Minimal Spanning Trees

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Outline

- MST problem
- Generic algorithm
- Prim and Kruskal

- Given a weighted graph $G = (V, E)$
  - with "weight" function $w : E \longrightarrow \mathbb{R}$

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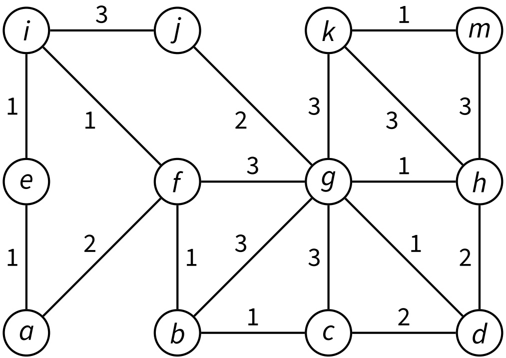
  - a ***tree***
- T "spans" the entire graph G (i.e., touches all vertexes)
  - a ***spanning tree***
- T's total weight of the tree is minimal

$$w(T) = \sum_{(u,v) \in T} w(u, v)$$

  - a ***minimum-weight spanning tree***, or "minimum spanning tree"

Example



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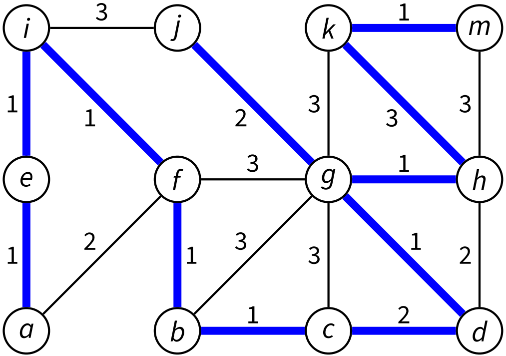

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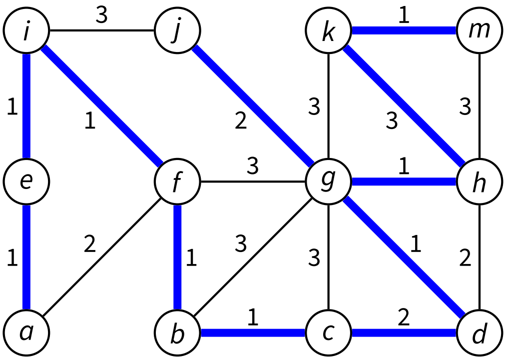

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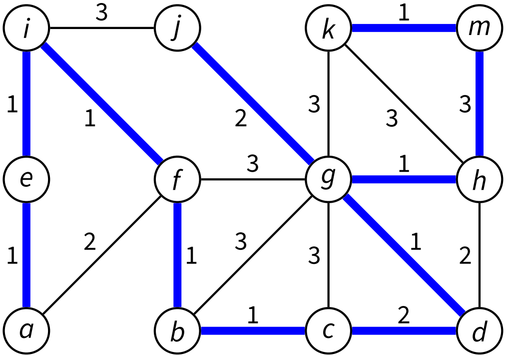

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- Does it work?

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GENERIC-MST(G, w)
A = ∅
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Refined Idea

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  - more or less the *definition* of a greedy algorithm

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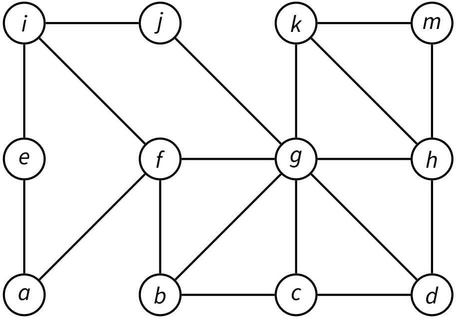

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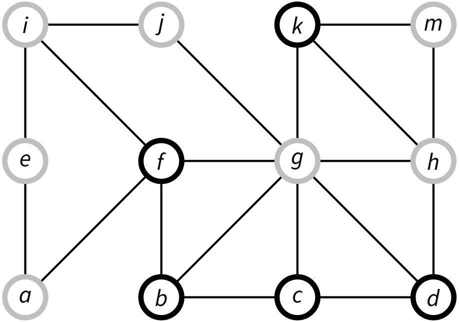

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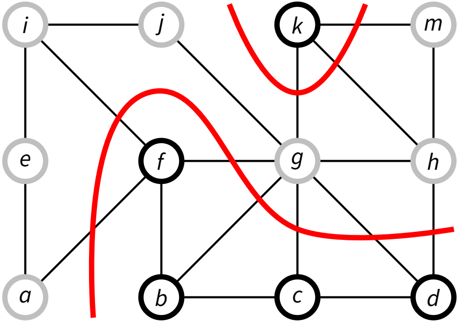

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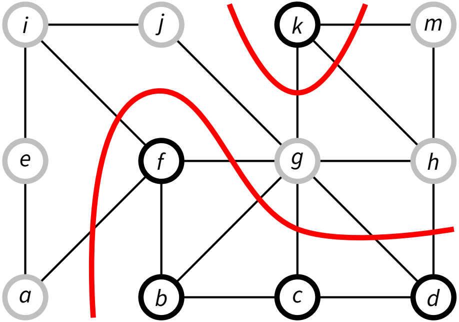

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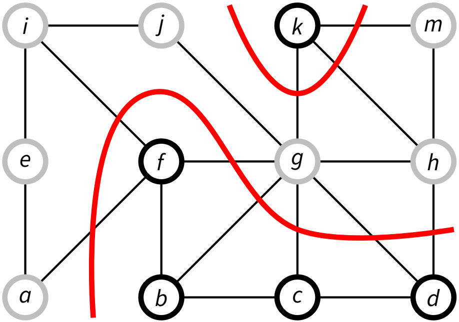

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- A cut $(S, V - S)$ *respects* a set of edges A if no edge in A crosses the cut

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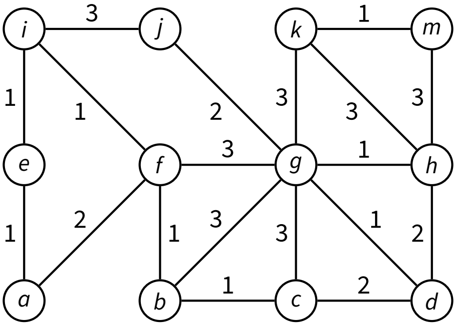

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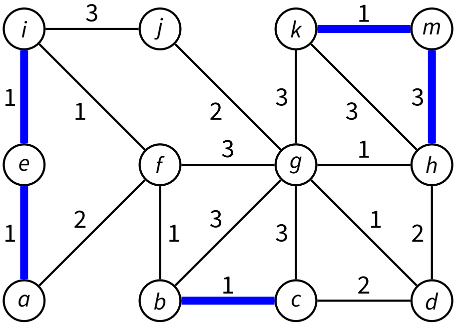

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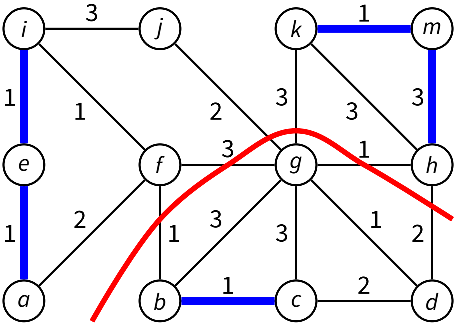

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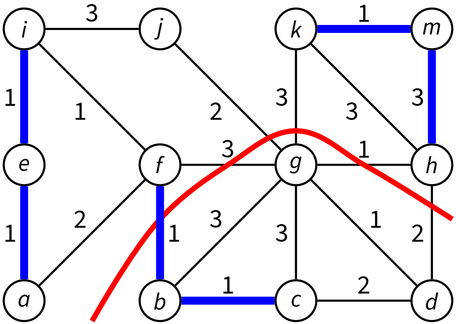

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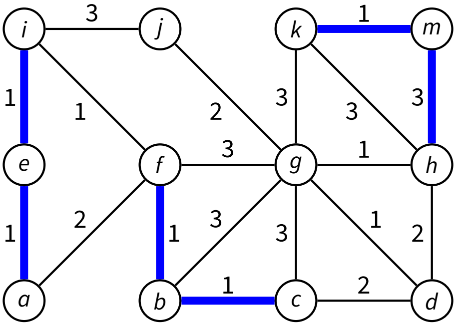

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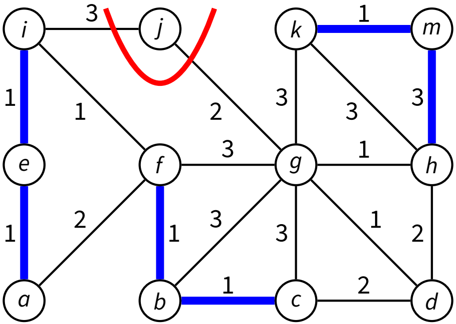

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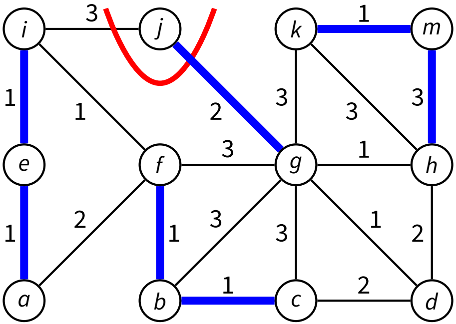

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Concrete Algorithms

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- Prim's algorithm (1957)
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- *Union*(x, y) joins the sets containing x and y

Kruskal's Algorithm

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  for each vertex v ∈ V(G)
      MAKE-SET(v)                    // disjoint-set data structure
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  for each edge (u, v) ∈ E, taken in non-decreasing order by w
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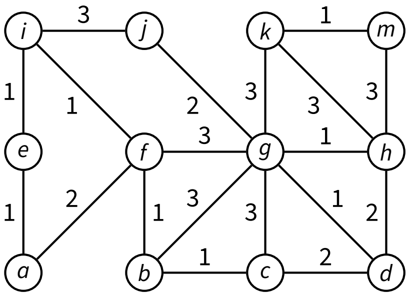

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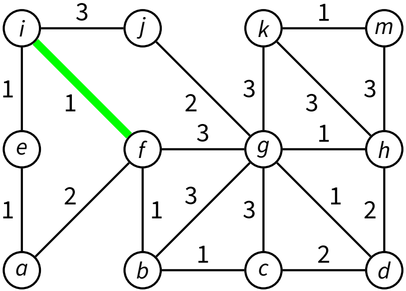

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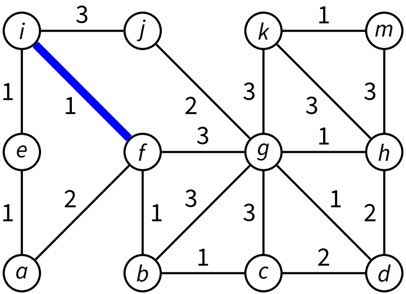

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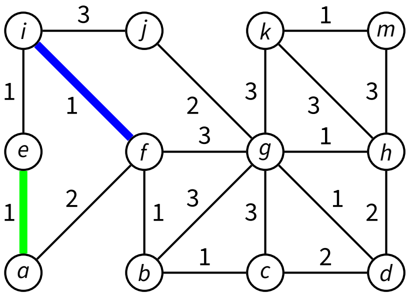

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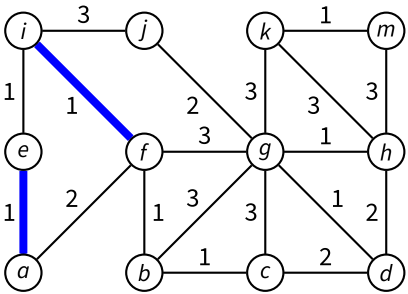

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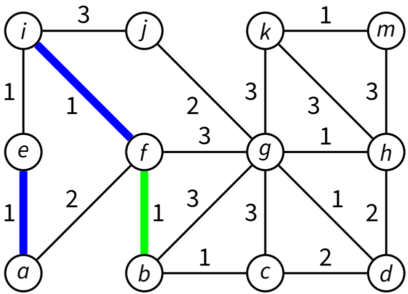

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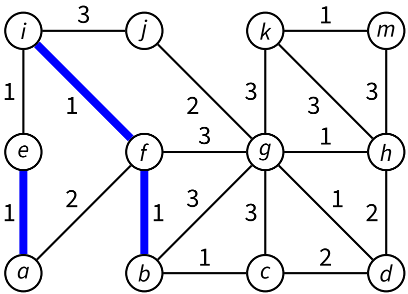

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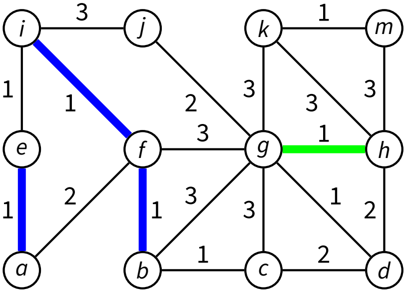

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$A = \emptyset$
for each vertex $v \in V(G)$
    MAKE-SET(v) // *disjoint-set* data structure
sort E in non-decreasing order by weight w
for each edge $(u, v) \in E$, taken in non-decreasing order by w
    if FIND-SET$(u) \neq$ **FIND-SET**(v)
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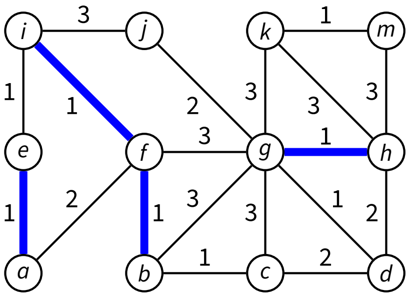

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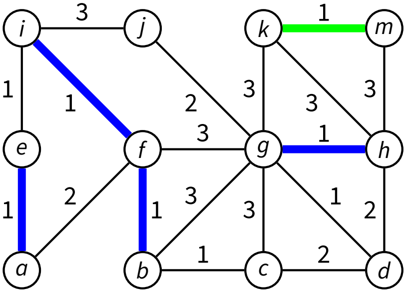

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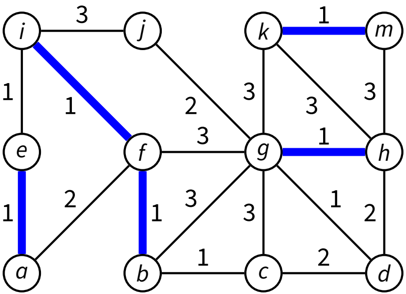

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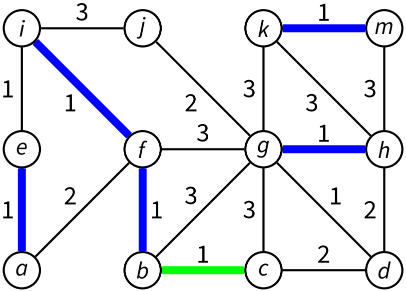

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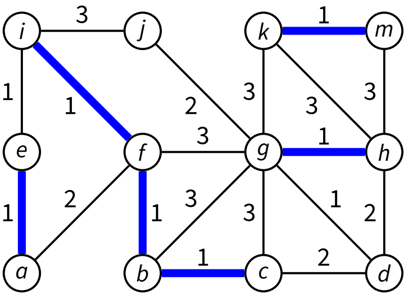

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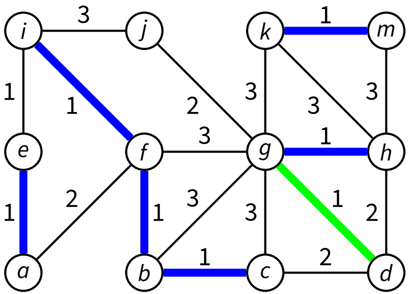

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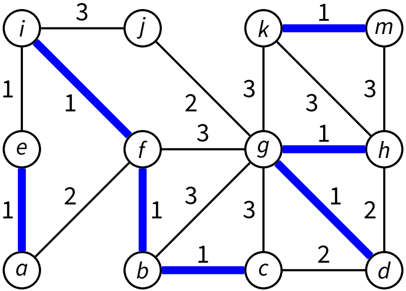

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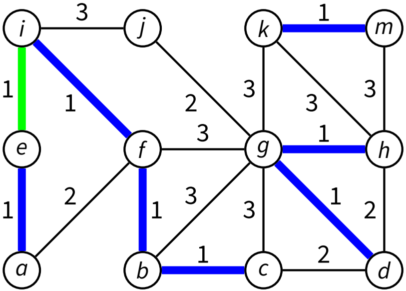

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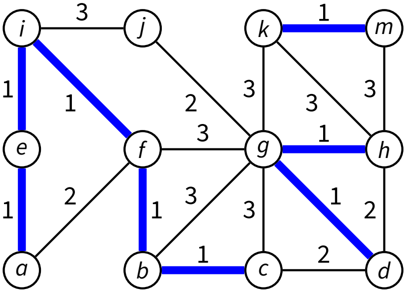

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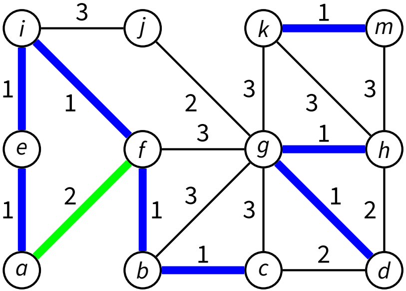

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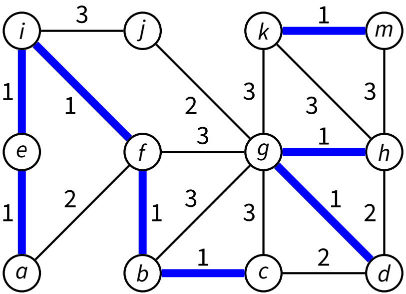

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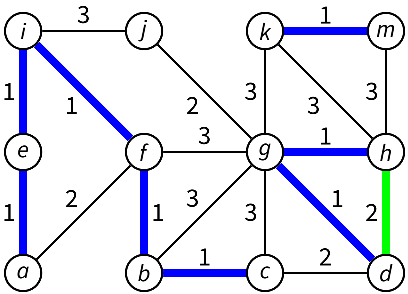

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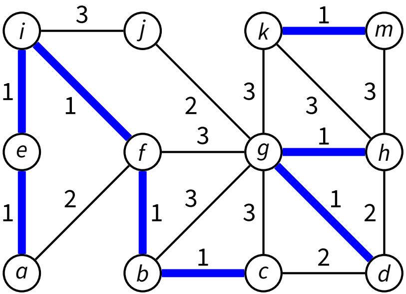

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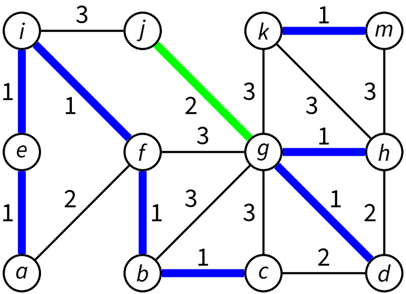

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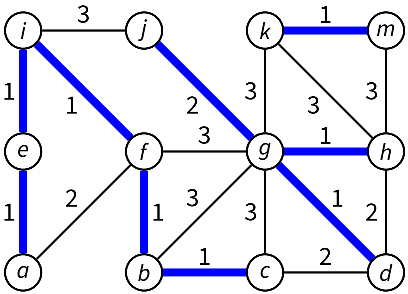

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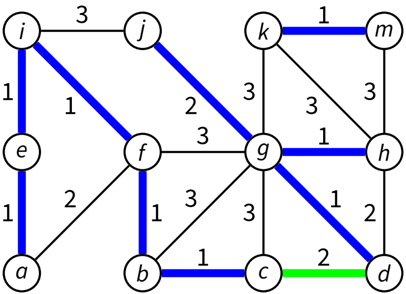

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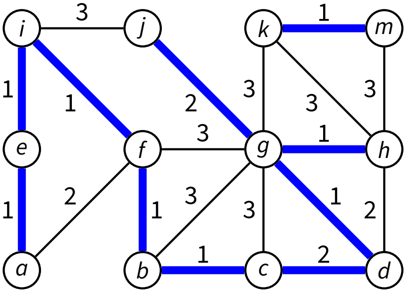

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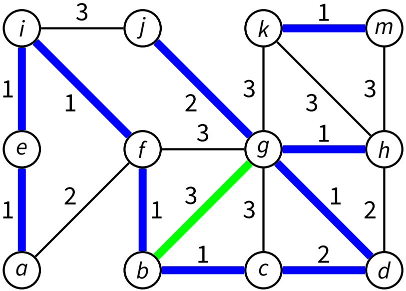

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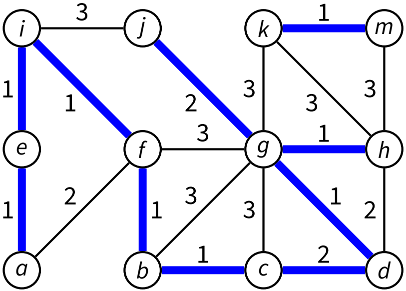

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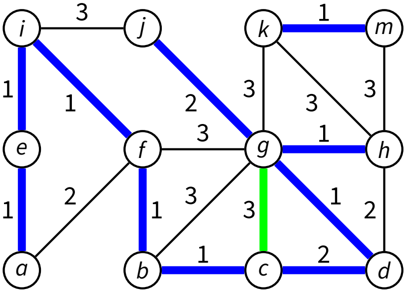

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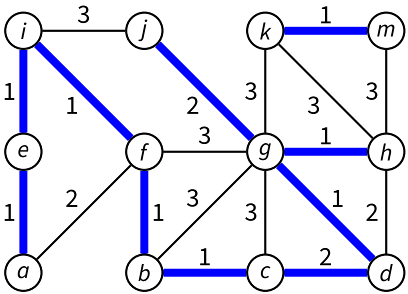

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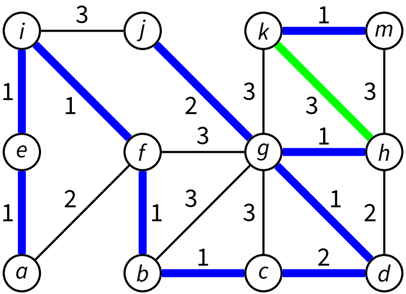

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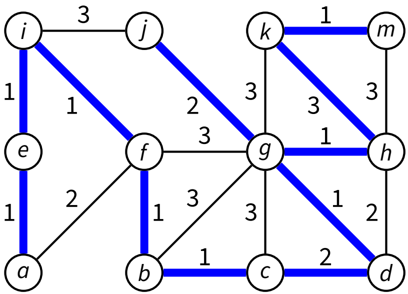

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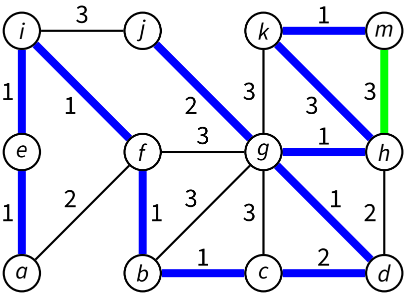

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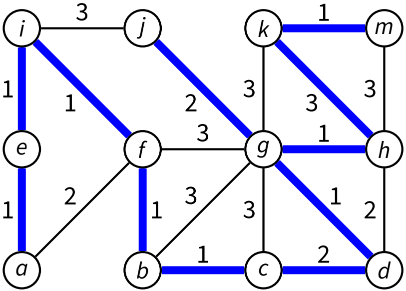

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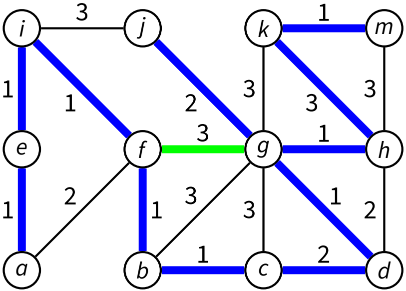

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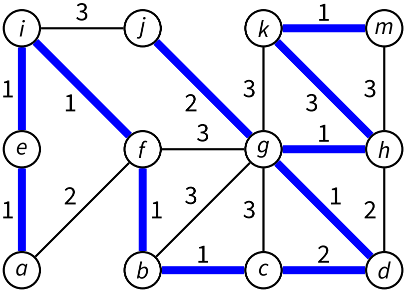

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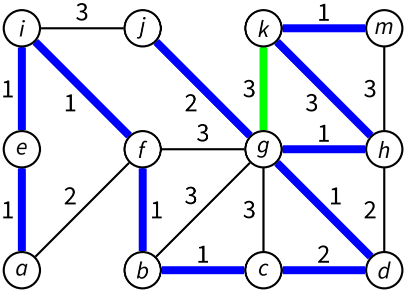

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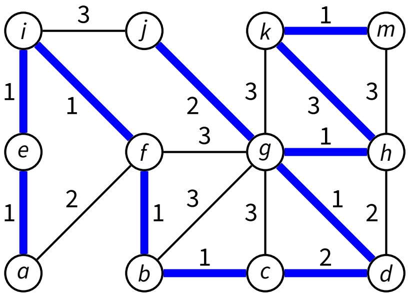

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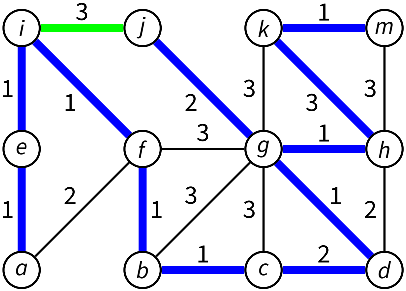

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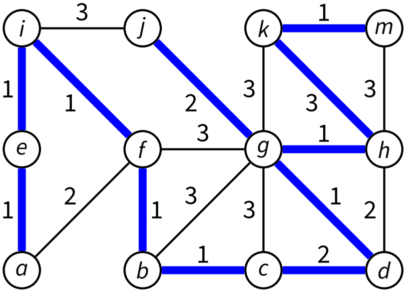

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- $|V|$ times **MAKE-SET** (loop of line 2–3)

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Prim's Algorithm

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- Variables storing values known at each iteration
  - T, tree being built
  - $W[v]$ best known cost/weight of connecting v with T
  - $P[v]$, node $u \in T$ such that the edge (u, v) is the least-cost edge connecting v with T

Prim's Algorithm

```
MST-PRIM(G, u, w)
T = (∅, ∅)
for each vertex v ∈ V(G)
    W[v] = ∞
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W[u] = 0
while V(T) ≠ V(G)
    find u ∉ V(T) such that W[u] is minimal
    T = T ∪ {u} // add u to T
    for all v ∈ Adj(u) \ V(T)
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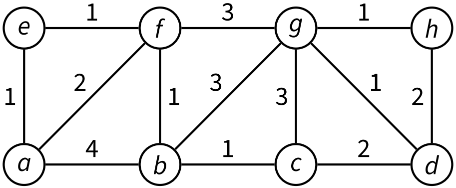

v	W	P	T
a	0		
b	∞		
c	∞		
d	∞		
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f	∞		
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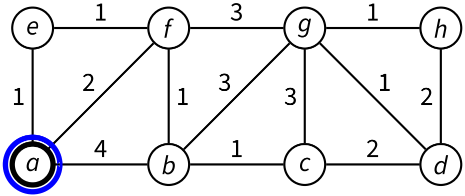

v	W	P	T
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b	∞		
c	∞		
d	∞		
e	∞		
f	∞		
g	∞		
h	∞		

Prim's Algorithm

```
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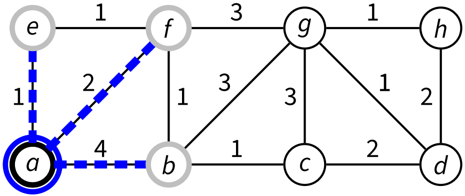
```



v	W	P	T
a	0		✓
b	4	a	
c	∞		
d	∞		
e	1	a	
f	2	a	
g	∞		
h	∞		

Prim's Algorithm

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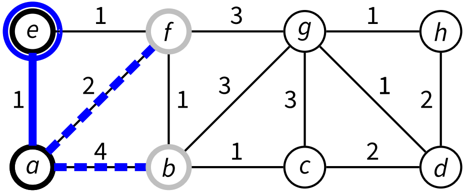

v	W	P	T
a	0		✓
b	4	a	
c	∞		
d	∞		
e	1	a	✓
f	2	a	
g	∞		
h	∞		

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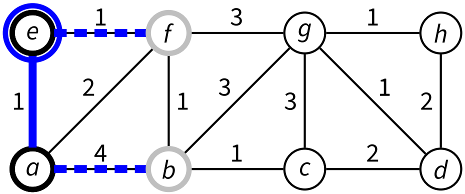

v	W	P	T
a	0		✓
b	4	a	
c	∞		
d	∞		
e	1	a	✓
f	1	e	
g	∞		
h	∞		

Prim's Algorithm

```
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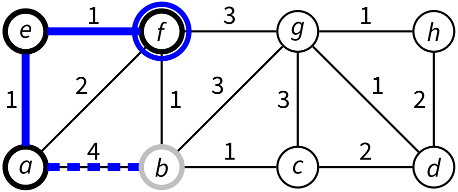

v	W	P	T
a	0		✓
b	4	a	
c	∞		
d	∞		
e	1	a	✓
f	1	e	✓
g	∞		
h	∞		

```
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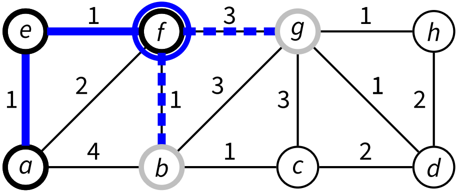

v	W	P	T
a	0		✓
b	1	f	
c	∞		
d	∞		
e	1	a	✓
f	1	e	✓
g	3	f	
h	∞		

Prim's Algorithm

```
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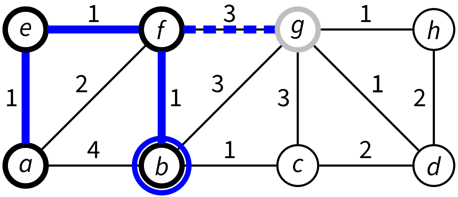

v	W	P	T
a	0		✓
b	1	f	✓
c	∞		
d	∞		
e	1	a	✓
f	1	e	✓
g	3	f	
h	∞		

```
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            W[v] = w(u, v)
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```

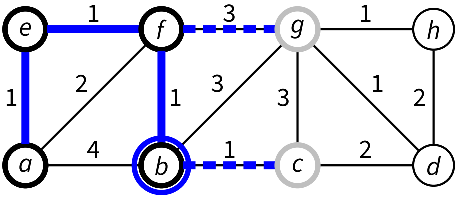

v	W	P	T
a	0		✓
b	1	f	✓
c	1	b	
d	∞		
e	1	a	✓
f	1	e	✓
g	3	f	
h	∞		

Prim's Algorithm

```
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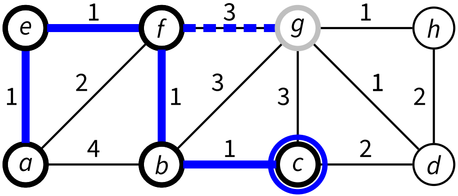

v	W	P	T
a	0		✓
b	1	f	✓
c	1	b	✓
d	∞		
e	1	a	✓
f	1	e	✓
g	3	f	
h	∞		

Prim's Algorithm

```
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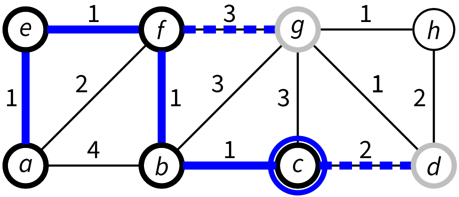
```



v	W	P	T
a	0		✓
b	1	f	✓
c	1	b	✓
d	2	c	
e	1	a	✓
f	1	e	✓
g	3	f	
h	∞		

Prim's Algorithm

```
MST-PRIM(G, u, w)
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    T = T ∪ {u} // add u to T
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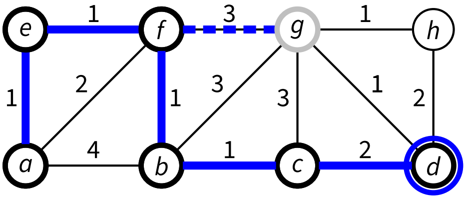
        if w(u, v) < W[v]
            W[v] = w(u, v)
            P[v] = u
```



v	W	P	T
a	0		✓
b	1	f	✓
c	1	b	✓
d	2	c	✓
e	1	a	✓
f	1	e	✓
g	3	f	
h	∞		

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```
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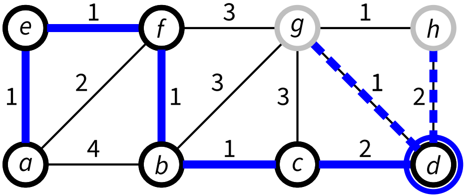

v	W	P	T
a	0		✓
b	1	f	✓
c	1	b	✓
d	2	c	✓
e	1	a	✓
f	1	e	✓
g	1	d	
h	2	d	

Prim's Algorithm

```
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        T = T ∪ {u} // add u to T
        for all v ∈ Adj(u) \ V(T)
                if w(u, v) < W[v]
                        W[v] = w(u, v)
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```
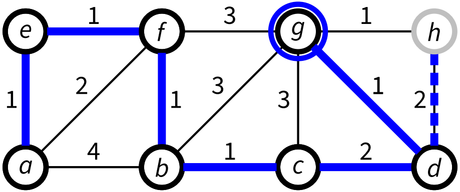
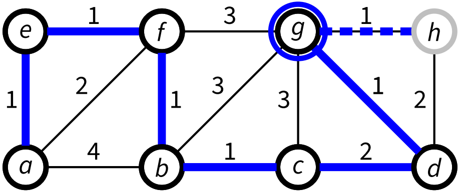



v	W	P	T
a	0		✓
b	1	f	✓
c	1	b	✓
d	2	c	✓
e	1	a	✓
f	1	e	✓
g	1	d	✓
h	2	d	

MST-PRIM(G, u, w)

$T = (\emptyset, \emptyset)$
for each vertex $v \in V(G)$
    $W[v] = \infty$
    $P[v] = \text{NIL}$
$W[u] = 0$
while $V(T) \neq V(G)$
    find $u \notin V(T)$ such that $W[u]$ is minimal
    $T = T \cup \{u\}$ // add u to T
    for all $v \in Adj(u) \setminus V(T)$
        if $w(u, v) < W[v]$
            $W[v] = w(u, v)$
            $P[v] = u$



v	W	P	T
a	0		✓
b	1	f	✓
c	1	b	✓
d	2	c	✓
e	1	a	✓
f	1	e	✓
g	1	d	✓
h	1	g	

Prim’s Algorithm

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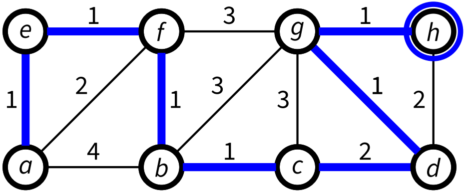

v	W	P	T
a	0		✓
b	1	f	✓
c	1	b	✓
d	2	c	✓
e	1	a	✓
f	1	e	✓
g	1	d	✓
h	1	g	✓

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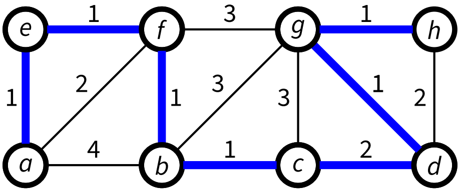

v	W	P	T
a	0		✓
b	1	f	✓
c	1	b	✓
d	2	c	✓
e	1	a	✓
f	1	e	✓
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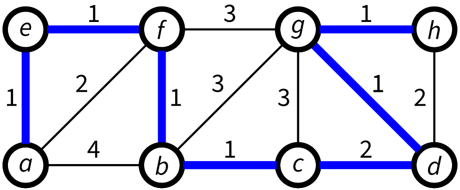

v	W	P	T
a	0		✓
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c	1	b	✓
d	2	c	✓
e	1	a	✓
f	1	e	✓
g	1	d	✓
h	1	g	✓