CSF Workshop on Generalized Barycentric Coordinates in Computer Graphics and Computational Mechanics

June 1–4, 2022
Monte Verità, Ascona, Switzerland

Book of Abstracts

Thursday, June 2

09:00–10:00 *Keynote Lecture* Auditorium

Combining optimal transportation theory and meshfree discretization in the simulation of advection and diffusion problems

Anna Pandolfi
Politecnico di Milano

Abstract. The optimal transportation meshfree (OTM) method has been developed for simu-lating general solid and fluid flows, including fluid-structure interaction [1]. The approach combines material-point sampling and max-ent meshfree interpolation [2] with concepts from optimal transportation theory. Innovative particle methods based on OTM have been developed recently for the solution of advection-diffusion problems, by approximating the density of the diffusive species by Dirac measures [3]. Relying on the optimal transport the-ory and in alternative to traditional schemes formulated in linear spaces, the method hybrid-izes elements of a Galerkin approximation with those of an updated Lagrangian approach. The time discretization of the diffusive step is based on the Jordan-Kinderlehrer-Otto (JKO) variational principle [4]. The JKO functional characterizes the evolution of the density as a competition between the Wasserstein distance (which penalizes departures from the initial conditions) and entropy (which tends to spread the density and it make uniform over the domain), and is regarded as a functional of an incremental transport map which rearranges the density over the time step. The resulting update is geometrically exact with respect to advection and volume. In applications, the JKO functional is discretized in space using one discretization for the density and another discretization for the incremental transport map. By exploiting the structure of the Euler-Lagrange equations, which are linear in the density, the density is treated as a measure and coherently approximated as a collection of Diracs. We apply the proposed method to the solution of advection-diffusion problems in three di-mensions, by initially excluding the presence of sources/sinks for the densities and the zero flux boundary conditions. Then we include the presence of sources, by defining a computa-tional way to introduce/remove material points to preserve an acceptable distribution of dis-crete particles. Then, for simple shapes of the boundaries, we include Neumann boundary conditions, by adding in-let and out-let boundary conditions in terms of density flux. Finally, we note that the general problem with mixed Neumann-Dirichelet boundary conditions can be reduced to a Neumann problem, by letting the flux be unknown over the boundary and computing the flux such that the Dirichket boundary condition is satisfied.A few examples will demonstrate the applicability of the approach to the different cases. The work has been developed in collaboration with Michael Ortiz (Caltech, CA and Bonn University, Germany) and Laurent Stainier (Ecole Centrale de Nantes, France).

[1] B. Li, F. Habbal, M. Ortiz. International Journal for Numerical Methods in Engineer-ing, 83:1541-1579, 2010.
[2] M. Arroyo, M. Ortiz. International Journal for Numerical Methods in Engineering, 65:2167-2202, 2006.
[3] L. Fedeli, A. Pandolfi, M. Ortiz. International Journal of Numerical Methods in Engineering, 112:1175-1193, 2017.
[4] R. Jordan, D. Kinderlehrer, F. Otto. Physica D, 107:265-271, 1997.

10:30–10:50 *Contributed Presentation* Auditorium

Scaled boundary polyhedral elements for high-performance computing

Chongmin Song
University of New South Wales

Abstract. Arbitrarily shaped star-convex polyhedral elements are constructed using the scaled boundary finite element method. The elements are highly complementary with octree algorithm for mesh generation. Considering the scaling and transformation of octree cells, the a limited number of unique cell patterns are identified. Leveraging the pre-computed element solutions of the unique cell patterns, an efficient algorithm is developed for a distributed computing environment. The performance is evaluated by numerical examples of static and dynamic problems of complex geometries and various practical applications.

10:50–11:10 *Contributed Presentation* Auditorium

Maximum-entropy collocation

Francesco Greco
Universitat Politècnica de Catalunya

Abstract. The presentation discusses the approximation of partial differential equations with a point collocation framework based on high-order local maximum-entropy schemes (HOLMES). These schemes are obtained from the entropy functional introduced by Sukumar for the construction of barycentric polygonal interpolants, combined with a locality term. The resulting basis functions are smooth but the extension to high-order compatibility conditions is obtained by removing the positivity constraint. Thanks to this smoothness, the strong form of the problem is directly imposed at the collocation points, reducing significantly the computational times with respect to the classical Galerkin formulation, which is normally used in the literature with linear maximum-entropy schemes. Remarkably, the resulting method is truly meshfree, since no background integration grid is needed. The validity of the proposed approach is discussed with supportive numerical applications, where it is also critically compared to the Galerkin formulation.

11:10–11:30 *Contributed Presentation* Auditorium

Barycentric coordinates in general dimensions

Andrew Gillette
Lawrence Livermore National Laboratory

Abstract. What are barycentric coordinates in general dimensions? Where do they have applications in computational mechanics? Why haven't they been used before? I will answer these questions by explaining how barycentric coordinates provide a deterministic interpolation method for unstructured numerical data in arbitrary dimensions of relevance to contemporary scientific machine learning pipelines. In particular, I will explain how the recently developed algorithm DelaunaySparse and accompanying open source software package have enabled a previously untenable approach to interpolating and assessing data from numerical simulations of micro-structured materials and inertial confinement fusion.

11:30–12:00 *Invited Presentation* Auditorium

PSGWC and PSMVC based on active curve

Chongyang Deng
Hangzhou Dianzi University

Abstract. Generalized barycentric coordinates (GBC) are widely used in computer graphics and related areas and there are a few kinds of GBC in the literature. We propose positive and smooth Gordon–Wixom coordinates (PSGWC), positive and smooth mean value coordinates (PSMVC) for the interior points of planar polygons. The basic idea of PSGWC, PSMVC is that we replace the polygon with an active curve for each inner point in the process of computing coordinates, and thus they can achieve the required continuity by adjusting the continuity order of the active curve.

14:00–15:00 *Keynote Lecture* Auditorium

Complex barycentric coordinates for injective harmonic maps

Ofir Weber
Bar-Ilan University

Abstract. Interactive shape deformation is a fundamental problem in computer graphics and geometry processing. In this talk, we will focus on planar harmonic shape deformation, and its relation to complex barycentric coordinates. We will see how complex barycentric coordinates are derived, and how they are used to generate planar conformal and harmonic maps. For deformation purposes, we are interested in maps that are locally injective, and have a bounded amount of geometric distortion. Computing such maps is a highly challenging task, since the underlying optimization problems are nonlinear nonconvex. Harmonic maps are special as their injectivity and distortion levels are controlled solely by boundary behavior.

15:30–15:50 *Contributed Presentation* Auditorium

Computing with Poisson elements on curvilinear polygons

Jeffrey Ovall
Portland State University

Abstract. We describe an approach for H1-conforming finite element computations on meshes that may include mesh cells that are quite general curvilinear polygons. The local spaces and basis functions are defined implicitly in terms of Poisson problems having polynomial source and boundary data. In the lowest order case, these basis functions are natural extensions of harmonic coordinates to curvilinear polygons. More generally, there are obvious connections with Virtual Element Methods (VEM), but this work is more of a natural evolution of Boundary Element Based Finite Element methods (BEM-FEM), in that we work more directly with bases, whose data (e.g., function values, normal derivatives) are accessed, as needed, using boundary integral equation techniques. After providing a few numerical illustrations of our approach, we describe some of the theoretical and computational challenges that arise when working with curved mesh cells, as well as our efforts to address them, highlighting some recent work on quadratures in this context.

15:50–16:10 *Contributed Presentation* Auditorium

A polygonal finite element formulation for modeling nearly incompressible materials

Mahmood Jabareen
Technion – Israel Institute of Technology

Abstract. The polygonal finite elements method has been drawing increasing attention during the last 15 years for modeling the response of structures. In comparison to the standard finite elements, polygonal finite elements offer greater flexibility in meshing arbitrary geometries and better description of certain materials (e.g., granular materials). On the other hand, the disadvantages of the polygonal finite element method loses the sparsity structure of the stiffness matrix and the need for a higher order numerical integration quadrature scheme to achieve high accuracy. Unlike the traditional finite element method, the construction of the shape functions in polygonal finite elements method is different, therefore, several approaches for constructing the shape functions for polygonal finite elements were proposed. Among these approaches are the Wachspress shape functions, which are rational interpolation functions providing C^0 continuity, the Laplace based coordinates and the maximum entropy coordinates among other approaches.

In the present study, the polygon finite element formulation will be advanced by using the three fields formulation. Generally speaking, the standard displacement formulation exhibits strong volumetric locking when modeling nearly incompressible materials (e.g., elastomers) or the isochoric plastic deformation. Thus, the mixed formulation is formulated using the polygon finite elements in order to eliminate the volumetric locking. Different example problems, including eigenvalue analysis, nonlinear patch test and other benchmark problems will be presented in this work.

16:10–16:30 *Contributed Presentation* Auditorium

Linear and quadratic shape functions for polygons and polyhedra

Astrid Bunge
TU Dortmund

Abstract. Being able to accurately solve PDEs on arbitrary polygonal/polyhedral meshes is a central goal and has been considered for various differential operators over the last years. In this talk I will present a simple approach for computing (piecewise) linear and quadratic basis functions for general polygons and polyhedra. The central idea is to (virtually) split each polygon/polyhedron into simplices, but to hide this refinement through a special prolongation operator in the matrix assembly stage. The resulting shape functions inherit the numerical properties of linear/quadratic shape functions for P1/P2 elements, reproduce the latter on triangles/tetrahedra, and thus generalize them to arbitrary polygon/polyhedra.

16:30–17:00 *Invited Presentation* Auditorium

A node-based uniform strain virtual element method for elastic and inelastic small deformation problems

Alejandro Ortiz-Bernardin
University of Chile

Abstract. A combined nodal integration and virtual element method is presented for elastic and inelastic small deformation problems, wherein the strain is averaged at the nodes from the strain of surrounding virtual elements. For the strain averaging procedure, a nodal averaging operator is constructed using a generalization to virtual elements of the node-based uniform strain approach for finite elements. The proposed technique is referred to as the node-based uniform strain virtual element method (NVEM). No additional degrees of freedom are introduced in this approach, thus resulting in a displacement-based formulation. A salient feature of the NVEM is that the state and history-dependent variables become nodal quantities just like displacements, which facilitates their tracking and postprocessing. Some benchmark problems will be presented to demonstrate that the NVEM is accurate, optimally convergent and devoid of volumetric locking.

17:00–17:20 *Contributed Presentation* Auditorium

Scaled boundary cubature for arbitrary dimensions: integration over polytopes and curved regions

Eric Chin
Lawrence Livermore National Laboratory

Abstract. This talk develops the scaled boundary cubature (SBC) scheme for integration of functions over polytopes and regions bounded by parametric curves and surfaces. The accuracy, efficiency, and ease of implementation of the SBC scheme in two dimensions [1] is retained over higher dimensions. Over three-dimensional domains, the SBC method reduces integration over a region bounded by m surfaces to integration over m regions. With proper definition of the normal to the bounding surfaces, the scheme is applicable to convex and nonconvex domains. Additionally, for star-convex domains, a tensor-product cubature rule with positive weights and integration points in the interior of the domain is obtained. If the integrand is homogeneous, the integration method reduces to the homogeneous numerical integration scheme [2]; however, the SBC scheme is more versatile since it is equally applicable to both homogeneous and non-homogeneous functions. We also introduce several methods which work with the SBC scheme in three dimensions for smoothing integrands with point singularities and near-singularities. When these methods are applied, highly efficient integration of weakly-singular functions is realized. We demonstrate the performance of the SBC method over a variety of integrals, which reveal its broad applicability and superior performance (in terms of time to generate a rule and accuracy per cubature point) when compared to existing methods of integration.

[1] E.B. Chin, N. Sukumar, Scaled boundary cubature scheme for numerical integration over planar regions with affine and curved boundaries, Computer Methods in Applied Mechanics and Engineering 380 (2021) 113796.
[2] E.B. Chin, N. Sukumar, An efficient method to integrate polynomials over polytopes and curved solids, Computer Aided Geometric Design 82 (2020) 101914.

Friday, June 3

09:00–10:00 *Keynote Lecture* Auditorium

X-MESH: An eXtreme Mesh deformation method to follow sharp physical interfaces

Jean-François Remacle
Université Catholique de Louvain

Abstract. We develop an innovative approach, X-MESH, to overcome a major difficulty associated with engineering analysis: we aim to provide a revolutionary way to track physical interfaces in finite element simulations using extreme deformation of the meshes. Unprecedented low computational cost, high robustness and accuracy are expected as the proposed approach is designed to avoid the pitfalls of the current methods, especially for topological changes. The key idea of X-MESH is to allow elements to deform up to zero measure. For example, a triangle can deform to an edge or even a point. This idea is rather extreme and totally revisits the interaction between the meshing community and the computational community who, for decades, have striven to interact through beautiful meshes. Different areas in fluid and solid mechanics as well as heat transfer are targeted. Interfaces will be either (i) material, i.e. attached to particles of matter (the interface between two immiscible fluids or the dry interface in a wetting and drying model) (ii) immaterial, i.e. migrating through the material (a solidification front, contact front, yield front in yield stress fluid flow or a crack front). In this presentation, we will focus both on the mathematical issues related to the use of zero-measure elements and on the eXtreme mesh deformation scheme that will be used to track physical interfaces. Two applications will be targeted: phase change Stefan model and two phase flows.

10:30–11:30 *Keynote Lecture* Auditorium

On Bézier and barycentric coordinates

Scott Schaefer
Texas A&M University

Abstract. We will explore the deep connection between barycentric coordinates and higher order parametric representations of curves, surfaces, and volumes in arbitrary dimension. In the case of curves, these curves are the well-known Bézier curves. The extension to convex surface patches is known as S-Patches. This restriction to convex domains was a function of the restrictions of Wachspress coordinates, the only generalized barycentric coordinates of the time. Today, with the advent of new barycentric coordinates, these restrictions no longer exist though the restricted view of S-patches persists. We will discuss S-Patches in their full generality afforded by modern barycentric coordinates with generalized domains and in arbitrary dimensions.

11:30–12:00 *Invited Presentation* Auditorium

Control point based multi-sided surfaces over curved, multi-connected domains

Tamás Várady
Budapest University of Technology and Economics

Abstract. We present a new control point based family of parametric surfaces that interpolate surface ribbons, i.e., boundary curves and cross-derivatives, given in Bézier or B-spline form. These patches are defined by a single surface equation and represent complex, free-from geometries, compatible with tensor-product Bézier or B-spline surfaces. The scheme is defined over a planar domain with curved edges that mimics the shape of the 3D boundary curves, and it is capable to handle strongly concave boundaries and periodic hole loops in the interior. The main topics of the talk are the following:

- algorithms for curved domain generation,
- methods to define local parameterizations using barycentric coordinates or harmonic functions,
- special blending functions associated with the control points of the ribbons,
- the composition of the surface patches,
- options to edit the interior of the patch.

The most important applications include curve network based design, hole filling (vertex blending), general lofting and the approximate representation of trimmed surfaces, particularly when watertight connections are important. The strength of the scheme is its flexibility to define complex shapes in a natural manner; its main weakness is that it cannot be exported in standard form. Several examples will be given to compare the difficulties of classical surfacing approaches and the benefits of the new multi-sided scheme.

[1] T. Várady, P. Salvi, Gy. Karikó, A Multi-sided Bézier patch with a simple control structure. Computer Graphics Forum, Vol. 35(2), pp. 307-317, 2016.

[2] T. Várady, P. Salvi, M. Vaitkus, Á. Sipos, Multi-sided Bézier surfaces over curved, multi-connected domains. Computer Aided Geometric Design, Vol. 78, #101828, 2020.

[3] M. Vaitkus, T. Várady, P. Salvi, Á. Sipos, Multi-sided B-spline surfaces over curved, multi-connected domains. Computer Aided Geometric Design, Vol. 89, #102019, 2021.

Saturday, June 4

09:00–10:00 *Keynote Lecture* Balint

On transfinite barycentric interpolation schemes and their applications

Alexander Belyaev
Heriot-Watt University

Abstract. Transfinite barycentric interpolations schemes are continuous counterparts of generalized barycentric coordinates which have numerous applications in computer graphics, geometric modelling, and computational mechanics. In this talk, I consider several transfinite barycentric interpolation schemes and discuss their links with boundary value problems for linear and nonlinear partial differential equations. In particular, links between transfinite barycentric interpolation and generalized double-layer potentials and distance function approximations are discussed.

10:30–11:00 *Invited Presentation* Balint

Semi-analytical computation of harmonic coordinates over arbitrary polytopes using scaled boundary finite element method

Sundararajan Natarajan
Indian Institute of Technology Madras

Abstract. In this talk, a displacement based Galerkin finite element formulation over arbitrary polytopes is presented with conforming shape functions constructed using harmonic coordinates. Typically, the harmonic coordinates are computed by solving the Laplace equation with appropriate boundary conditions. Although this is robust, it is computationally intensive, because the Laplace equation has to be solved as many times as there are number of vertices in the polytope. Here, we propose to compute the harmonic coordinates semi-analytically using the scaled boundary finite element method (SBFEM). The SBFEM shares the advantages of the finite element method (FEM) and the boundary element method (BEM). Similar to BEM, only the boundary is discretized and like FEM, does not require Green's function. The shape functions of arbitrary polytopes can be computed in one step. The computed shape functions satisfy all the necessary properties as required by the FE framework. The versatility and robustness is demonstrated through benchmark problems.

11:00–11:30 *Invited Presentation* Balint

Polytopal composite finite elements: Implementation and applications

Hung Nguyen-Xuan
Ho Chi Minh City University of Technology

Abstract. Polytopal elements, including polygonal elements for a two-dimensional case and polyhedral elements for a three-dimensional case, are adopted to mesh material domains with high complexity. In our recent work [1], polytopal composite finite elements (PCEs), which were proposed to model the mechanical behavior of compressible and incompressible materials, could pass the patch test and satisfy the inf-sup stability with high accuracy. The PCEs were constructed with a polynomial projection of compatible strain fields through the least-squares approximation. For incompressible materials, volumetric locking arises when the Poison's ratio close to being 0.5, leading to the over stiffening of incompressible material elements. The volumetric locking phenomenon occurs when a fully integrated element (e.g., Q4 elements with 4 quadrature points or H8 elements with 8 quadrature points) is employed. One well-known solution to the volumetric locking is to couple pressure unknowns together with displacement unknowns. However, this way requires additional unknowns in the system of linear equations, thus require large storage and cost expensively. Here PCEs were purely based on displacement formulation without the additional unknowns and could overcome the over stiffening to accurately model incompressible materials. Then, PCEs have been applied to topology optimization [2,3,4], fracture [5], limit analysis, breakwater structures, etc., to show its feasible potential in computational mechanics.

[1] H. Nguyen-Xuan, K.N. Chau, K.N. Chau, Polytopal composite finite elements, Comput. Methods Appl. Mech. Eng. 355 (2019) 405-437.

[2] Van-Nam Hoang, Hoang B Nguyen, H. Nguyen-Xuan, Explicit topology optimization of nearly incompressible materials using polytopal composite elements, Advances in Engineering Software, 149, 102903, 2020

[3] Van-Nam Hoang, Trung Pham, Duc Ho, H. Nguyen-Xuan, Robust multiscale design of incompressible multi-materials under loading uncertainties, Engineering with Computer, 38, 875-890, 2022

[4] Nam V Nguyen, H Nguyen-Xuan, Jaehong Lee, Polygonal composite elements for stress-constrained topology optimization of nearly incompressible materials, European Journal of Mechanics-A/Solids, 94, 104548.

[5] Hai D. Huynh, S Nataraja, H. Nguyen-Xuan, Xiaoying Zhuang, Polygonal composite finite elements for modeling concrete fracture based on nonlocal damage models, Computational Mechanics, 66, 1257-1274, 2020 (2020) 2378-2390.

11:30–12:00 *Invited Presentation* Balint

Penalty-free discontinuous Galerkin method

Jan Jaśkowiec
Cracow University of Technology

Abstract. A new approach for the discontinuous Galerkin (DG) method is presented, where, contrary to other DG methods, no penalty nor stabilization parameter is needed. Thus the method is called penalty-free DG. In this method, the trial and test function belong to the broken Sobolev space, in which, generally, the functions are discontinuous on the mesh skeleton and do not satisfy the Dirichlet boundary conditions. However, a subset can be distinguished in this space, where the functions are continuous and attain the Dirichlet values on the domain boundary and the approximate solution is searched in this subset. In the numerical application of the PFDG, the augmented consistency subset is applied in which some small deviation of the consistency conditions is allowed. In this approach, all the advantages of the DG method are preserved without the necessity of applying any stability parameters or numerical fluxes. The scaled boundary cubature scheme is used to integrate polynomial basis functions over polygons and polyhedra. Numerical examples for Poisson equation, elasticity, and biharmonic problem are presented to demonstrate the accuracy and convergence of the method.

14:00–15:00 *Keynote Lecture* Balint

GBCs and the conjugate basis for both element-based and element-free solution of PDEs

Joe Bishop
Sandia National Laboratories

Abstract. The use of generalized barycentric coordinates (GBC) is now widespread in computational mechanics, in particular for the numerical solution of PDEs. GBCs can be used to construct shape functions for conforming polyhedral finite elements and basis functions for element-free methods. A key challenge for the use of GBCs in the solution of PDEs is the construction of an efficient and consistent quadrature scheme for integrating the weak form that results in a stable discretization without having to resort to the use of artificial stabilization techniques. This is particularly important when coarse discretizations are used which is typical in engineering practice. Development of an efficient quadrature scheme is especially crucial for applications in nonlinear solid mechanics where material constitutive models can be extremely expensive. A gradient projection scheme is proposed for use in both polyhedral element and element-free methods that ensures the necessary integration consistency. The correction scheme involves projecting the gradient of the shape functions (polyhedral element) or basis functions (element-free) to the conjugate (dual) basis. With this construction, a natural pairing of the covariant and contravariant components appears in the weak form of the PDE that induces a quadrature scheme for nonlinear problems.

The use of GBCs in the element-free context provides opportunities for rapid discretization of PDEs. For example, in mechanical engineering applications, domains of interest are typically geometrically complex containing numerous small boundary features such as fillets, blends, and chamfers. These features are typically manually removed *a priori* to facilitate a conventional meshing process. A hybrid mesh/element-free method is proposed in which a fine-scale triangulation of the domain is used to first discretize the fully featured geometry, but an element-free discretization is used to approximate the PDE solution. The fine-scale triangulation facilitates the construction of the element-free basis functions by using manifold geodesics. The fine-scale triangulation also facilitates the gradient projection scheme. The element-free basis can be adapted through refinement or coarsening without the need for geometric considerations. While developed for PDEs in H^1, the methodology should be applicable for PDEs in H(div) and H(curl) as well.

Verification problems are presented for elasticity for both the element-based and element-free methods along with several applications in nonlinear solid mechanics including large-deformation plasticity.

15:30–15:50 *Contributed Presentation* Balint

Biinvariant barycentric coordinates on Lie groups and homogeneous spaces

Jan Hakenberg
GRZ Technologies

Abstract. My presentation consists of two parts:

1) I review weighted averages in non-linear geodesic spaces. Under certain assumptions, the weighted average exists and is uniquely defined. Applications are curve subdivision, and filtering.
2) I present several barycentric coordinate formulas that are invariant under all symmetry operations of the non-linear space. Applications are classification, function interpolation, deformation, and domain transfer.

The special Euclidean group $SE(2)$, that encodes the position and heading of a mobile robot in the plane, as well as the 2-dimensional sphere S^2 (not a Lie group, but a homogeneous space) serve as illustrations of the concepts.

15:50–16:10 *Contributed Presentation* Balint

Multi-sided generalizations of the Coons patch

Péter Salvi
Budapest University of Technology and Economics

Abstract. Hole filling, i.e., generating a continuous surface with given transfinite boundary constraints, is an important topic of computer-aided design, with practical applications such as curvenet-based design, or the creation of vertex blends. When there are exactly four boundary curves, the Coons patch provides a simple solution. Most of the time, however, a more general approach is needed.

In this talk, I am going to explore different ways of generalizing the Coons patch to an arbitrary number of sides. These methods vary in their continuity and shape editing capabilities, but all of them make use of generalized barycentric coordinates over a convex polygonal domain.

16:10–16:30 *Contributed Presentation* Balint

Circumscribed quadrics in barycentric coordinates

Marc Alexa
TU Berlin

Abstract. Barycentric coordinates admit a simple but little known representation of circumspheres and Steiner ellipsoids as bilinear forms. The approach extends naturally to more than $d+1$ points in d-dimensional space using generalized barycentric coordinates.

16:30–17:00 *Invited Presentation* Balint

Hyperbolic barycentric coordinates and applications

Aziz Ikemakhen
Cadi Ayyad University

Abstract. Barycentric coordinates are a fundamental tool in computer graphics and geometry processing. A variety of ways have been proposed for constructing such coordinates on the Euclidean plane. The spherical barycentric coordinates are also developed. In my talk, I will define hyperbolic barycentric coordinates (HBC) that describe the position of a point in the hyperbolic plane with respect to the vertices of a given geodesic polygon. We construct explicitly three kinds of HBC, namely hyperbolic Wachspress, mean values and discrete harmonic coordinates. These coordinates have properties which resemble those of the planar ones, and they are invariant by the Lorentzian transformations. Furthermore, we figure out the HBC on the Poincaré disk model. The HBC associated to a point in a hyperbolic triangle are unique. We develop two expressions of these coordinates, taking into account the parameters of a point inside the triangle. In addition, we exploit these coordinates to define a parameterization of a surface-mesh with boundary into the Poincaré disk, and we show some examples. This hyperbolic parameterization extends that of the planar one, known as Tutte's embedding. Furthermore, we demonstrate the efficiency of these coordinates by giving other applications. Namely, hyperbolic deformation and rapid shape morphing.