A Geometric Discretization Scheme for Incompressible Elasticity

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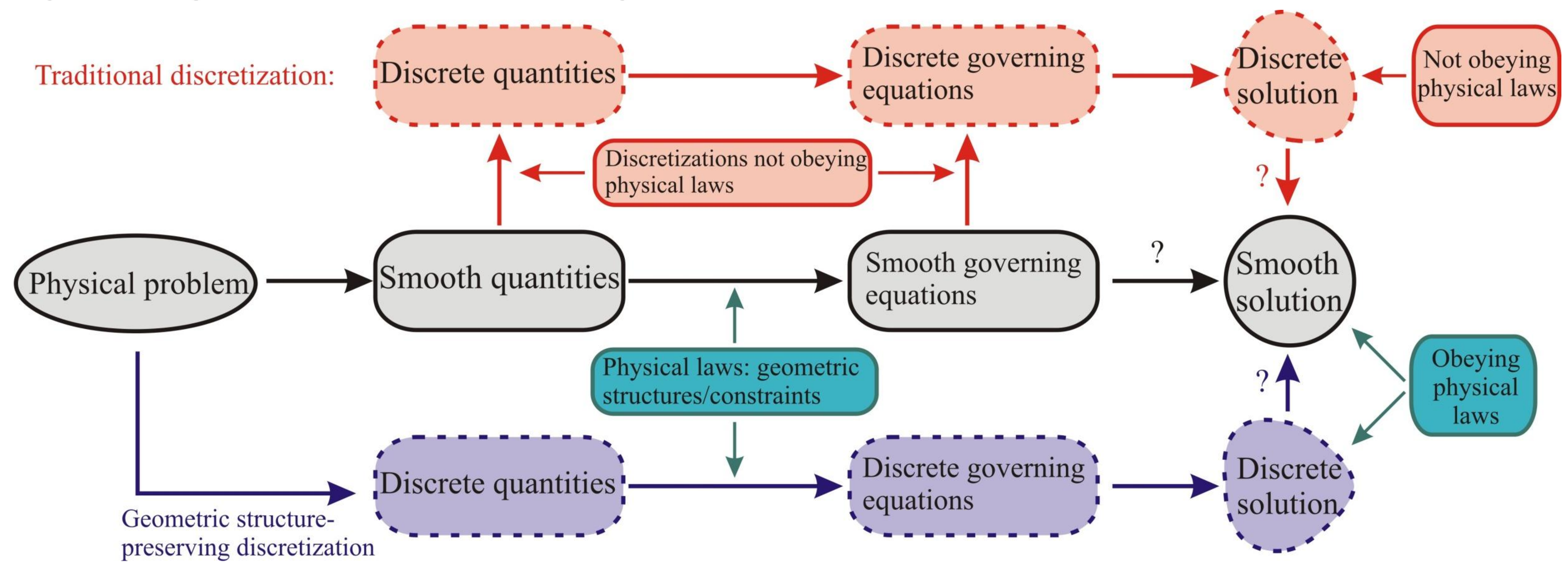
1. Introduction

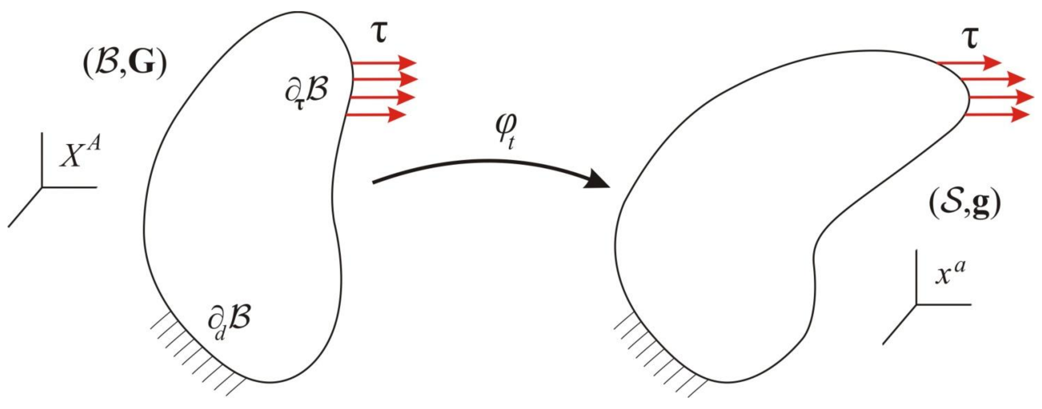
Finding robust numerical schemes for solving incompressible elasticity problems has been of great interest due to its important applications, e.g. modeling soft tissues of the human body. In this work, using some techniques from discrete exterior calculus (DEC), we develop a structure-preserving scheme for incompressible elasticity. The main feature of this method is that the physical structure of the problem is preserved after discretization. We observe that this method is free from numerical artifacts, e.g. volume locking. For more details, see [1] and references therein.

2. Geometric Structure-Preserving Schemes (GSPS)

GSPS can be considered as discrete physical theories rather than discretizations of smooth problems. They are physically more realistic, e.g. preserve conservation laws or variational structures. Moreover, they can give better accuracy with the same numerical cost.



3. Incompressible Linear Elasticity



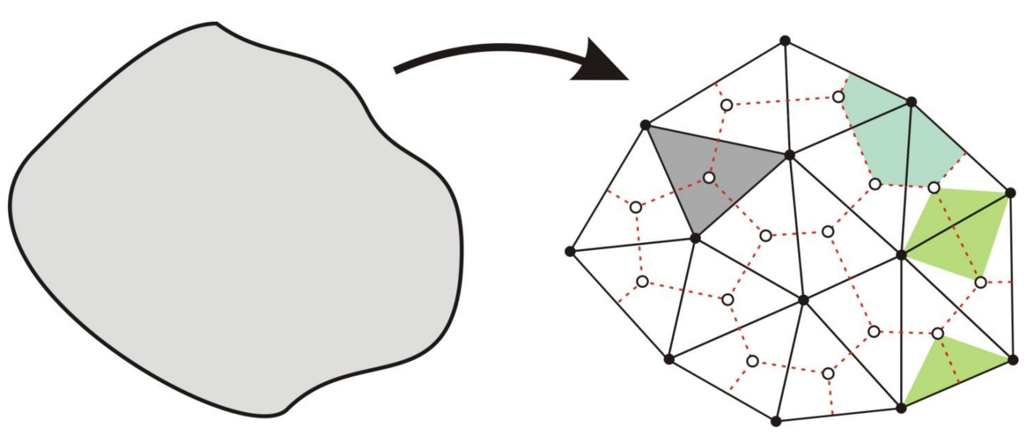
Linear elasticity is the linearization of nonlinear elasticity with respect to a reference motion $\overset{o}{\varphi}$. The unknown is a "displacement" field, i.e. a two-point tensor field $\mathbf{U}:\mathcal{B}\to T\overset{o}{\varphi}(\mathcal{B})$. If we have $\overset{o}{\varphi}=Id_{\mathcal{B}}$, the incompressibility condition reads

$$\mathrm{Div}\,\mathbf{U}=\mathrm{div}\,\mathbf{u}=0.$$

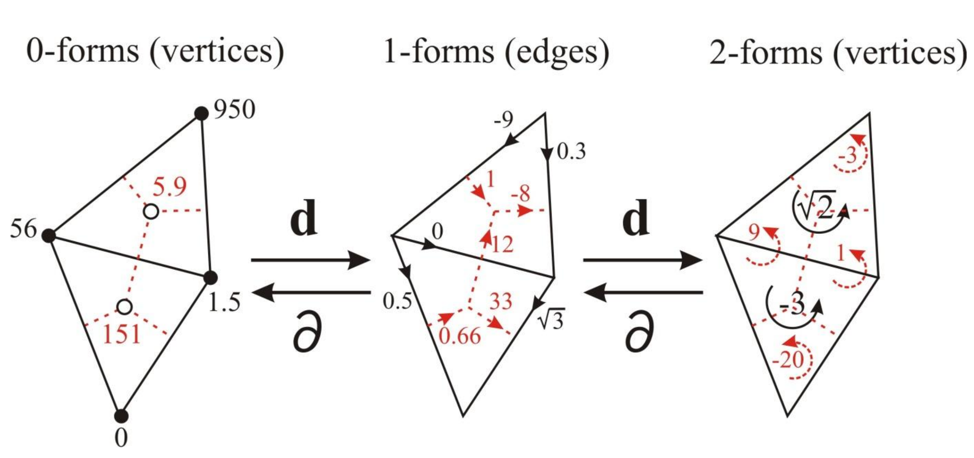
Incompressible linear elasticity can be obtained from the variational principle without using Lagrange multipliers [1]. The strong form of the governing equations is:

$$\rho\ddot{\mathbf{u}}=\rho\mathbf{b}+\mathrm{div}(2\mu\mathbf{e}^{\sharp}-p\mathbf{g}^{\sharp})\quad \text{in } \mathcal{B},$$
$$\boldsymbol{\tau}=\langle 2\mu\mathbf{e}^{\sharp}-p\mathbf{g}^{\sharp},\mathbf{n}^{\flat}\rangle\quad \text{on } \partial_{\tau}\mathcal{B}.$$

4. Discrete Exterior Calculus (DEC)



• Discrete chain and cochain complexes:



• One can define discrete exterior derivative, discrete Hodge star, and discrete flat operators. Using these discrete operators, one can define discrete divergence and discrete Laplace-Beltrami operators.

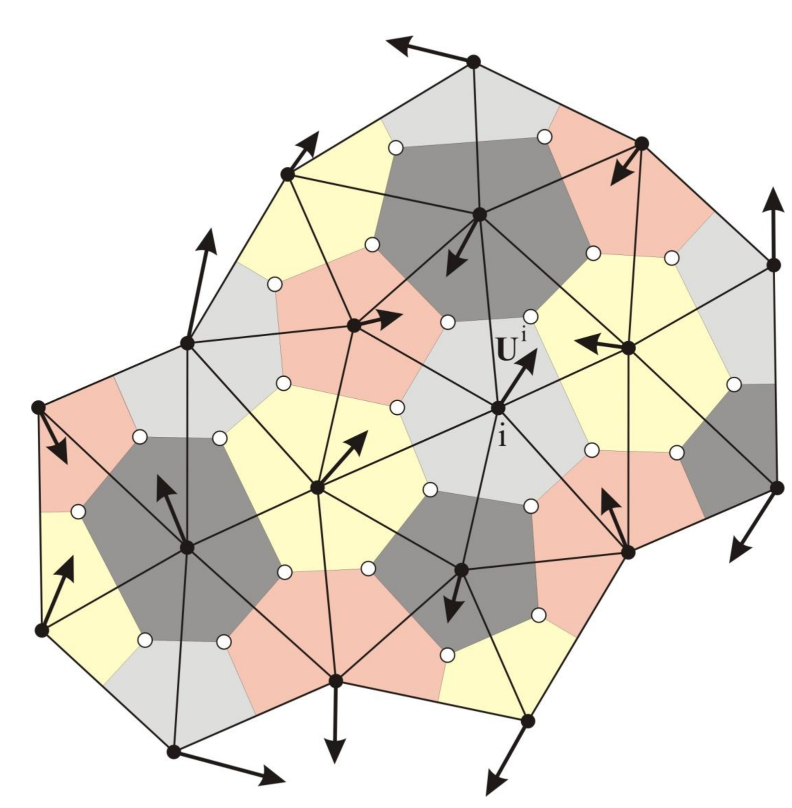
5. Discrete Incomp. Linear Elasticity

Idea: Similar to the smooth case, one can obtain the governing equations of discrete incompressible elasticity by extremizing a "discrete Lagrangian" over "discrete divergence-free" displacement fields. The Primary unknown is displacement:

$$\boldsymbol{X}_{n\mathsf{P}_h\times 1}=\left\{\mathbf{U}^{1}_{n\times 1}\cdots\mathbf{U}^{\mathsf{P}_h}_{n\times 1}\right\}^{\mathsf{T}}$$

• Divergence-free displacements satisfy:

$$\mathbb{I}^{h}_{\mathsf{D}_h\times(n\mathsf{P}_h)}\boldsymbol{X}_{(n\mathsf{P}_h)\times 1}=\mathbf{0}$$



6. Discrete Governing Equations

Theorem. *Let K_h be a 2-dimensional well-centered primal mesh such that $|K_h|$ is a simply-connected set. Then, the associated incompressibility matrix $\mathbb{I}^h$ is full ranked.*

• Discrete Lagrangian:

$$L^d=\frac{1}{2}\dot{\boldsymbol{X}}^{\mathsf{T}}\mathbf{M}\dot{\boldsymbol{X}}-\boldsymbol{X}^{\mathsf{T}}\mathbf{S}\boldsymbol{X}+\mathbf{F}\cdot\boldsymbol{X}+L^d_e$$

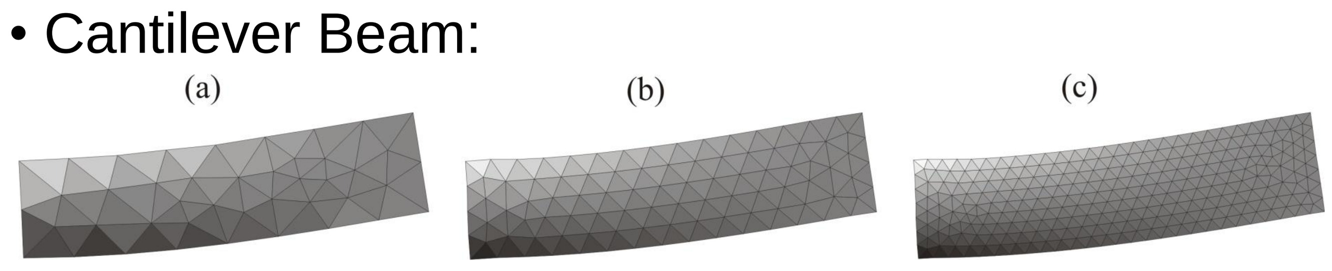
• Discrete governing equations (Euler-Lagrange equations of the discrete Lagrangian):

$$\mathbf{M}\ddot{\boldsymbol{X}}+\left(\mathbf{S}+\mathbf{S}^{\mathsf{T}}\right)\boldsymbol{X}-\mathbf{F}=\boldsymbol{\Lambda}$$

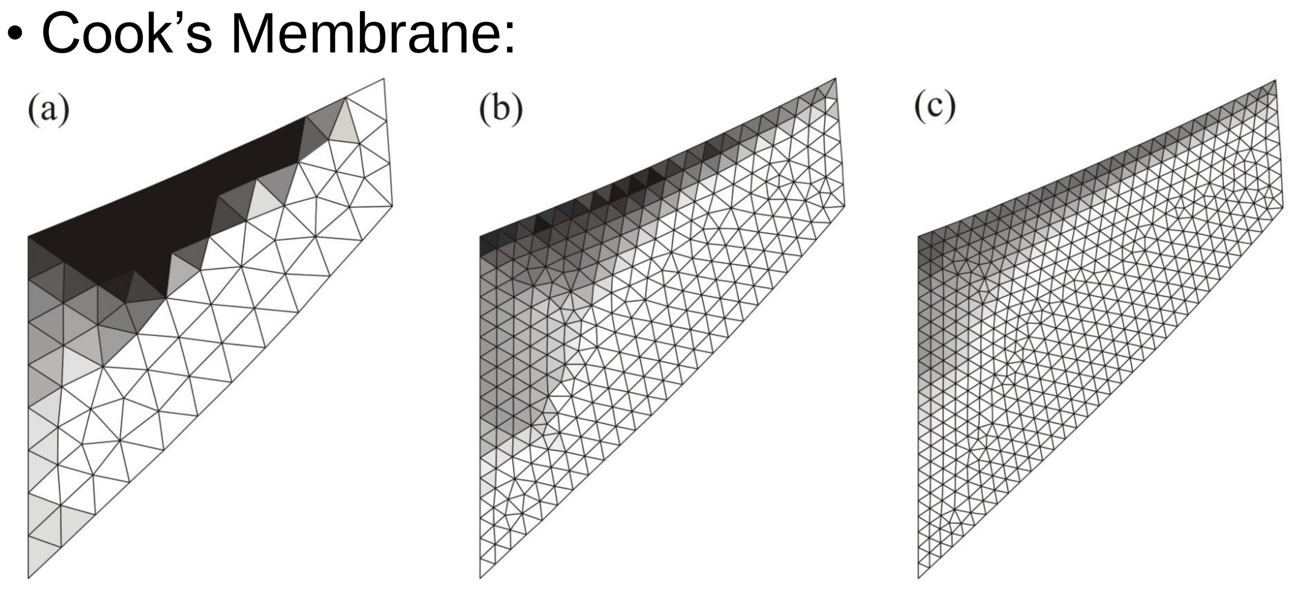
• $\boldsymbol{\Lambda}\in\mathrm{Ker}(\mathbb{I})^{\perp}$ is the discrete pressure gradient. Pressure is a dual zero-form: a geometric justification for the known fact that using different function spaces for displacement and pressure is crucial in incompressible linearized elasticity.

7. Numerical Examples

• Cantilever Beam:



• Cook's Membrane:



8. Discussions

• **Respecting geometric structures can solve some of our problems!**

• By construction free of numerical artifacts. Speed is comparable to the mixed finite element formulations.

• Can be extended to fluid mechanics and nonlinear elasticity.

Acknowledgement

AFOSR – Grant No. FA9550-10-1-0378

NSF – Grant No. CMMI 1042559

Reference

[1] A. Angoshtari and A. Yavari, A geometric structure-preserving discretization scheme for incompressible linear elasticity, under review, 2012.