Divide-and-Conquer Algorithms

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Outline

- Merging (or set union)
- Searching
- Sorting
- Multiplying
- Computing the *median*

Merging (Set Union)

- **Input:** sequences $A = \langle a_1, a_2, \ldots, a_n \rangle$ and $B = \langle b_1, b_2, \ldots, b_m \rangle$

  Output: a sequence $X = \langle x_1, x_2, \ldots, x_\ell \rangle$ such that

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- **Example:**

  $A = \langle 34, 7, 11, 31, 14, 51, 8, 21, 10 \rangle$

  $B = \langle 51, 21, 14, 15, 27, 31, 2 \rangle$

  $X =$

Merging (Set Union)

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  $B = \langle 51, 21, 14, 15, 27, 31, 2 \rangle$

  $X = \langle 34, 7, 11, 31, 14, 51, 8, 21, 10, 15, 27, 2 \rangle$

A Simple Merge Algorithm

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  - iterate through every position i, first through A, and then B
  - output a_i if a_i is not in $\langle a_1, a_2, \ldots, a_{i-1} \rangle$
  - output b_i if b_i is not in $\langle a_1, a_2, \ldots, a_n, b_1, b_2, \ldots b_{i-1} \rangle$

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MergeSimple(A, B)
for i = 1 to length(A)
        if not Find(A[1 . . i − 1], A[i])
                output A[i]
for i = 1 to length(B)
        if not Find(A, B[i]) and not Find(B[1 . . i − 1], B[i])
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Complexity

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```

let $n = length(A) + length(B)$

$$T(n) = \sum_{i=1}^{length(A)} T_{\textsc{Find}}(i) + \sum_{i=1}^{length(B)} \left(T_{\textsc{Find}}(i) + T_{\textsc{Find}}(length(A))\right)$$

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$$T(n) = \sum_{i=1}^{n} T_{\textbf{FIND}}(i)$$

- **Input:** a sequence *A* and a value *key*
  Output: TRUE if *A* contains *key*, or FALSE otherwise

Searching

- **Input:** a sequence *A* and a value *key*
  Output: TRUE if *A* contains *key*, or FALSE otherwise

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FIND(A, key)
for i = 1 to length(A)
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                return TRUE
return FALSE
```

```
FIND(A, begin, end, key)
for i = begin to end
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- The complexity of **FIND** is

$$T(n) = O(n)$$

- **Input:** a sequence *A* and a value *key*

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Searching in a List

- **Input:** a sequence *A* and a value *key*

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FINDINLIST(A, key)
item = first(A)
while item ≠ last(A)
    if value(item) == key
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Complexity of MergeSimple

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first = 1
last = length(A)
while first ≤ last
        middle = ⌈(first + last)/2⌉
        if A[middle] == key
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Binary Search

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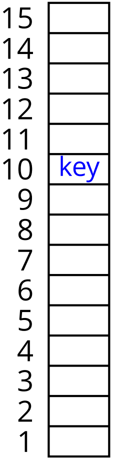

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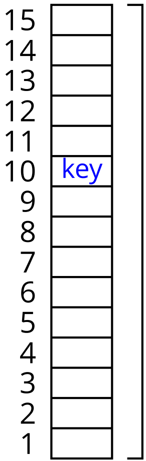

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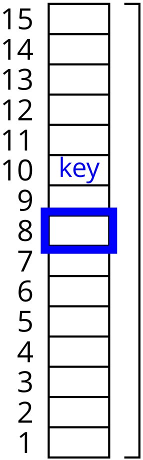

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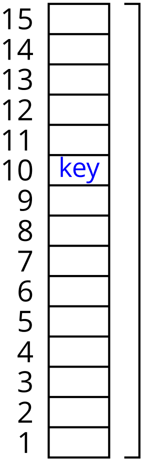

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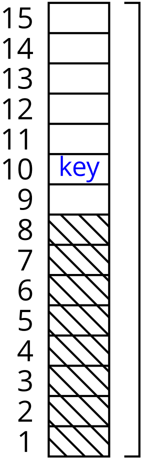

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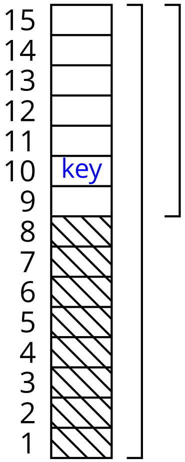

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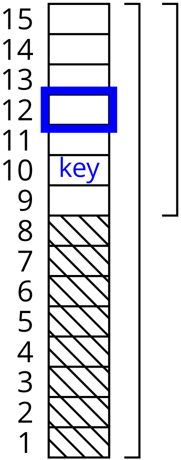

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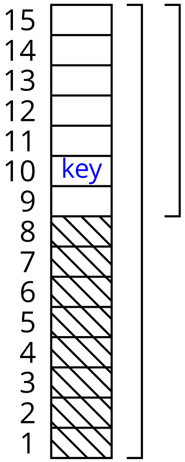

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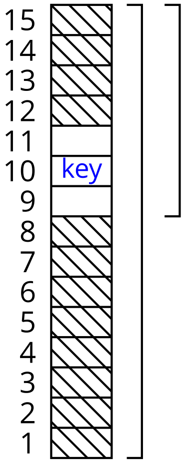

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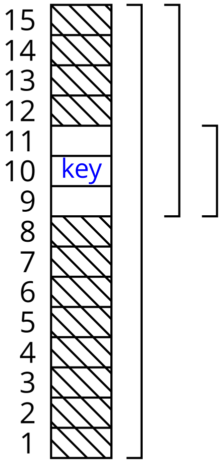

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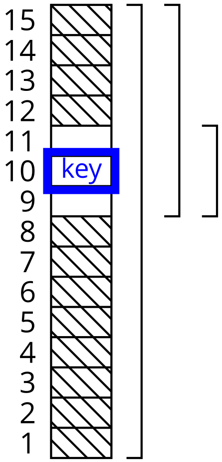

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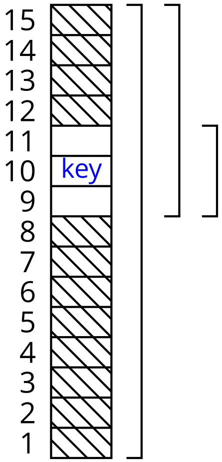

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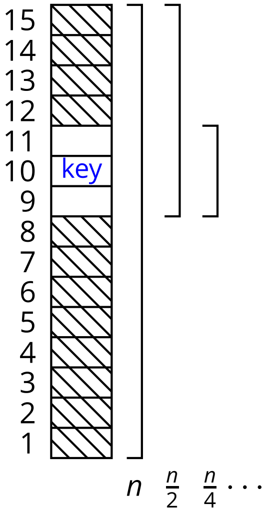

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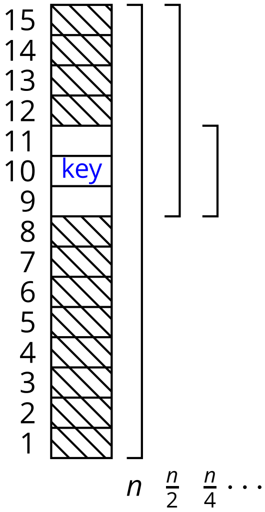

$$T(n) = O(\log n)$$

- A slightly different problem:

  Input: two *sorted* sequences $A = \langle a_1, a_2, \ldots, a_n \rangle$ and $B = \langle b_1, b_2, \ldots, b_m \rangle$, where $a_1 \leq a_2 \leq \ldots \leq a_n$ and $b_1 \leq b_2 \leq \ldots \leq b_m$

  Output: a sequence $X = \langle x_1, x_2, \ldots, x_\ell \rangle$ such that

  - every element of A appears once in X
  - every element of B appears once in X
  - every element of X appears in A or in B or in both

A Better Merge Algorithm

```
MergeSimple2(A, B)
for i = 1 to length(A)
        if not BinarySearch(A[1 . . i − 1], A[i])
                output A[i]
for i = 1 to length(B)
        if not BinarySearch(A, B[i])
        and not BinarySearch(B[1 . . i − 1], B[i])
                output B[i]
```

```
MergeSimple2(A, B)
for i = 1 to length(A)
    if not BinarySearch(A[1 .. i − 1], A[i])
        output A[i]
for i = 1 to length(B)
    if not BinarySearch(A, B[i])
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```

$$T(n) = \sum_{i=1}^{n} O(\log i) =$$

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for i = 1 to length(A)
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```

$$T(n) = \sum_{i=1}^{n} O(\log i) = O(n \log n)$$

A Better Merge Algorithm

```
MERGESIMPLE2(A, B)
for i = 1 to length(A)
        if not BINARYSEARCH(A[1 . . i − 1], A[i])
                output A[i]
for i = 1 to length(B)
        if not BINARYSEARCH(A, B[i])
        and not BINARYSEARCH(B[1 . . i − 1], B[i])
                output B[i]
```

$$T(n) = \sum_{i=1}^{n} O(\log i) = O(n \log n)$$

Better than $O(n^2)$, but can we do even better than $O(n \log n)$?

An Even Better Merge Algorithm

- *Intuition:* A and B are *sorted*
  e.g.
  $A = \langle 3, 7, 12, 13, 34, 37, 70, 75, 80 \rangle$
  $B = \langle 1, 5, 6, 7, 34, 35, 40, 41, 43 \rangle$

- *Intuition:* A and B are *sorted*
  e.g.
  $A = \langle 3, 7, 12, 13, 34, 37, 70, 75, 80 \rangle$
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  so just like in **BinarySearch** I can avoid looking for an element x if the *first* element I see is $y > x$

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  $A = \langle 3, 7, 12, 13, 34, 37, 70, 75, 80 \rangle$
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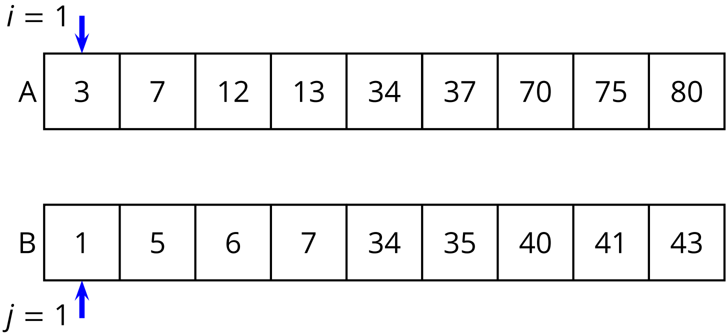
  so just like in **BinarySearch** I can avoid looking for an element x if the *first* element I see is $y > x$

- High-level algorithm strategy
  - step through every position i of A and every position j of B
  - output a_i and advance i if $a_i \leq b_j$ or if j is beyond the end of B
  - output b_j and advance j if $a_i \geq b_j$ or if i is beyond the end of A

A	3	7	12	13	34	37	70	75	80

B	1	5	6	7	34	35	40	41	43

MERGE Algorithm



Output:

MERGE Algorithm

$i = 1$

A	3	7	12	13	34	37	70	75	80

B	1	5	6	7	34	35	40	41	43

$j = 1$

Output:

MERGE Algorithm

$i = 1$

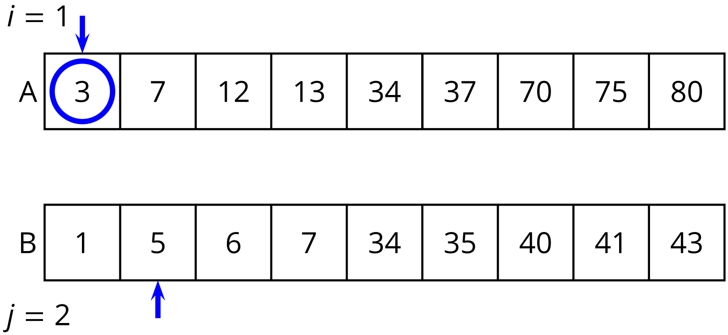
A	3	7	12	13	34	37	70	75	80

B	1	5	6	7	34	35	40	41	43

$j = 2$

Output: 1

MERGE Algorithm



Output: 1

$i = 2$

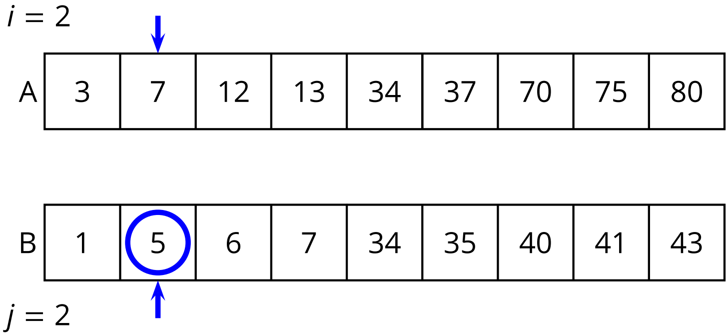
A	3	7	12	13	34	37	70	75	80

B	1	5	6	7	34	35	40	41	43

$j = 2$

Output: 1 3

MERGE Algorithm



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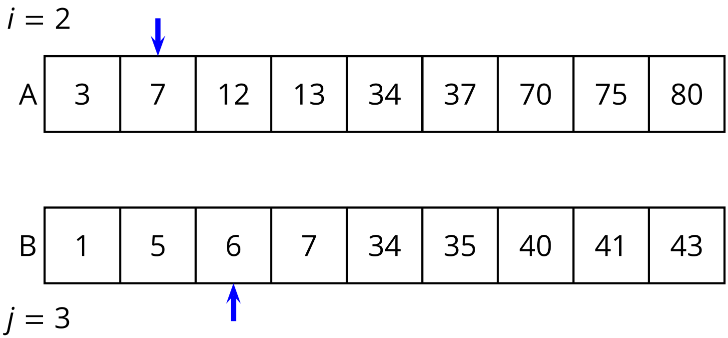

Output: 1 3 5

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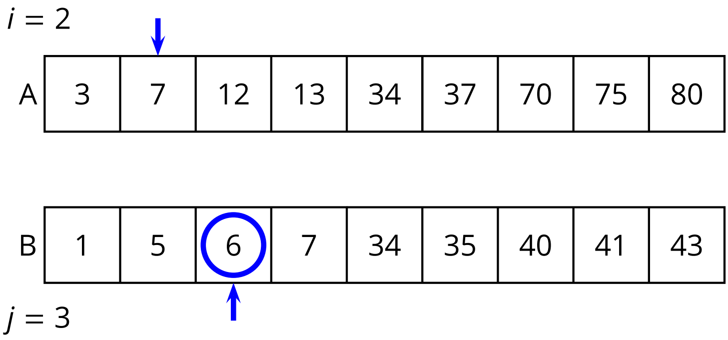

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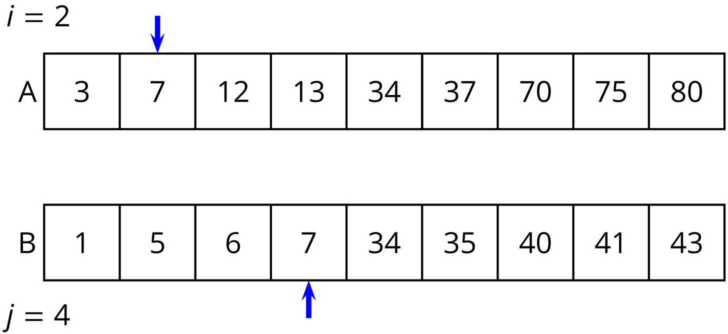
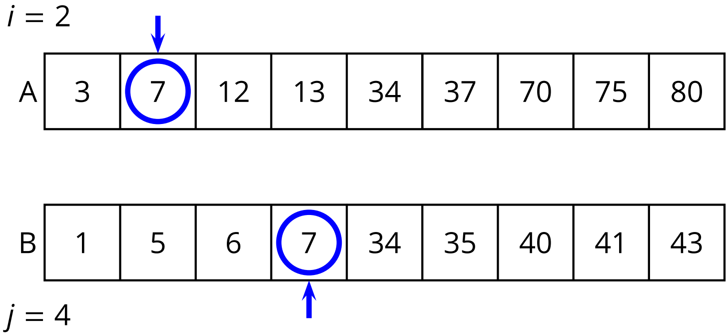


Output: 1 3 5 6



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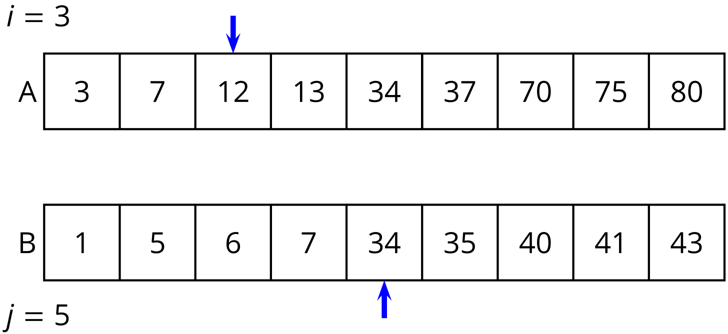
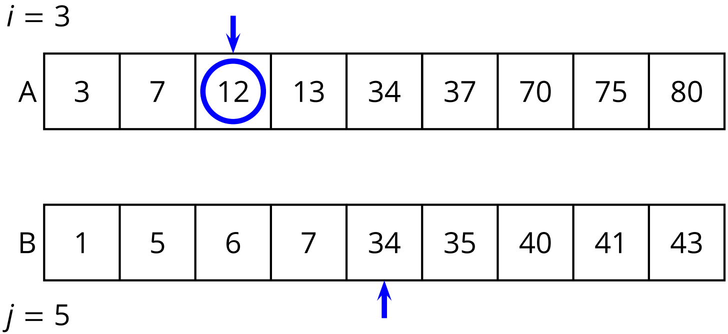


Output: 1 3 5 6 7



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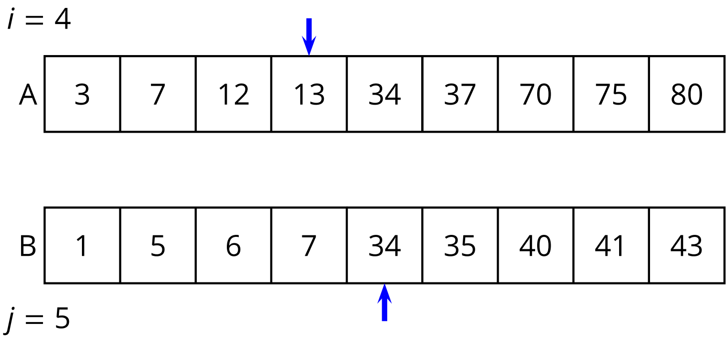

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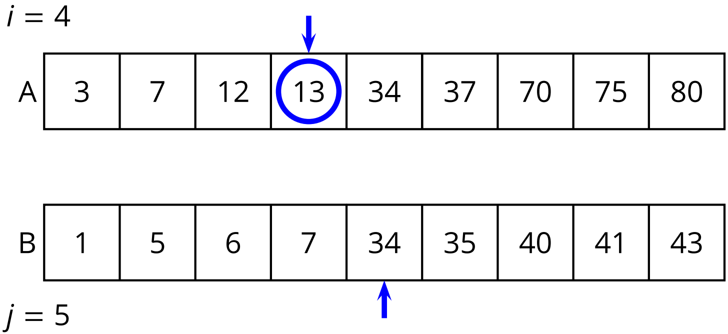

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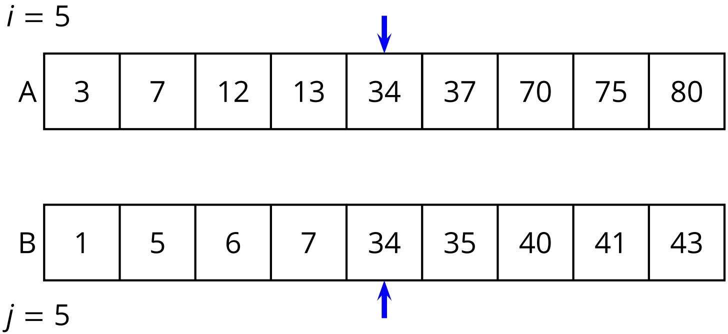

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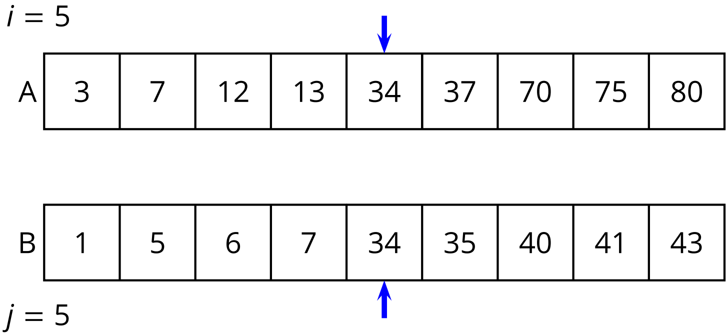

Output: 1 3 5 6 7 12 13. . .

Merge Algorithm (2)

```
MERGE(A, B)
i, j = 1
X = ∅
while i ≤ length(A) or j ≤ length(B)
        if i > length(A)
                X = X ∘ B[j]            // appends B[j] to X
                j = j + 1
        elseif j > length(B)
                X = X ∘ A[i]
                i = i + 1
        elseif A[i] < B[j]
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- This algorithm is incorrect! (Exercise: fix it)

Complexity of MERGE

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$$T(n) = \Theta(n)$$

- Can we do better? No!
  - we have to output $n = \mathit{length}(A) + \mathit{length}(B)$ elements

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Using MERGE

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  - this suggests a recursive algorithm

Merge Sort

```
MergeSort(A)
if length(A) == 1
        return A
m = ⌊length(A)/2⌋
A_L = MergeSort(A[1 . . m])
A_R = MergeSort(A[m + 1 . . length(A)])
return Merge(A_L, A_R)
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- Complexity analysis

$$T(n) = T_{\text{divide}} + \sum_{i=1}^{k} T(|A_i|) + T_{\text{combine}}$$

we will analyze this formula another time...

A Divide-and-Conquer Merge

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MERGER(A, B)
if length(A) == 0
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- Can we do better?

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- Can we do better? No! (We knew that already)

Divide-and-Conquer Multiplication

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  $x = \boxed{X_L}\,\boxed{X_R}$ and $y = \boxed{Y_L}\,\boxed{Y_R}$

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  $x = \boxed{X_L}\boxed{X_R}$ and $y = \boxed{Y_L}\boxed{Y_R}$

  which means $x = 2^{\ell/2}x_L + x_R$ and $y = 2^{\ell/2}y_L + y_R$, so…

$$\begin{aligned} xy &= (2^{\ell/2}x_L + x_R)(2^{\ell/2}y_L + y_R) \\ &= 2^{\ell}x_Ly_L + 2^{\ell/2}(x_Ly_R + x_Ry_L) + x_Ry_R \end{aligned}$$

  we reduced the problem of multiplying two numbers of ℓ bits into the problem of multiplying *four* numbers of $\ell/2$ bits…

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Only 3 multiplications: x_Ly_L, $(x_L + x_R)(y_R + y_L)$, and x_Ry_R

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$$T(\ell) = 3T(\ell/2) + O(\ell)$$

which, as we will see, leads to a much better complexity

$$T(\ell) = O(\ell^{\log_2 3}) = O(\ell^{1.59})$$

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- Is it correct? Yes
- How long does it take? $T(n) = T_{\text{MergeSort}}(n) = O(n \log n)$
- Can we do better? Let's try *divide-and-conquer*...

Computing the Median (2)

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    the 6th smallest element of A—a.k.a. $select(A, 6)$—is 8

k-Smallest Element

- Idea: we split the sequence A in three parts based on a *chosen value* $v \in A$
  - A_L contains the set of elements that are *less than v*
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It is the 2nd smallest value of A_R

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We use $select(A, k)$ to denote the k-smallest element of A

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  - so, ideally we should pick $v = median(A)$, but...
- We pick *a random element of A*

Selection Algorithm

```
SELECTION(A, k)
v = A[random(1 . . . |A|)]
A_L, A_v, A_R = ∅
for i = 1 to |A|
        if A[i] < v
                A_L = A_L ∪ A[i]
        elseif A[i] == v
                A_v = A_v ∪ A[i]
        else A_R = A_R ∪ A[i]
if k ≤ |A_L|
        return SELECTION(A_L, k)
elseif k > |A_L| + |A_v|
        return SELECTION(A_R, k − |A_L| − |A_v|)
else return v
```