Upscaling for the Laplace problem using a Discontinuous Galerkin Method

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Problem

We consider the Laplace problem on the open unit square $\Omega = (0,1)^2$.

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

Meshes

We define two meshes on Ω: a coarse mesh $\mathcal{T}_H$ and a fine mesh $\mathcal{T}_h$. We use a $N \times N$ regular cartesian mesh of the unit square.

$$\mathcal{T}_H = \{K^i\}_{i=1}^{N^2}, \quad \bigcup_{i=1}^{N^2} K^i = \Omega$$

Each coarse cell K^i is subdivided with a $M \times M$ regular cartesian submesh.

$$\mathcal{T}_h^i = \{K_j^i\}_{j=1}^{M^2}, \quad \bigcup_{j=1}^{M^2} K_j^i = K^i$$

The fine mesh is the reunion of all submeshes.

$$\mathcal{T}_h = \bigcup_{i=1}^{N^2} \mathcal{T}_h^i$$

Polynomial spaces

We define four polynomial spaces. The coarse and the fine polynomial spaces V_H and V_h are classical Discontinuous Galerkin discretisation spaces.

$$V_H = \left\{ v \in L^2(\Omega) \;\middle|\; v|_{K^i} \in \mathbb{Q}_1(K^i)\ \forall K^i \in \mathcal{T}_H \right\}$$
$$V_h = \left\{ v \in L^2(\Omega) \;\middle|\; v|_{K_j^i} \in \mathbb{Q}_1(K_j^i)\ \forall K_j^i \in \mathcal{T}_h \right\}$$

The space $V_{H,h}^i$ contains functions of V_h with support in the coarse cell K^i satisfying a Dirichlet boundary condition on ∂K^i.

$$V_{H,h}^i = \left\{ v \in V_h \;\middle|\; \text{supp } v \subset K^i \text{ and } v|_{\partial K^i} = 0 \right\}$$

We set $V_{H,h} = \bigoplus_{i=1}^{N^2} V_{H,h}^i$. The upscaling discretisation space is then defined as the direct sum $V_{ups} = V_H \oplus V_{H,h}$. The following inclusions hold

$$V_H \subset V_{ups} \subset V_h.$$

Discontinuous Galerkin method

We note $\mathcal{F}_h^{int}$ and $\mathcal{F}_h^{ext}$ the set of internal and external edges of the fine mesh $\mathcal{T}_h$. We also note $\mathcal{F}_h = \mathcal{F}_h^{int} \cup \mathcal{F}_h^{ext}$ the set of all edges. If $u \in V_h$ and $v \in (V_h)^2$ are scalar and vector functions, the jump of u and the mean of v through an internal edge $e = \partial K \cap \partial J \in \mathcal{F}_h^{int}$ are defined as

$$[[u]]_e = u_K|_e - u_J|_e, \quad \{\{v\}\}_e = \frac{v_K|_e + v_J|_e}{2} \cdot n_K.$$

For an external edge $e = \partial K \cap \partial\Omega \in F_h^{ext}$, we set $[[u]]_e = u_K|_e$ and $\{\{v\}\} = v_K|_e \cdot n$. We use the Internal Penalty Discontinuous Galerkin bilinear form defined by Arnold et al. (2002), for $u, v \in V_h$ as

$$a_h(u,v) = \sum_{K \in \mathcal{T}_h} \int_K \nabla u \cdot \nabla v dx - \sum_{e \in \mathcal{F}_h} \int_e [[u]]\{\{\nabla v\}\} ds - \sum_{e \in \mathcal{F}_h} \int_e \{\{\nabla u\}\}[[v]] ds + \sum_{e \in \mathcal{F}_h} \int_e \gamma [[u]][[v]] ds.$$

We are lead to the following variational problem. Find $u \in V_{ups}$ satsifying

$$a_h(u,v) = L(v) = \int_\Omega f v dx \quad \forall v \in V_{ups}. \tag{2}$$

Upscaling algorithm

Following the idea presented in Arbogast et al. (1998), we seek the solution $u \in V_{ups}$ of (2) under the form $u = \bar{u} + \hat{u}$ with $\bar{u} \in V_H$ and $\hat{u} \in V_{H,h}$. Then, the variational formulation (2) is equivalent to

$$\begin{cases} a_h(\bar{u} + \hat{u}, \bar{v}) = L(\bar{v}) & \forall \bar{v} \in V_H \\ a_h(\bar{u} + \hat{u}, \hat{v}) = L(\hat{v}) & \forall \hat{v} \in V_{H,h}. \end{cases}$$

$$a_h(\hat{u}, \hat{v}) = L(\hat{v}) - a_h(\bar{u}, \hat{v}) \quad \forall \hat{v} \in V_{H,h}$$

We have $\hat{u} = \sum_{i=1}^{N^2} \hat{u}^i$ with $\hat{u}^i \in V_{H,h}^i$. We show that $a_h(\hat{\phi}^i, \hat{\phi}^j) = 0$ if $(\hat{\phi}^i, \hat{\phi}^j) \in V_{H,h}^i \times V_{H,h}^j$ with $i \neq j$, therefore we have

$$a_h(\hat{u}^i, \hat{v}^i) = L(\hat{v}^i) - a_h(\bar{u}, \hat{v}^i) \quad \forall \hat{v}^i \in V_{H,h}^i$$

This equation provides us $\hat{u}^i(\bar{u})$ and then $\hat{u}(\bar{u})$ in terms of $\bar{u}$. We eventualy obtain the upscaling system

$$\begin{cases} a_h(\bar{u} + \hat{u}(\bar{u}), \bar{v}) = & L(\bar{v}) & \forall \bar{v} \in V_H \\ a_h(\hat{u}^i, \hat{v}^i) = & L(\hat{v}^i) - a_h(\bar{u}, \hat{v}^i) & \forall \hat{v}^i \in V_{H,h}^i \end{cases}$$

Matricial form

Using the notation presented in Minkoff et al. (2006), we note K the stiffness matrix. The matricial form of the variational problem (2) is $KU = F$. We apply the decomposition

$$\begin{pmatrix} K_{cc} & K_{cf} \\ K_{cf}^T & K_{ff} \end{pmatrix} \begin{pmatrix} U_c \\ U_f \end{pmatrix} = \begin{pmatrix} F_c \\ F_f \end{pmatrix}$$

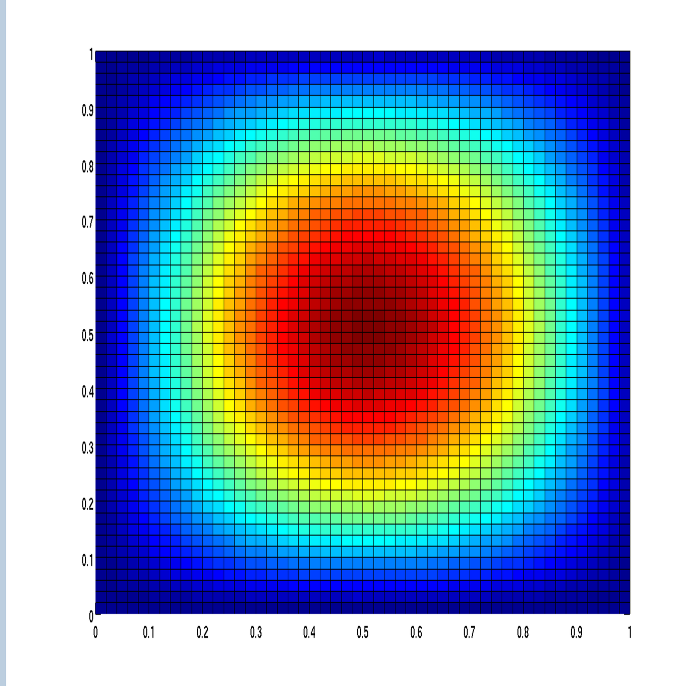
where K_{cc} and K_{ff} are the stiffness matrix on the spaces V_H and $V_{H,h}$. Using a Schur complement, we obtain

$$\begin{aligned} (K_{cc} - K_{cf}^T K_{ff}^{-1} K_{cf}) U_c &= F_c - K_{cf}^T K_{ff}^{-1} F_f \\ U_f &= K_{ff}^{-1}(F_f - K_{cf} U_c) \end{aligned}$$

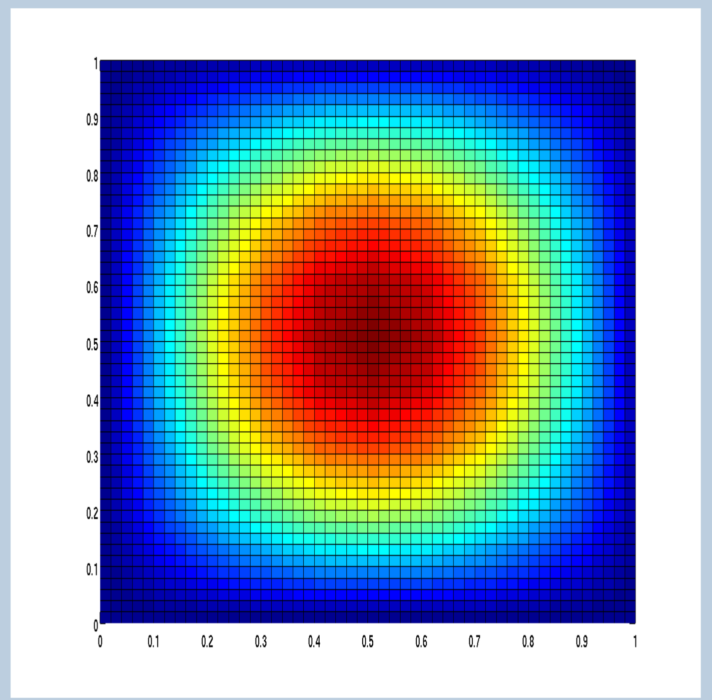
The matrix K_{ff} is block diagonal. The diagonal blocks are the matrices K_{ff}^i ($i = [|1, N^2|]$). Each block K_{ff}^i is associated with the space $V_{H,h}^i$.

Example ($N = 10$, $M = 5$)

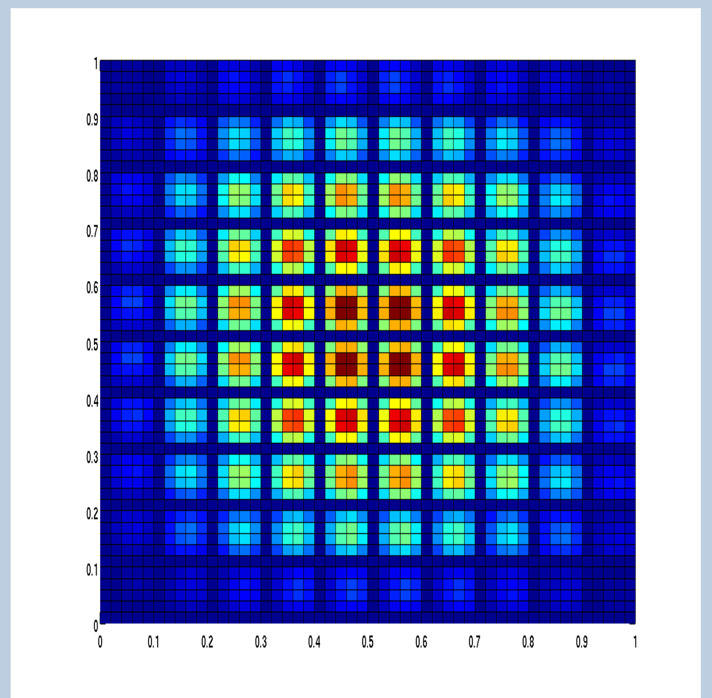
Full solution $u = \bar{u} + \hat{u}$



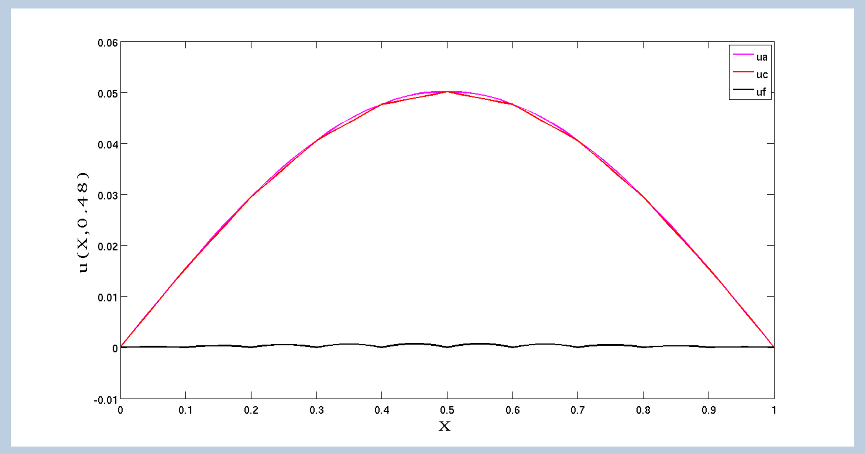
Coarse part $\bar{u}$



Fine part $\hat{u}$



Slice at $Y = 0.48$



Reference

H. Barucq, T. Chaumont-Frelet, J. Diaz, V. Peron. Upscaling for the Laplace problem using a Discontinuous Galerkin Method. *J. of Computational and Applied Mathematics*, to appear 2012.