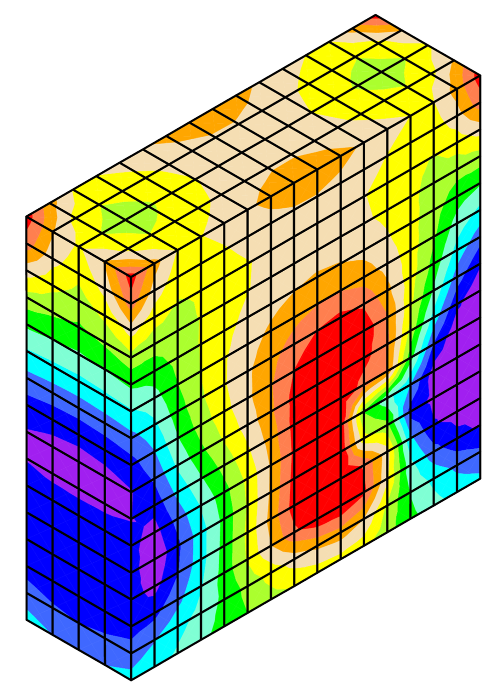
- AMG used as preconditioner to CG. Table shows number of iterations and time to reduce residual norm by a magnitude of $1e-7$.



Case	Direct	CG-brute		AMG		XAMG	
	time(s)	nits	time(s)	nits	time(s)	nits	time(s)
1	0.11	-	-	58	0.15	25	0.07
2	0.13	-	-	80	0.25	27	0.09
3	4.16	51	2.49	50	0.58	22	0.29
4	5.44	192	4.17	69	0.86	29	0.42
5	77.76	102	11.88	70	3.04	23	1.17
6	100.05	191	17.03	101	4.82	27	1.49
7	292.88	77	25.43	57	4.46	25	2.23
8	345.65	78	25.90	64	4.99	26	2.32
9	313.90	-	-	138	12.25	35	3.45

Fracture in Elasto-Plastic Material:

- In this case, each solve involves multiple sub-iterations (plits) to converge stress to yield-surface.



Load	CG-brute			AMG			XAMG		
step	plits	cgits	time	plits	cgits	time	plits	cgits	time
1	3	501	7.20	2	77	2.26	2	31	2.23
2	3	501	14.17	2	77	4.30	2	31	4.20
3	3	501	21.15	2	77	6.33	2	31	6.15
4	3	501	28.13	2	77	8.36	2	31	8.11
5	4	1006	39.07	4	153	12.34	4	59	11.94
6	10	1445	60.96	10	475	22.90	10	183	21.66
7	10	1767	84.53	10	499	33.64	10	192	31.44
8	12	3582	120.17	12	665	47.01	12	251	43.35
9	24	3181	171.29	24	774	69.52	23	306	64.70
10	26	3654	227.72	26	921	94.57	26	357	88.93

Summary & Conclusions

- AMG applied to transformed XFEM system works extremely well.
- This method does not require invasive modifications of AMG.
- Transformation involves additional expense, however relatively low because of simple block sparsity structure of $\boldsymbol{G}$.
- Sophisticated implementations need not explicitly compute and store $\boldsymbol{G}$, computations could be done on-the-fly.

References

[1] B. Hiriyur, R.S. Tuminaro, H. Waisman, E.G. Boman, D.E. Keyes, *A quasi-algebraic multigrid approach to fracture problems based on extended finite element methods*, SIAM SISC, [2012]

[2] L. Berger-Vergiat, H. Waisman, B. Hiriyur, R.S. Tuminaro, D.E. Keyes, *Inexact Schwarz-AMG preconditioners for crack problems modeled by extended finite element methods*, IJNME, [2012]

[3] A. Gerstenberger, R.S. Tuminaro, *An algebraic multigrid approach to solve XFEM-based fracture problems*, IJNME, [2012]

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