Analysis of Insertion Sort

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Outline

- Sorting
- Insertion Sort
- Analysis

- **Input:** a sequence $A = \langle a_1, a_2, \ldots, a_n \rangle$

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Sorting

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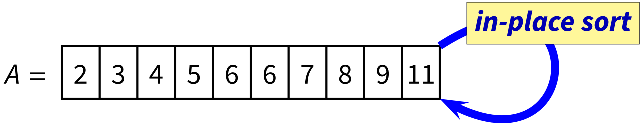

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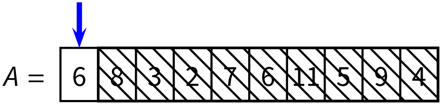

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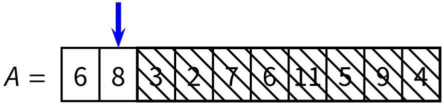

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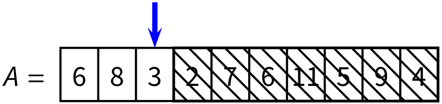

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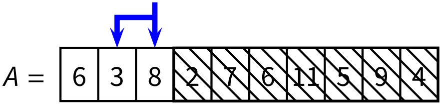

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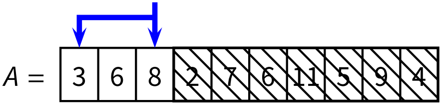

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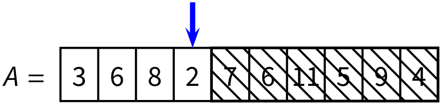

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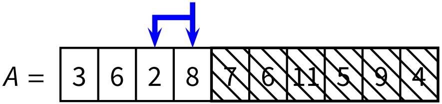

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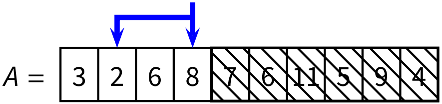

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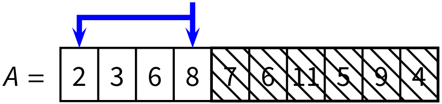

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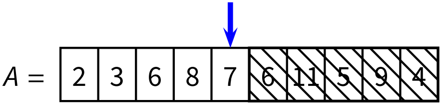

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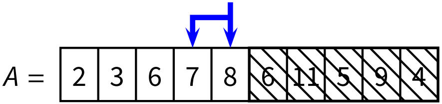

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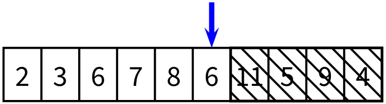

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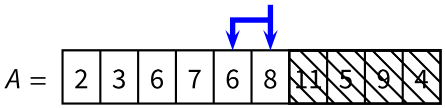

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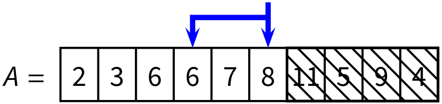

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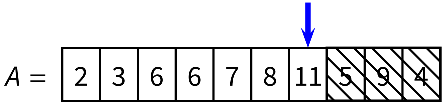

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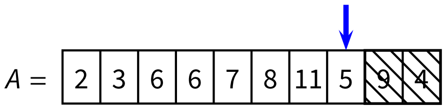

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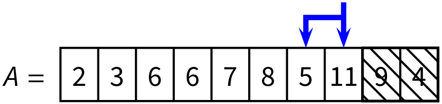

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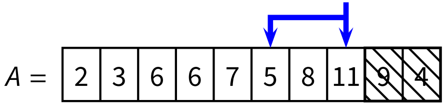

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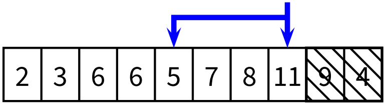
$A =$ 

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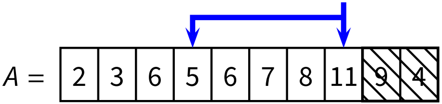

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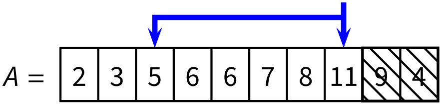

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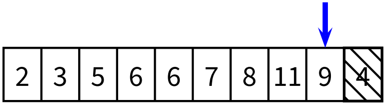

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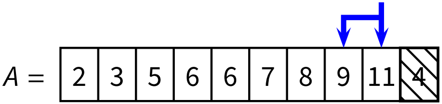

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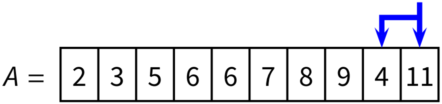

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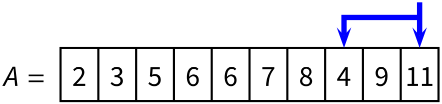

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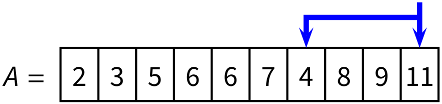

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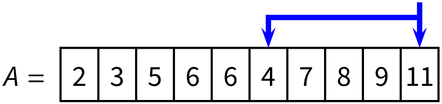

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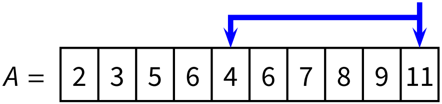

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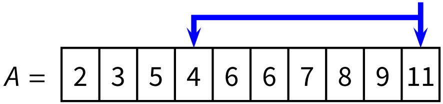

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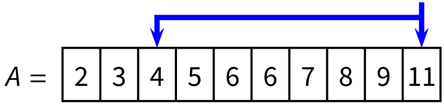

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Insertion Sort (2)

```
INSERTION-SORT(A)
for i = 2 to length(A)
    j = i
    while j > 1 and A[j − 1] > A[j]
        swap A[j] and A[j − 1]
        j = j − 1
```

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- Is **Insertion-Sort** *correct?*
- What is the time complexity of **Insertion-Sort**?
- Can we do better?

Complexity of INSERTION-SORT

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Complexity of INSERTION-SORT

```
INSERTION-SORT(A)
1  for i = 2 to length(A)
2      j = i
3      while j > 1 and A[j − 1] > A[j]
4          swap A[j] and A[j − 1]
5          j = j − 1
```

- Outer loop (lines 1–5) runs exactly $n - 1$ times (with $n = \mathit{length}(A)$)
- What about the inner loop (lines 3–5)?
  - best, worst, and average case?

Complexity of INSERTION-SORT (2)

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- **Best case:**

Complexity of INSERTION-SORT (2)

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- **Best case:** the inner loop is *never* executed
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- **Worst case:** the inner loop is executed exactly $j - 1$ times for every iteration of the outer loop
  - what case is this?

Complexity of INSERTION-SORT (3)

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- We want ***a formal proof of correctness***
  - does not seem straightforward…

The Logic of Algorithmic Steps

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```

Example 2: (branching)

```
SORTTWO(A)
// must sort (in-place) an array of 2 elements
if A[1] > A[2]
      t = A[1]
      A[1] = A[2]
      A[2] = t
```

Example 3: (nested branching)

```
MaxThree(A)
// find the maximum value in an array of 3 elements
if A[1] > A[2]
        if A[2] > A[3]
                return A[1]
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else if A[3] > A[2]
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```

Is this algorithm correct?

Prove it!

Example 5: (third variant)

```
MAXTHREE(a, b, c)
  // find the maximum among 3 values
  if a > b and a > c
      return a
  if b > c
      return b
  else return c
```

Example 5: (third variant)

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Problem: what happens when we have ***loops?***

Loop Invariants

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  - C must remain true *through* the loop
  - C is relevant to the problem definition: we use C at the end of the loop to prove the correctness of the result
- Then, we only need to prove that the algorithm terminates

Loop Invariants (2)

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- Formulation: this is where we try to be smart
  - *the invariant must reflect the structure of the algorithm*
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  - *the invariant must reflect the structure of the algorithm*
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- Proof of validity (i.e., that *C* is indeed a loop invariant): typical *proof by induction*
  - ***initialization:*** we must prove that
    the invariant C is true before entering the loop
  - ***maintenance:*** we must prove that
    if *C is true at the beginning of a cycle* ***then*** *it remains true after one cycle*

Loop Invariant for Insertion-Sort

```
INSERTION-SORT(A)
for i = 2 to length(A)
    j = i
    while j > 1 and A[j − 1] > A[j]
        swap A[j] and A[j − 1]
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- The main idea is to insert $A[i]$ in $A[1\,..\,i-1]$ so as to maintain a *sorted subsequence* $A[1\,..\,i]$

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- ***Invariant:*** (outer loop) *the subarray* $A[1\,..\,i-1]$ *consists of the elements originally in* $A[1\,..\,i-1]$ *in sorted order*

Loop Invariant for INSERTION-SORT (2)

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- **Initialization:** $j = 2$, so $A[1\,..\,j-1]$ is the single element $A[1]$
  - $A[1]$ contains the original element in $A[1]$
  - $A[1]$ is trivially sorted

Loop Invariant for Insertion-Sort (3)

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Insertion-Sort(A)
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- **Maintenance:** informally, if $A[1\,..\,i-1]$ is a permutation of the original $A[1\,..\,i-1]$ and $A[1\,..\,i-1]$ is sorted (invariant), then *if* we enter the inner loop:
  - shifts the subarray $A[k\,..\,i-1]$ by one position to the right
  - inserts *key*, which was originally in $A[i]$ at its proper position $1 \leq k \leq i-1$, in sorted order

Loop Invariant for INSERTION-SORT (4)

```
INSERTION-SORT(A)
for i = 2 to length(A)
    j = i
    while j > 1 and A[j − 1] > A[j]
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- **Termination: INSERTION-SORT** terminates with $i = length(A) + 1$; the invariant states that
  - $A[1 \mathinner{..} i-1]$ is a permutation of the original $A[1 \ldots i-1]$
  - $A[1 \mathinner{..} i-1]$ is sorted

  Given the termination condition, $A[1 \mathinner{..} i-1]$ is the whole A

  So **INSERTION-SORT** is *correct!*

- You are given a problem *P* and an algorithm *A*
  - *P* formally defines a *correctness* condition
  - assume, for simplicity, that *A* consists of one loop

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   - prove that C holds right before the first execution of the first instruction of the loop

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4. **Termination** (for all valid inputs)
   - prove that the loop terminates, with some exit condition X
5. Prove that $X \wedge C \Rightarrow P$, which means that A is correct

Exercise: Analyze Selection-Sort

```
SELECTION-SORT(A)
n = length(A)
for i = 1 to n − 1
    smallest = i
    for j = i + 1 to n
        if A[j] < A[smallest]
            smallest = j
    swap A[i] and A[smallest]
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- Correctness?
  - loop invariant?
- Complexity?
  - worst, best, and average case?

Exercise: Analyze Bubblesort

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BUBBLESORT(A)
  for i = 1 to length(A)
      for j = length(A) downto i + 1
          if A[j] < A[j − 1]
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Exercise: Analyze Bubblesort

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