More on Sorting: Quick Sort and Heap Sort

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Outline

- Another divide-and-conquer sorting algorithm
- The *heap*
- Heap sort

Sorting Algorithms Seen So Far

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Algorithm		**Complexity**		**In place?**
	worst	*average*	*best*	

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MERGE-SORT				

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Insertion-Sort	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n)$	yes
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Merge-Sort	$\Theta(n \log n)$	$\Theta(n \log n)$	$\Theta(n \log n)$	no

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??		$\Theta(n \log n)$		***yes***
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- *Basic step:* partition A in three parts based on a *chosen value* $v \in A$
  - A_L contains the set of elements that are *less than* v
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- *Can we partition A* ***in place****?*

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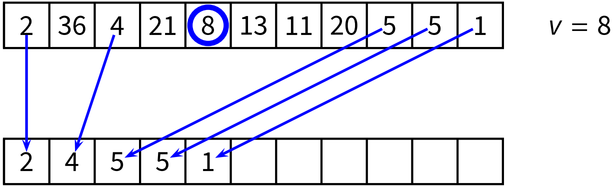

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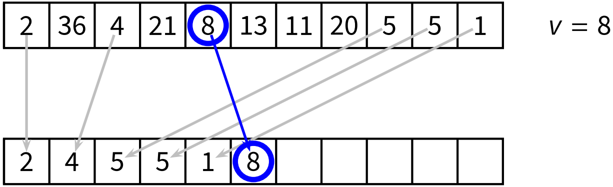

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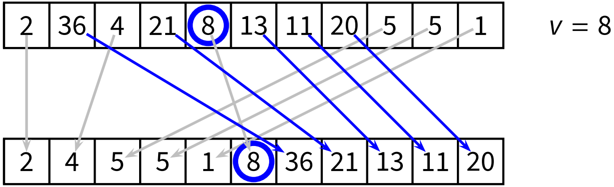

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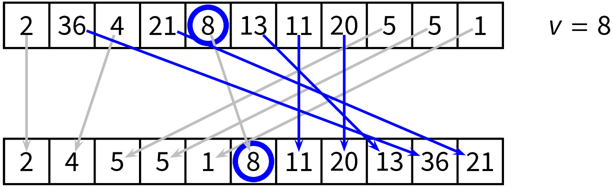

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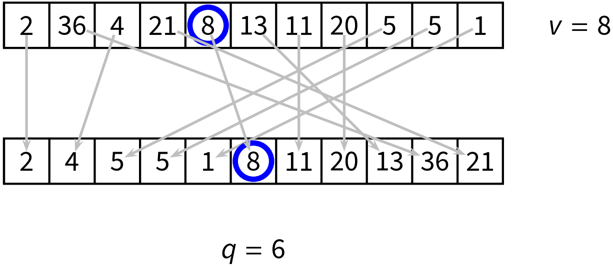

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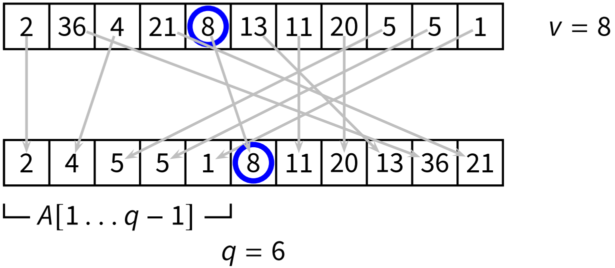

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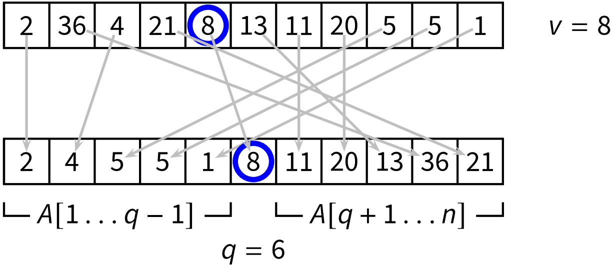

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```
QuickSort(A, begin, end)
if begin < end
    q = Partition(A, begin, end)
    QuickSort(A, begin, q − 1)
    QuickSort(A, q + 1, end)
```

- Start with $q = 1$
  - i.e., *assume all elements are greater than the pivot*
- Scan the array left-to-right, starting at position 2
- If an element $A[i]$ is less than or equal to pivot, then swap it with the current q position and shift q to the right

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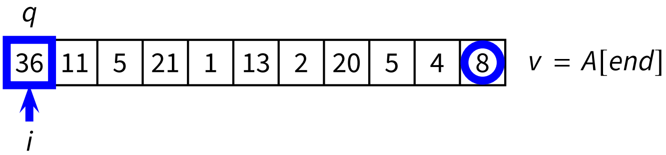

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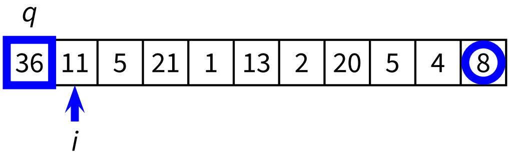

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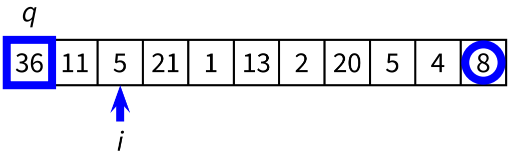

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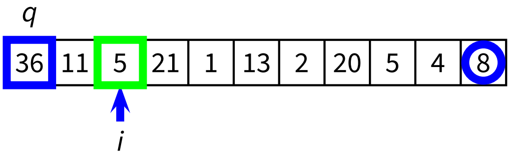

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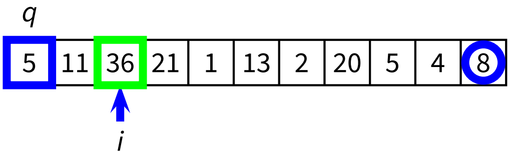

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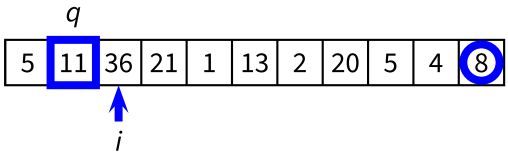

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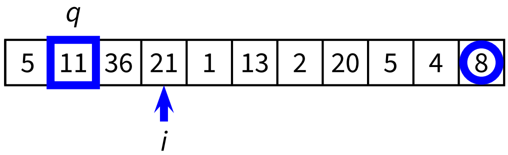

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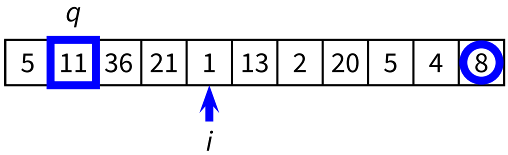

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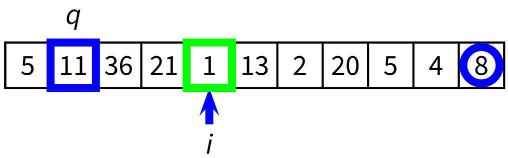

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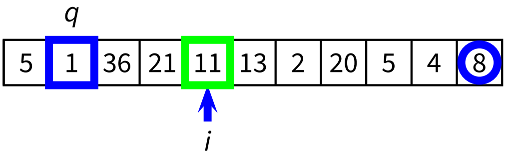

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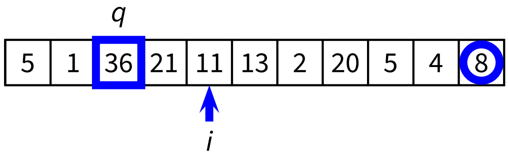

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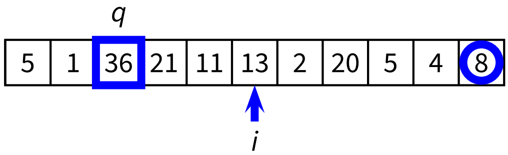

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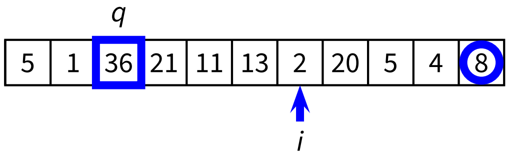

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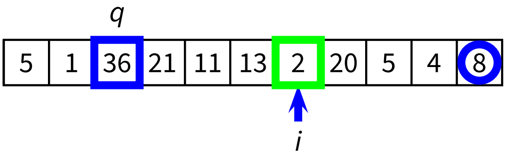

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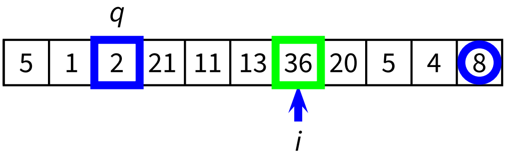

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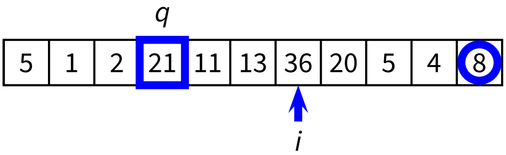

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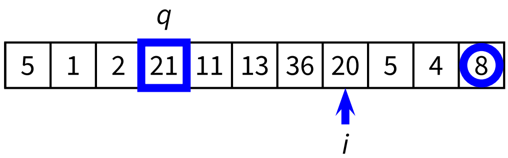

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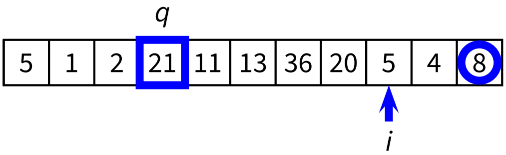

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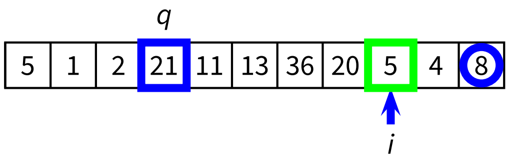

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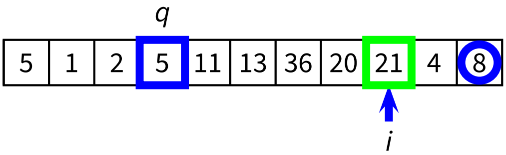

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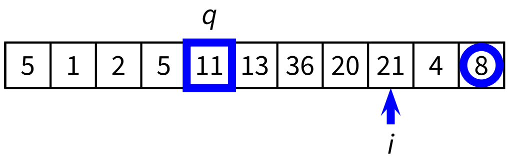

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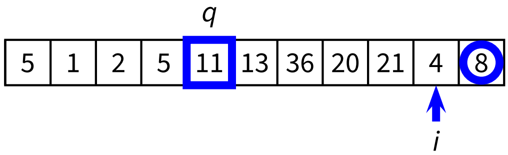

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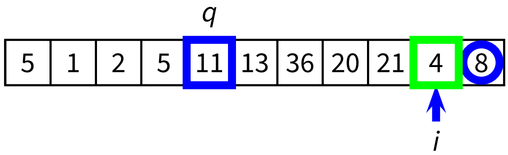

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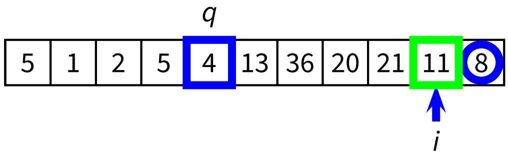

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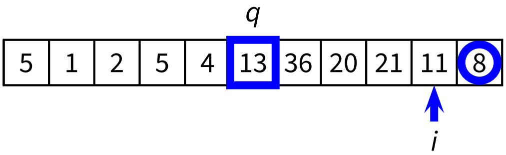

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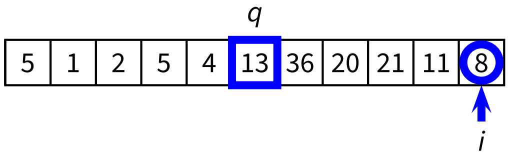

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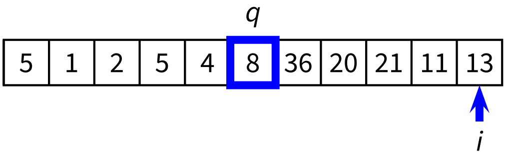

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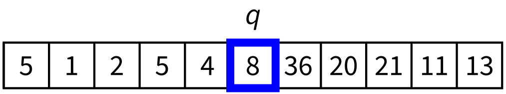

Complete QuickSort Algorithm

```
PARTITION(A, begin, end)
q = begin
v = A[end]
for i = begin to end
    if A[i] ≤ v
        swap A[i] and A[q]
        q = q + 1
return q − 1
```

```
QUICKSORT(A, begin, end)
if begin < end
    q = PARTITION(A, begin, end)
    QUICKSORT(A, begin, q − 1)
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Complexity of PARTITION

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$$T(n) = \Theta(n)$$

Complexity of QUICKSORT

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QUICKSORT(A, begin, end)
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Complexity of QuickSort

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QuickSort(A, begin, end)
if begin < end
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```

- Worst case

```
QUICKSORT(A, begin, end)
1  if begin < end
2       q = PARTITION(A, begin, end)
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```

- Worst case
  - $q = begin$ or $q = end$

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  - $q = begin$ or $q = end$
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Complexity of QuickSort (2)

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QuickSort(A, begin, end)
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  - $q = \lceil n/2 \rceil$

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$$T(n) = \Theta(n \log n)$$

Sorting Algorithms Seen So Far

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Algorithm	Complexity			In place?
	worst	*average*	*best*	
INSERTION-SORT	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n)$	yes
SELECTION-SORT	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$	yes
MERGE-SORT	$\Theta(n \log n)$	$\Theta(n \log n)$	$\Theta(n \log n)$	no

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Selection-Sort	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$	yes
Merge-Sort	$\Theta(n \log n)$	$\Theta(n \log n)$	$\Theta(n \log n)$	no
QuickSort				

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QuickSort	$\Theta(n^2)$	$\Theta(n \log n)$	$\Theta(n \log n)$	yes
??	$\Theta(n \log n)$			***yes***

Binary Heap

- Our first real ***data structure***

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- Interface

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Binary Heap: Structure

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- Conceptually a full binary tree

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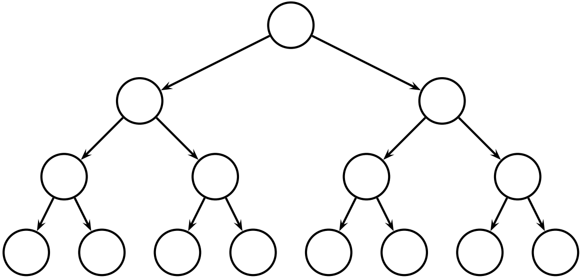

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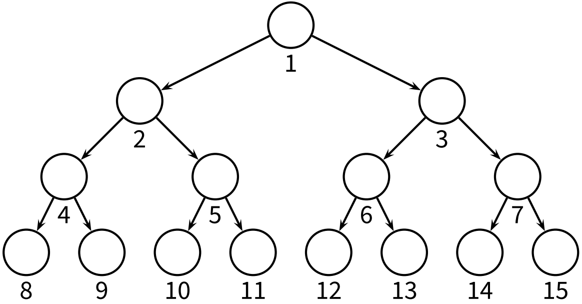

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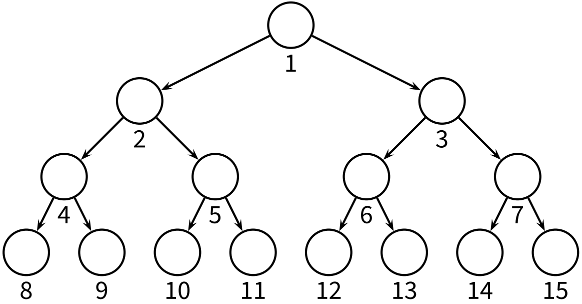

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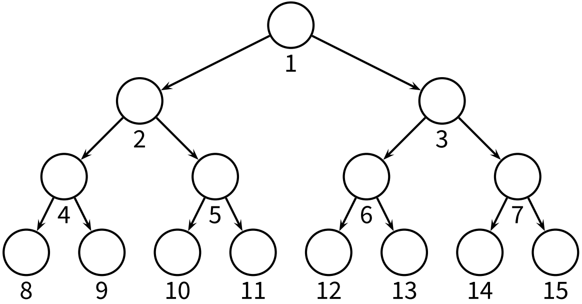

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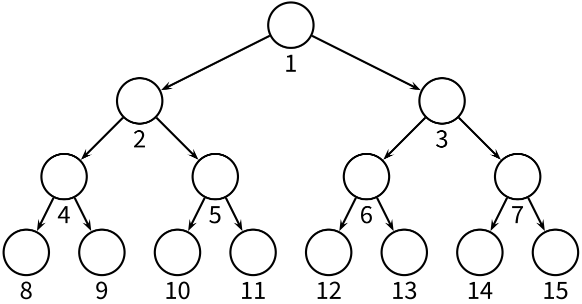

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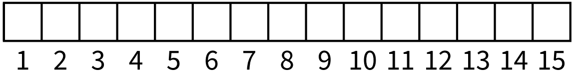

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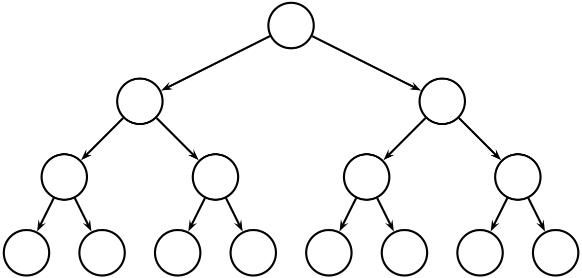

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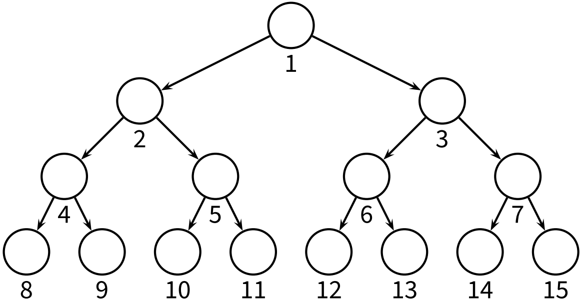

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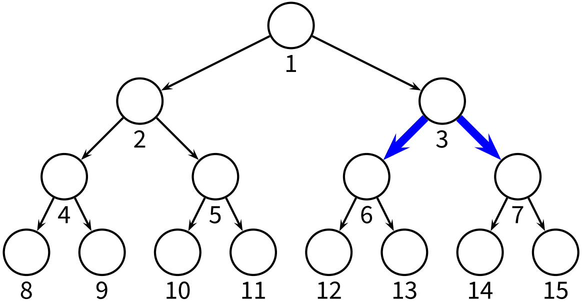

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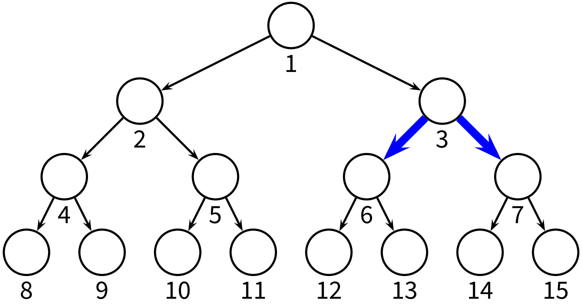


Parent(i)

return $\lfloor i/2 \rfloor$

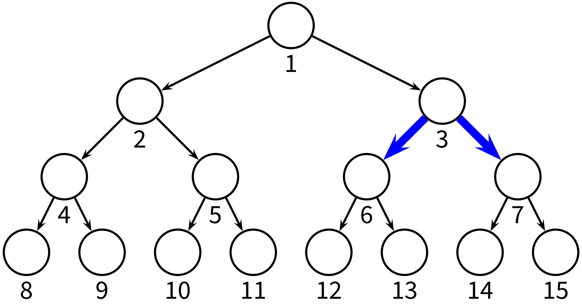
Left(i)

return $2i$

Right(i)

return $2i + 1$

Binary Heap: Properties



PARENT(i)

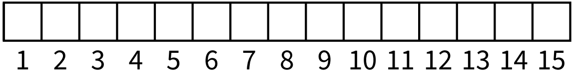
  return $\lfloor i/2 \rfloor$

LEFT(i)

  return $2i$

RIGHT(i)

  return $2i + 1$



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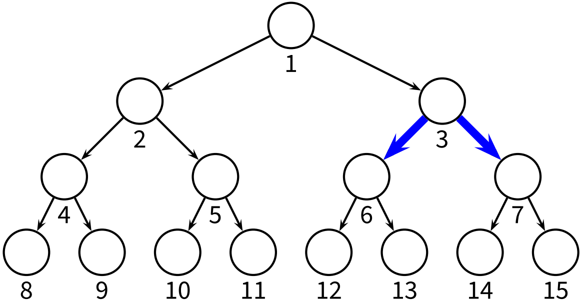

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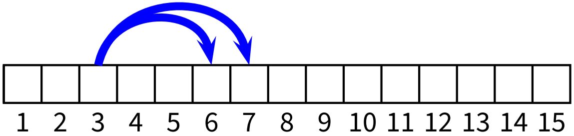

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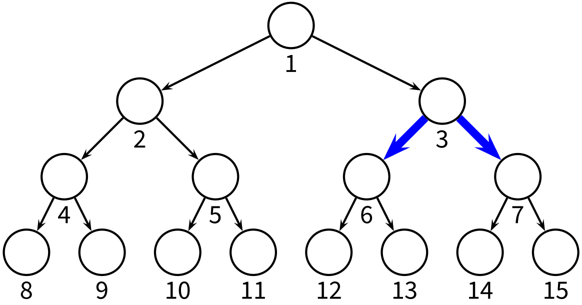

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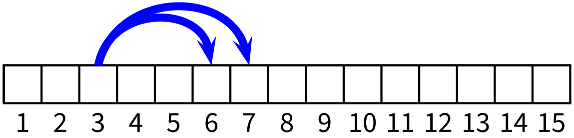

- ***Max-heap property:*** for all $i > 1$ $A[\textbf{Parent}(i)] \geq A[i]$

Example

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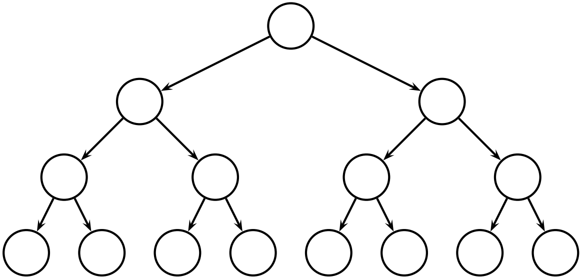

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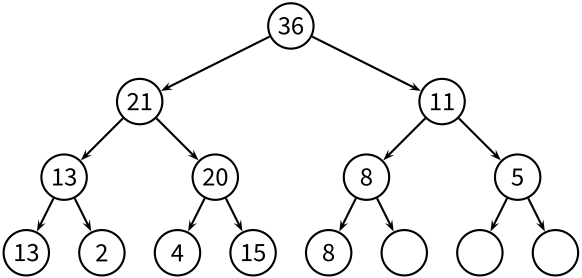

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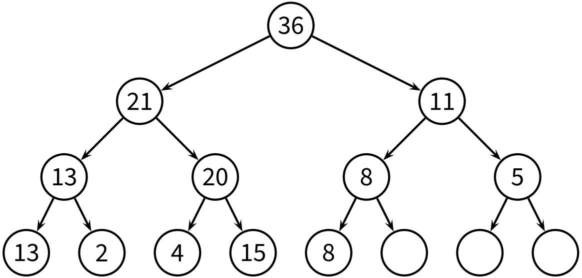

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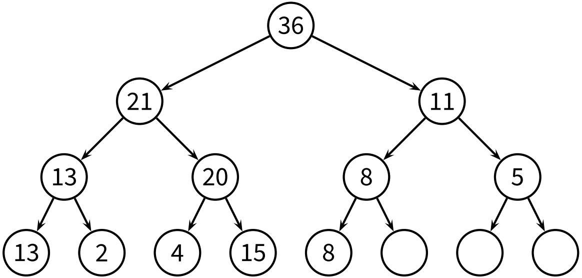

- Where is the max element?

- How can we implement **Heap-Extract-Max**?

Heap-Extract-Max

- **HEAP-EXTRACT-MAX** procedure
  - extract the max key
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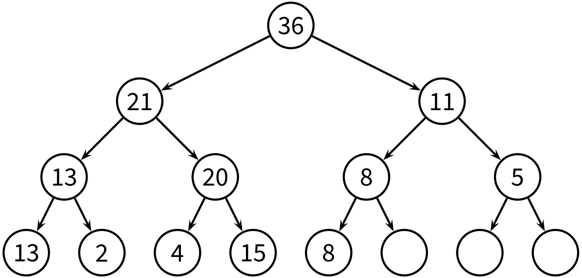

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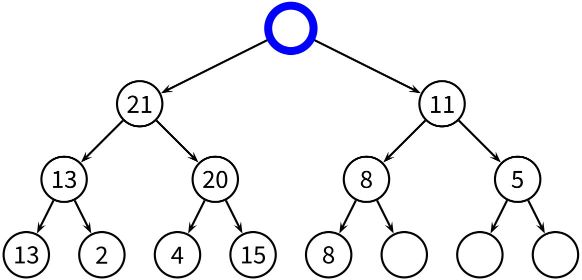

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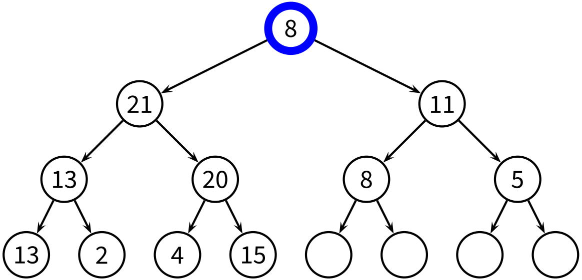

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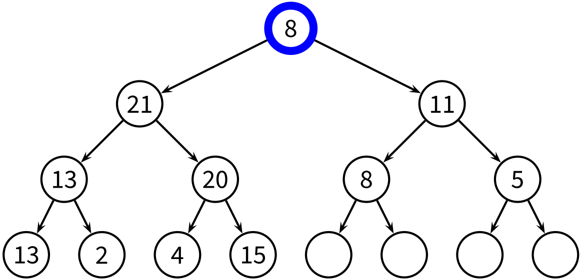

- Now we have two subtrees where the *max-heap property* holds

- **Max-Heapify**(A, i) procedure
  - *assume:* the *max-heap property* holds in the subtrees of node *i*
  - *goal:* rearrange the heap to maintain the *max-heap property*

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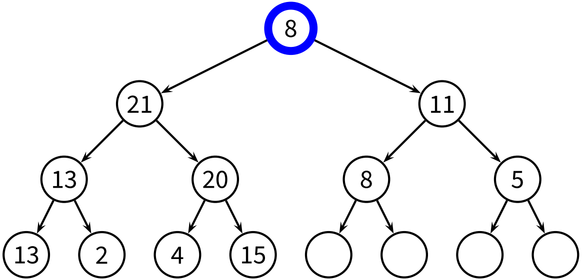

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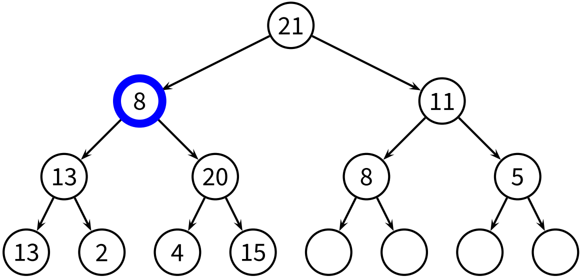

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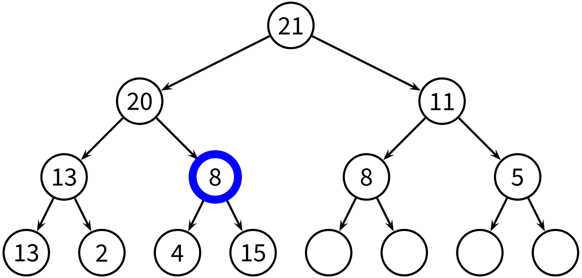

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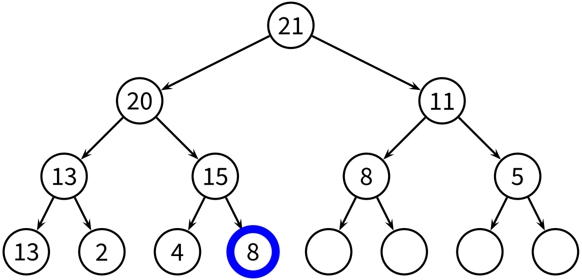

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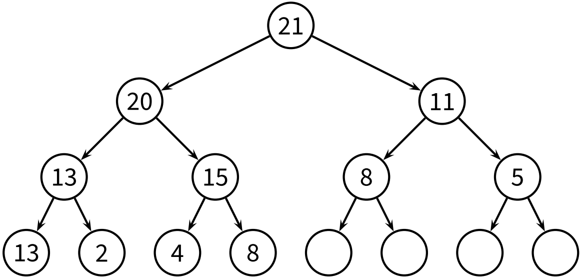

Max-Heapify

```
MAX-HEAPIFY(A, i)
l = LEFT(i)
r = RIGHT(i)
if l ≤ A.heap-size and A[l] > A[i]
        largest = l
else largest = i
if r ≤ A.heap-size and A[r] > A[largest]
        largest = r
if largest ≠ i
        swap A[i] and A[largest]
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$$T(n) = \Theta(\log n)$$

Building a Heap

Building a Heap

BUILD-MAX-HEAP(A)

$A.\mathit{heap\text{-}size} = \mathit{length}(A)$
for $i = \lfloor \mathit{length}(A)/2 \rfloor$ **downto** 1
    MAX-HEAPIFY(A, i)

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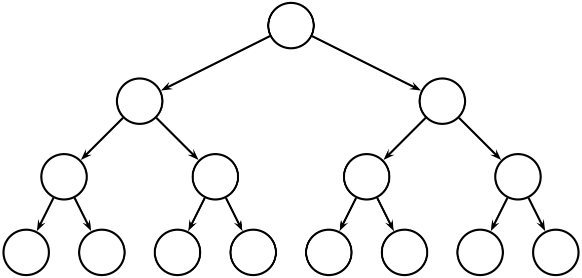

BUILD-MAX-HEAP(A)

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1  A.heap-size = length(A)
2  for i = ⌊length(A)/2⌋ downto 1
3      MAX-HEAPIFY(A, i)
```
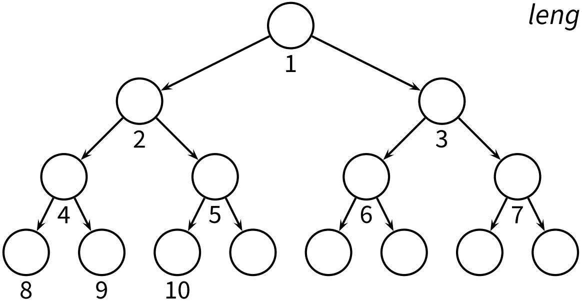

$length(A) = 10$



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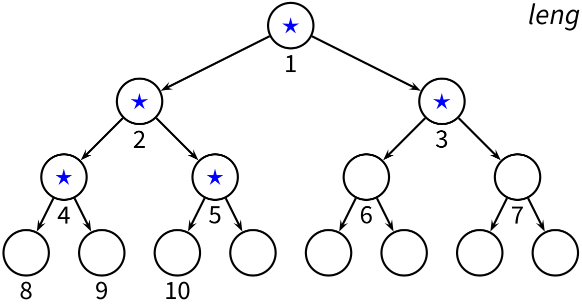

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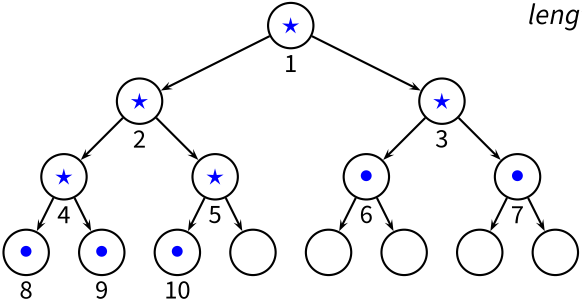

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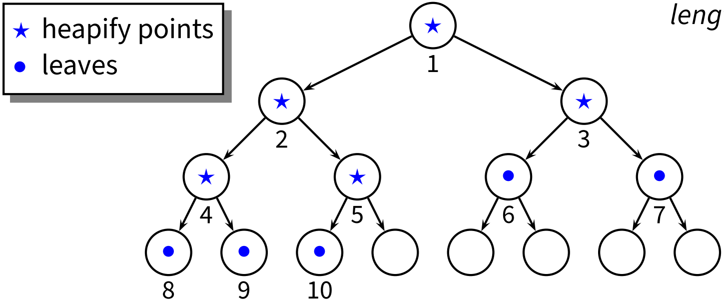

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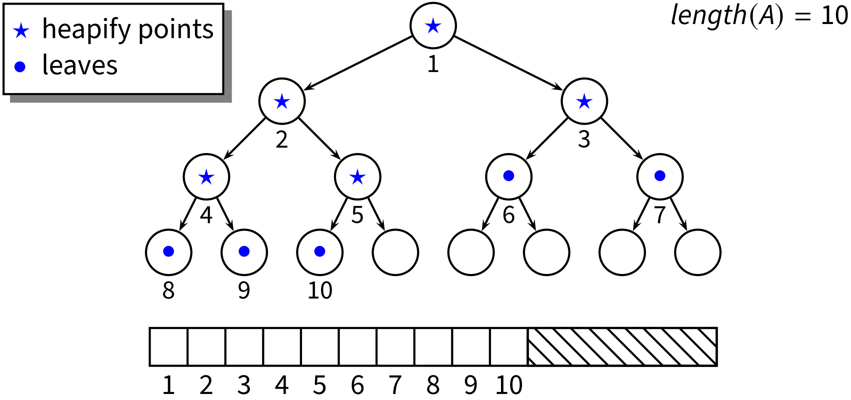

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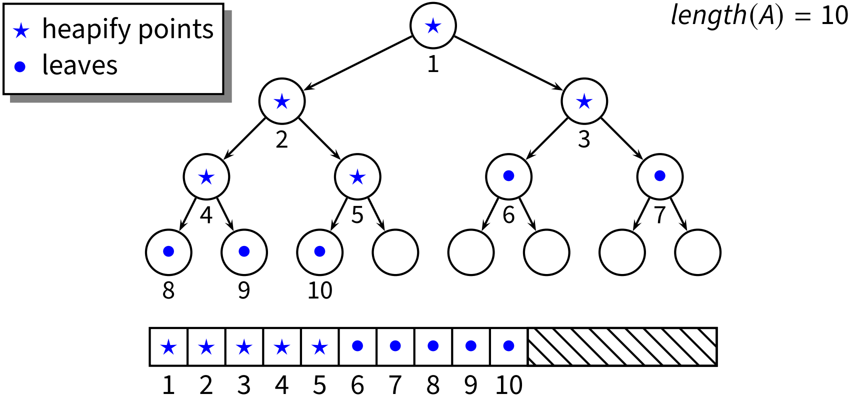

Heap Sort

- Idea: we can use a heap to sort an array

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HEAP-SORT(A)
BUILD-MAX-HEAP(A)
for i = length(A) downto 1
    swap A[i] and A[1]
    A.heap-size = A.heap-size − 1
    MAX-HEAPIFY(A, 1)
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- What is the complexity of **Heap-Sort**?

$$T(n) = \Theta(n \log n)$$

- Benefits
  - in-place sorting; worst-case is $\Theta(n \log n)$

Summary of Sorting Algorithms

Algorithm	Complexity			In place?
	worst	*average*	*best*	
INSERTION-SORT				

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INSERTION-SORT	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n)$	yes
SELECTION-SORT	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$	yes
MERGE-SORT	$\Theta(n \log n)$	$\Theta(n \log n)$	$\Theta(n \log n)$	no
QUICK-SORT				

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