Randomized Algorithms

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Outline

- Examples
- Las Vegas and Monte Carlo algorithms
- More examples

Remember Quick-Sort?

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  - $A[1 \dots q-1]$ contain elements that are *less than or equal to* v
  - $A[q] = v$ is the “pivot” value ($v \in A$)
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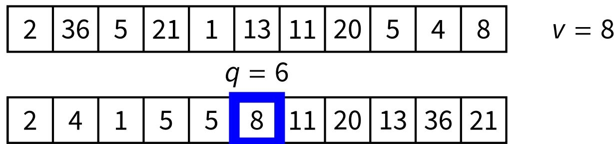

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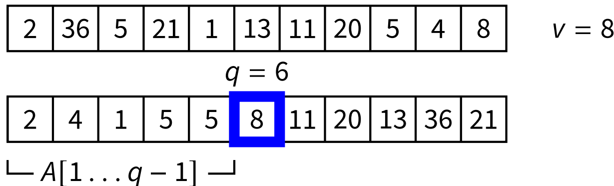

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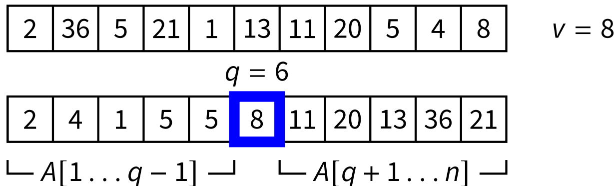

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```
PARTITION(A, begin, end)
1  q = begin
2  v = A[end]  // we deterministically choose v
3  for i = begin to end
4        if A[i] ≤ v
5              swap(A[i], A[q])
6              q = q + 1
7  return q − 1
```

Complexity of QuickSort

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  - the partition transforms P of size n into *two* problems P of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil - 1$, respectively; so $T(n) = 2T(n/2) + \Theta(n)$

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- *Worst case:* $q = 1$ or $q = n$ ← **not so improbable!**
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A Randomized Solution

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- Simple

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RANDOMIZED-PARTITION(A, begin, end)
1   i = RANDOM(begin, end)
2   swap(A[i], A[end])
3   return PARTITION(A, begin, end)
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Other Examples

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```
TREE-RANDOMIZED-INSERT(t, z)
 1  if t = NIL
 2      return z
 3  r = uniformly random value from {1, ..., size(t) + 1}
 4  if r = 1                          // Pr[r = 1] = 1/(size(t) + 1)
 5      size(z) = size(t) + 1
 6      return TREE-ROOT-INSERT(t, z)
 7  if key(z) < key(t)
 8      left(t) = TREE-RANDOMIZED-INSERT(left(t), z)
 9  else right(t) = TREE-RANDOMIZED-INSERT(right(t), z)
10  size(t) = size(t) + 1
11  return t
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- However, we can do a lot more with randomized algorithms...

Simple Example

- Problem: *find a zero bit*
  - *Input:* an array A, of n bits, containing more or less the same number of 1s and 0s (*hamming weight* is roughly $n/2$)
  - *Output:* i such that $A[i] = 0$, or NIL if none exists

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1  for i = 1 to |A|
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  - any *deterministic* search strategy is vulnerable

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- Take one

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- ***Las Vegas***

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- ***Monte Carlo***

Simple Monte Carlo Algorithm

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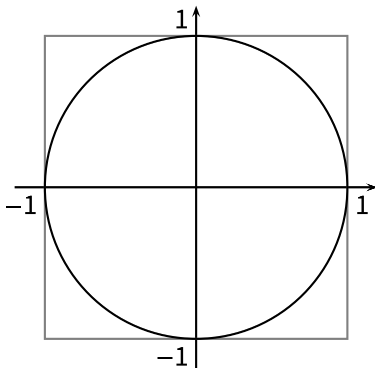
- Problem: *compute the surface of the unit disc*
  - you don't know the value of π—you may not even know that $S = \pi r^2$, but you know that a point

$$(x, y) \quad \text{is} \begin{cases} \textit{outside} \text{ the unit disc if} & x^2 + y^2 > 1 \\ \textit{inside} \text{ the unit disc if} & x^2 + y^2 \leq 1 \end{cases}$$



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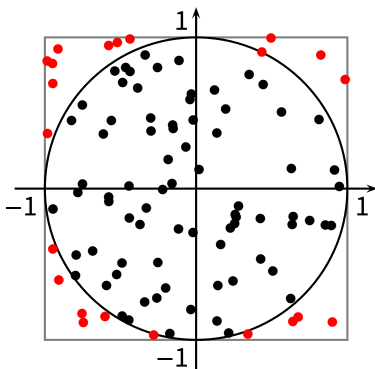

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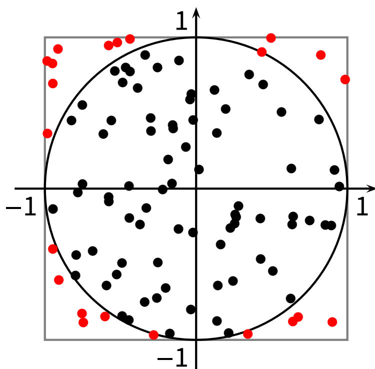

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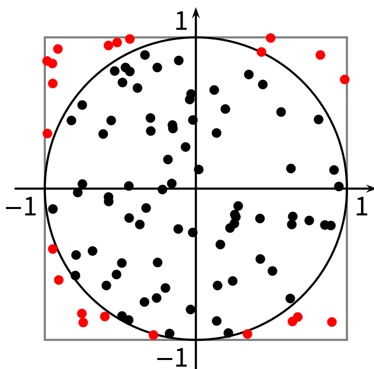

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$S = 4 \times 78/100 = 3.12$

Monte Carlo π

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2  for j = 1 to n
3      x = RANDOM(−1, 1)
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- It is also easy to think about a better (adaptive) *stopping condition* other than going through n loops