

Red-Black Trees

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Summary on Binary Search Trees

■ Binary search trees

- ▶ embody the *divide-and-conquer* search strategy
- ▶ **SEARCH**, **INSERT**, **MIN**, and **MAX** are $O(h)$, where h is the *height of the tree*
- ▶ in general, $h(n) = \Omega(\log n)$ and $h(n) = O(n)$
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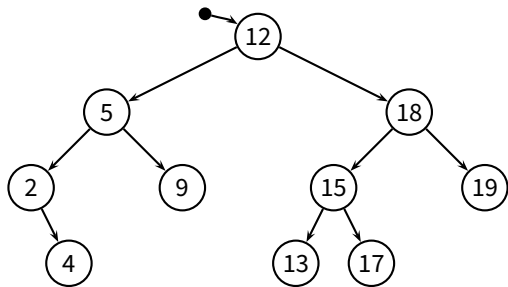
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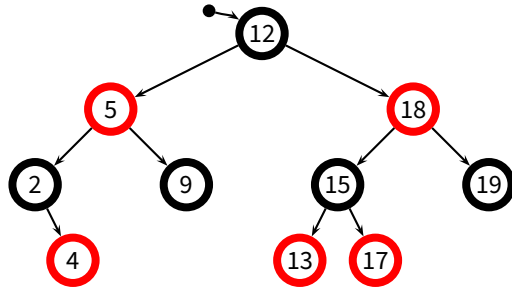
■ Problem

- ▶ worst-case scenario is unlikely but still possible
- ▶ simply bad cases are even more probable

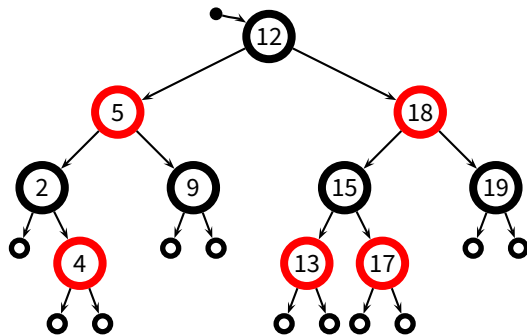
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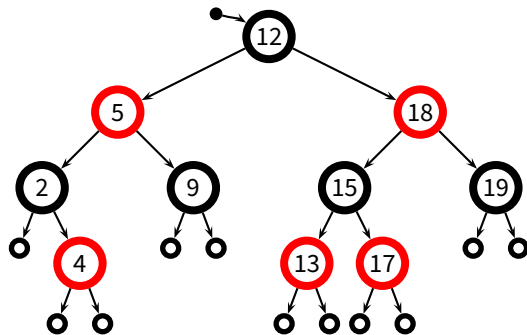


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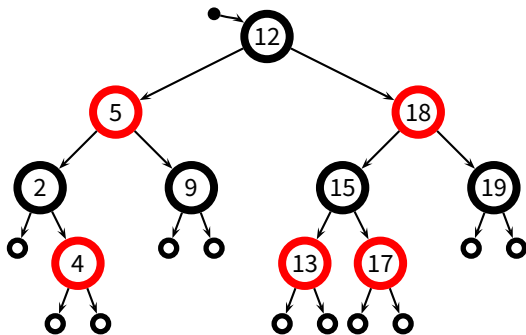


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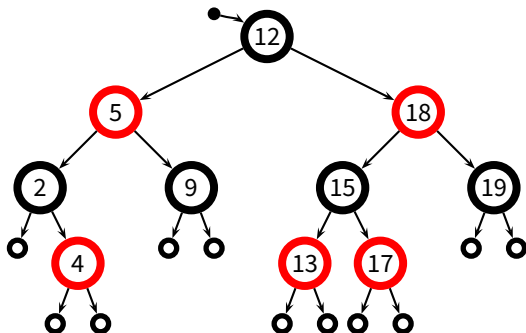


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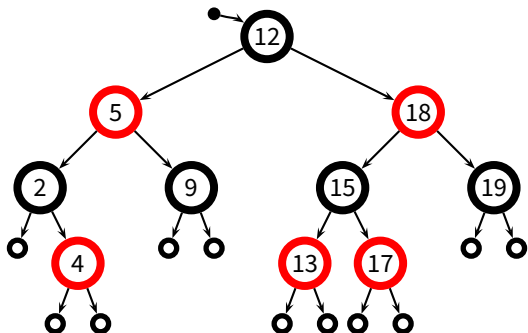
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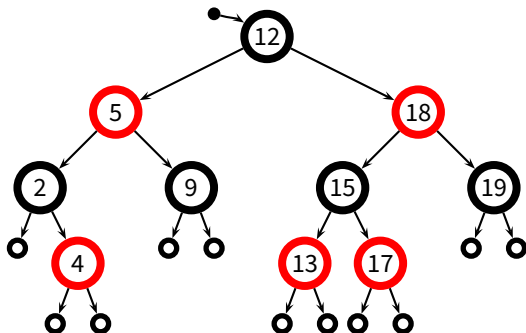
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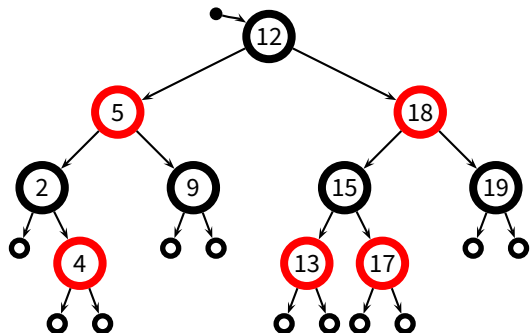
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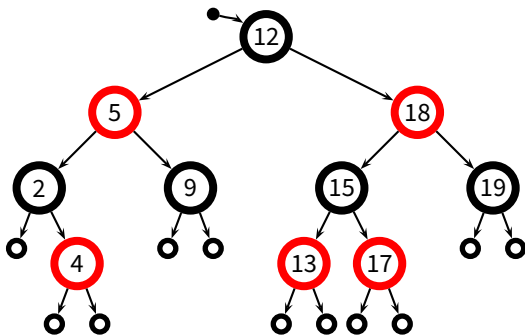
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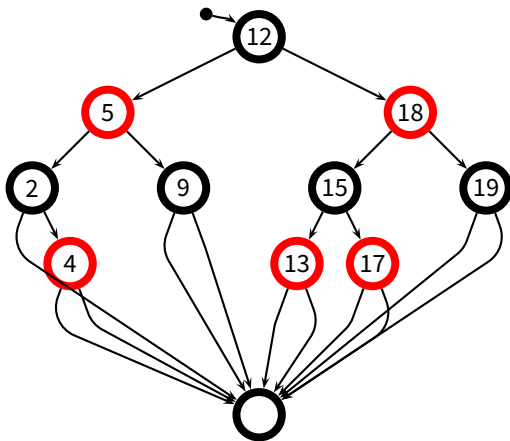


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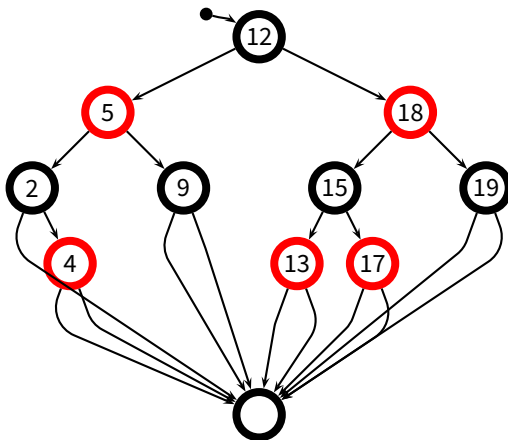
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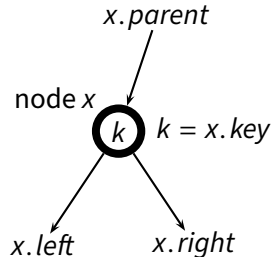
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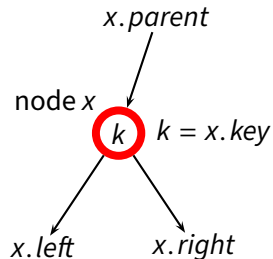


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- ▶ $x.color \in \{\text{RED}, \text{BLACK}\}$ is the color of node x



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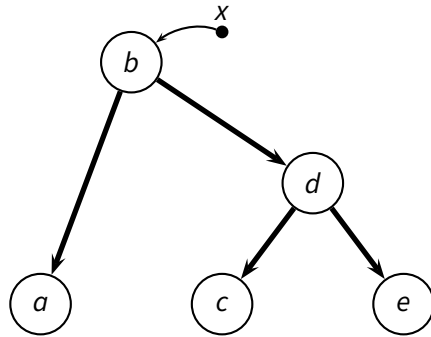
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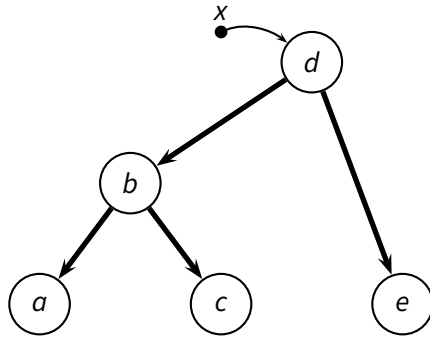
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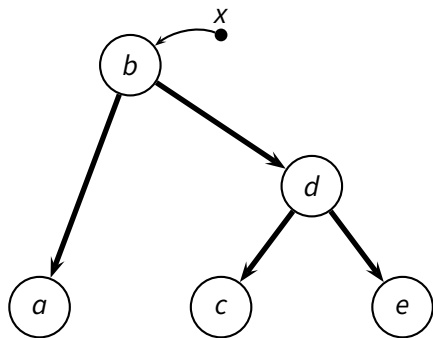
- A red-black tree works as a binary search tree for search, etc.
- So, the complexity of those operations is $T(n) = O(h)$, that is

$$T(n) = O(\log n)$$

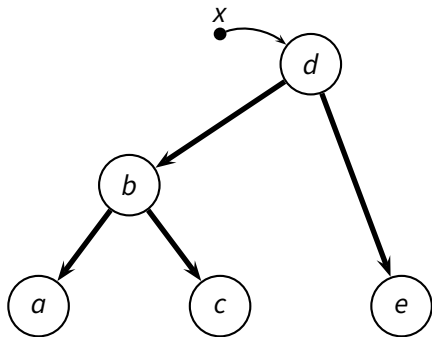
- ▶ which is also the *worst-case* complexity







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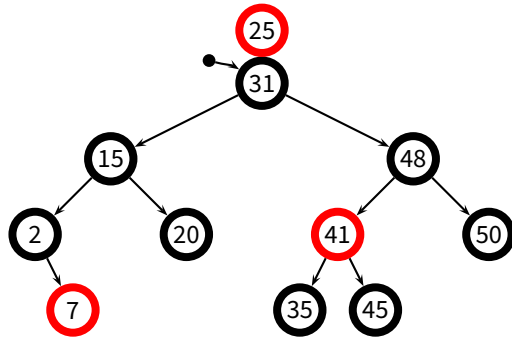
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- *General strategy*
 1. insert z as in a binary search tree
 2. color z **red** so as to preserve property 5
 3. *fix the tree* to correct possible violations of property 4

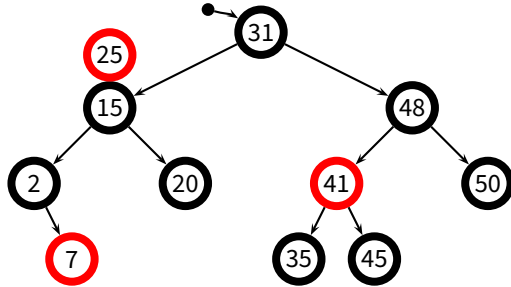
RB-INSERT(T, z)

```
1   $y = T.nil$ 
2   $x = T.root$ 
3  while  $x \neq T.nil$ 
4       $y = x$ 
5      if  $z.key < x.key$ 
6           $x = x.left$ 
7      else  $x = x.right$ 
8   $z.parent = y$ 
9  if  $y == T.nil$ 
10      $T.root = z$ 
11  else if  $z.key < y.key$ 
12      $y.left = z$ 
13  else  $y.right = z$ 
14   $z.left = z.right = T.nil$ 
15   $z.color = RED$ 
16  RB-INSERT-FIXUP( $T, z$ )
```

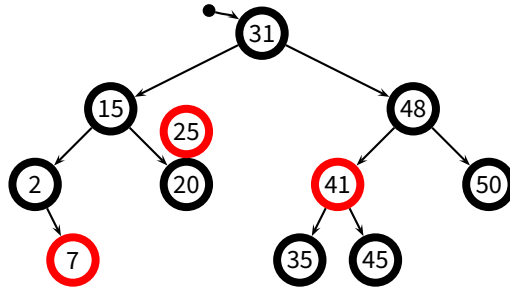
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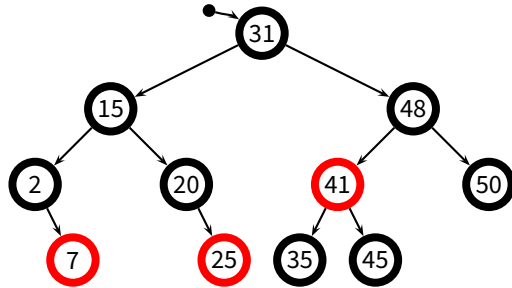
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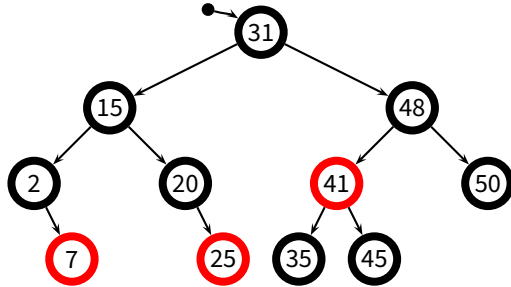
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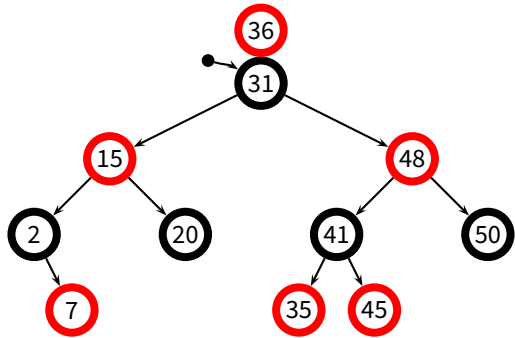
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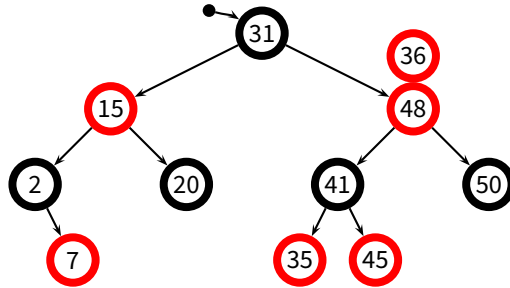
- z's parent is **black**, so no fixup needed

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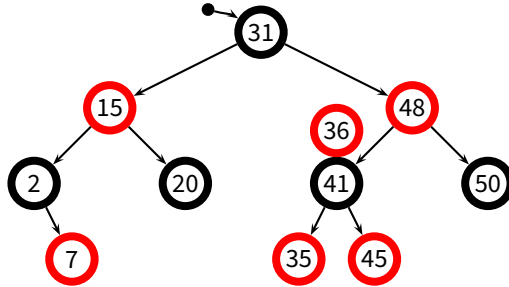
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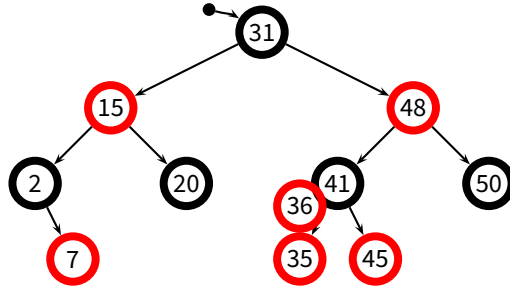
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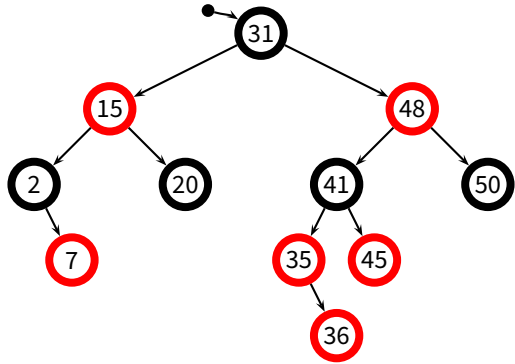
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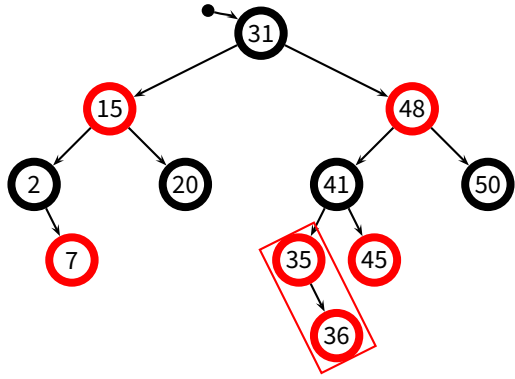
Red-Black Insertion (3)



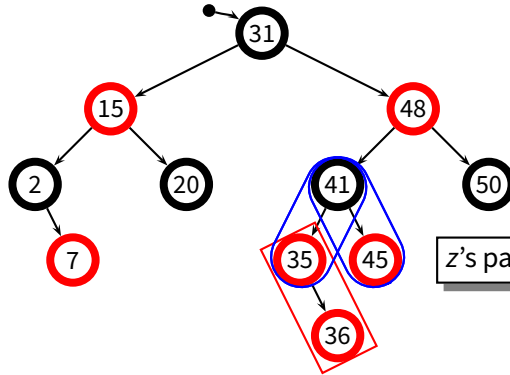
Red-Black Insertion (3)



Red-Black Insertion (3)

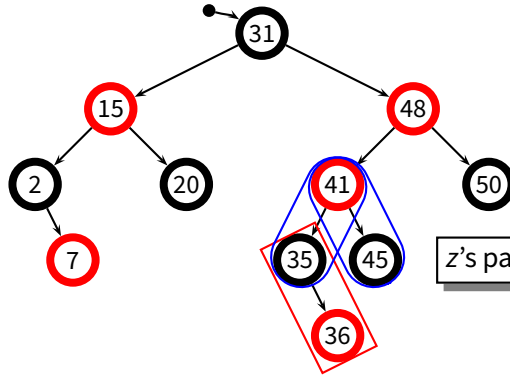


Red-Black Insertion (3)

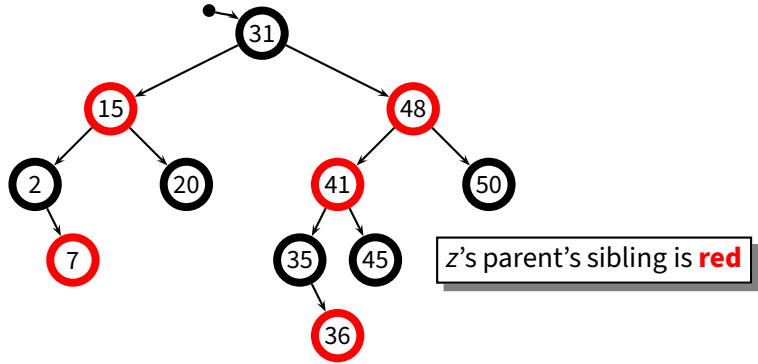


z's parent's sibling is **red**

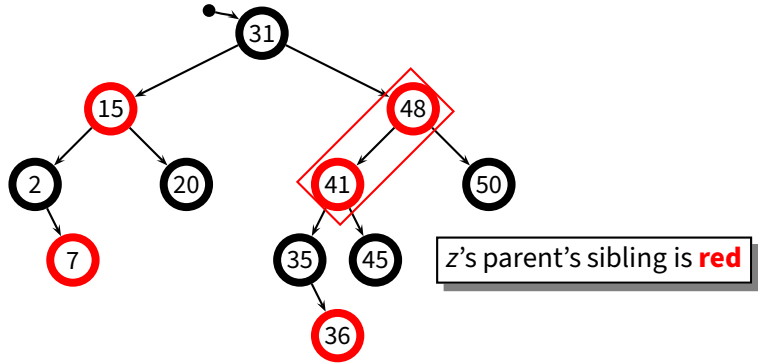
Red-Black Insertion (3)



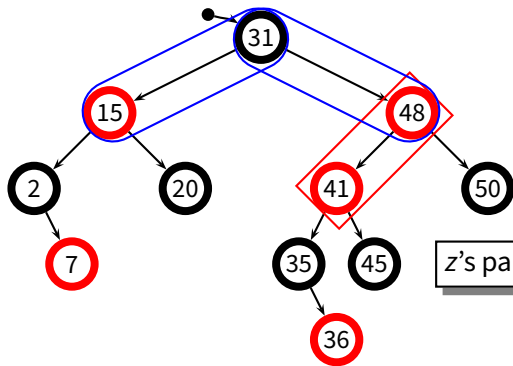
Red-Black Insertion (3)



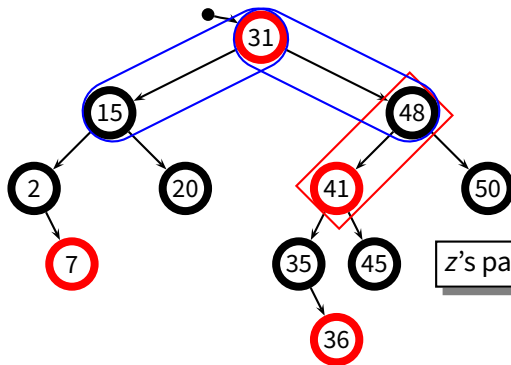
Red-Black Insertion (3)



Red-Black Insertion (3)

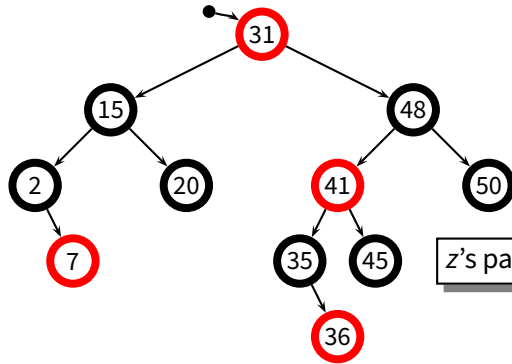


Red-Black Insertion (3)



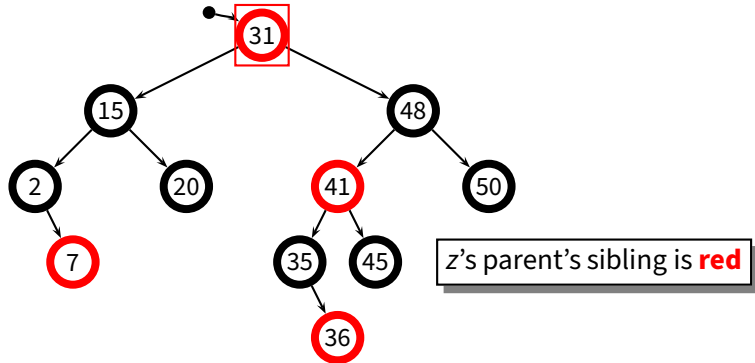
z's parent's sibling is **red**

Red-Black Insertion (3)

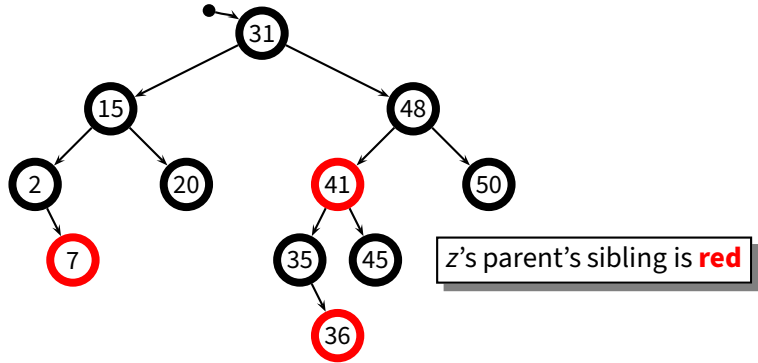


z's parent's sibling is **red**

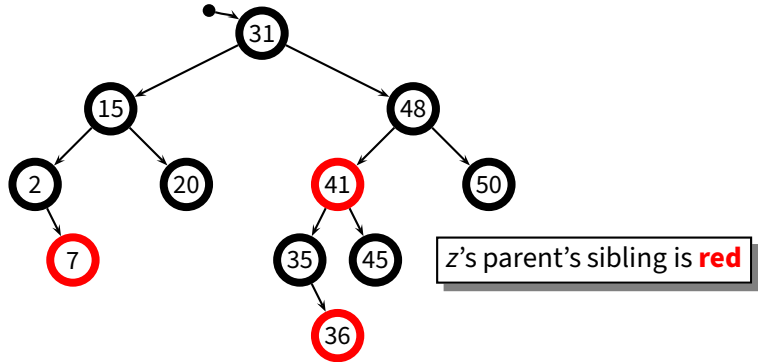
Red-Black Insertion (3)



Red-Black Insertion (3)

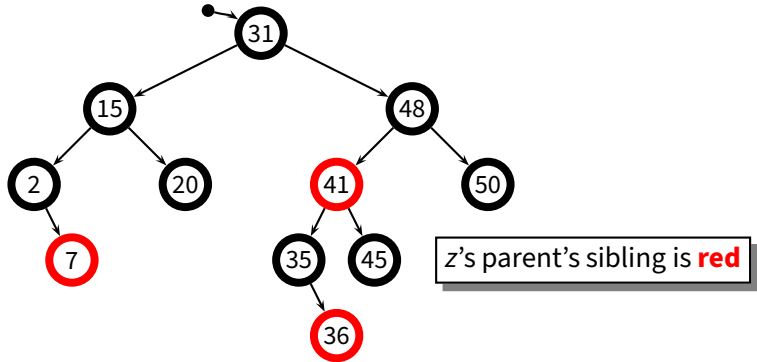


Red-Black Insertion (3)



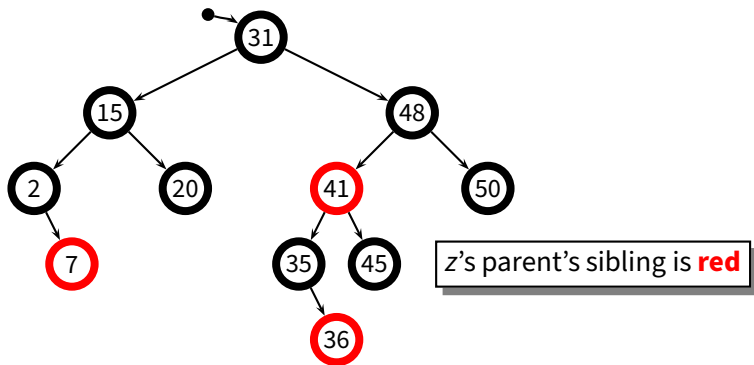
- A **black** node can become **red** and transfer its **black** color to its two children

Red-Black Insertion (3)



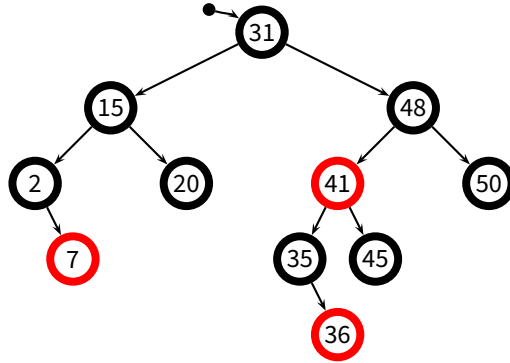
- A **black** node can become **red** and transfer its **black** color to its two children
- This may cause other **red-red** conflicts, so we iterate...

Red-Black Insertion (3)

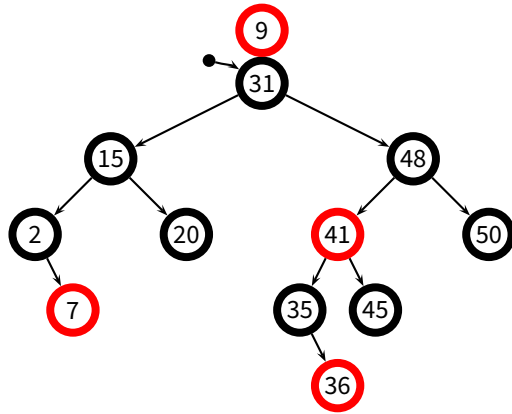


- A **black** node can become **red** and transfer its **black** color to its two children
- This may cause other **red-red** conflicts, so we iterate...
- The root can change to **black** without causing conflicts

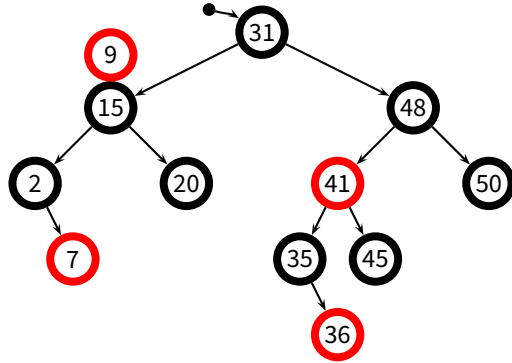
Red-Black Insertion (4)



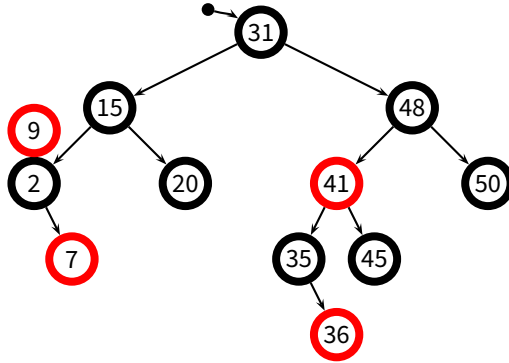
Red-Black Insertion (4)



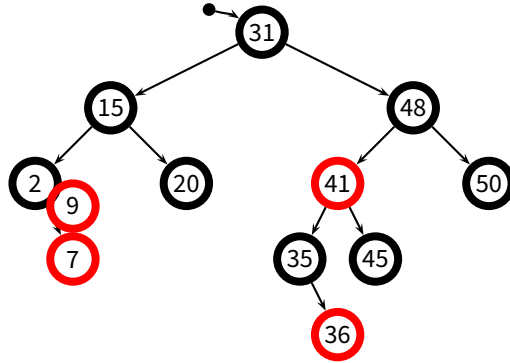
Red-Black Insertion (4)



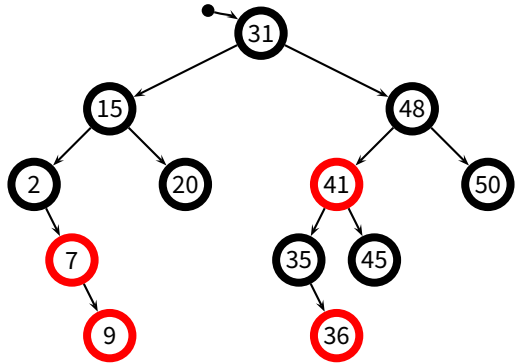
Red-Black Insertion (4)



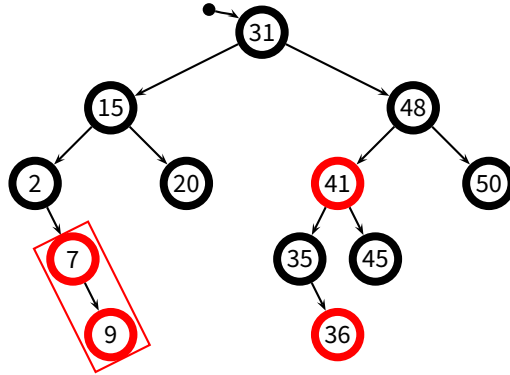
Red-Black Insertion (4)



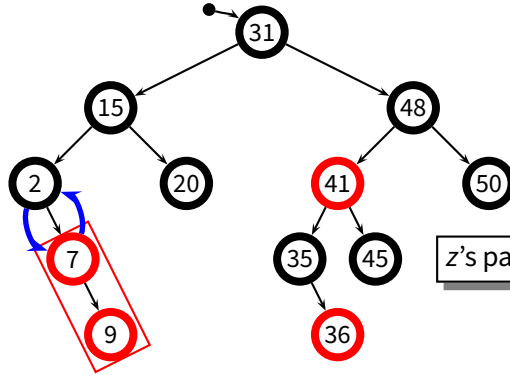
Red-Black Insertion (4)



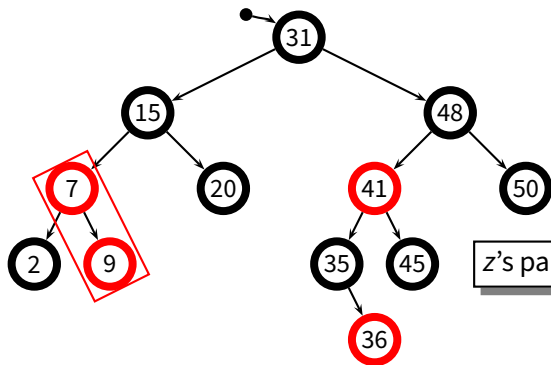
Red-Black Insertion (4)



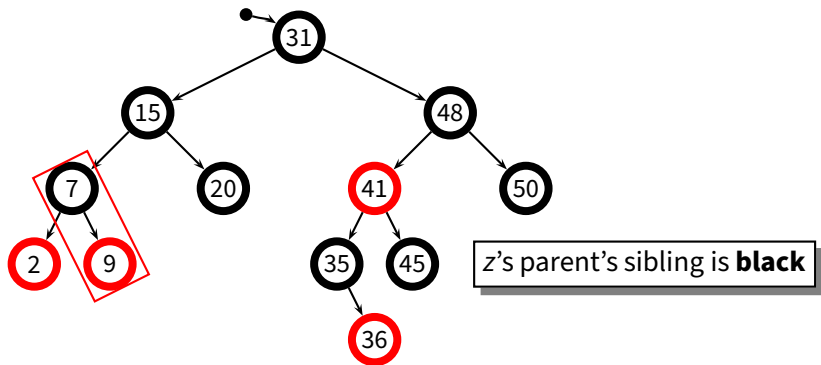
Red-Black Insertion (4)



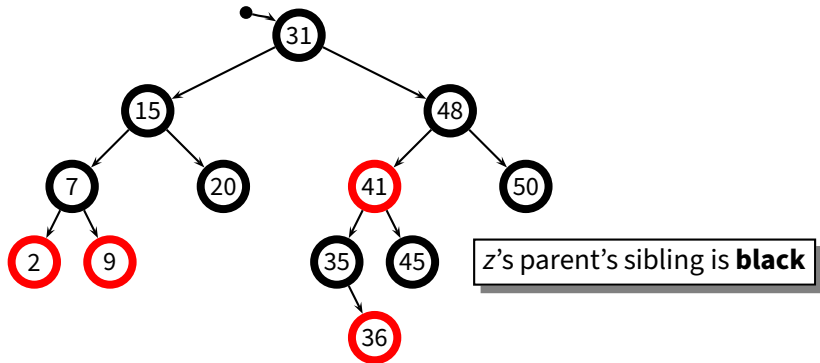
Red-Black Insertion (4)



Red-Black Insertion (4)

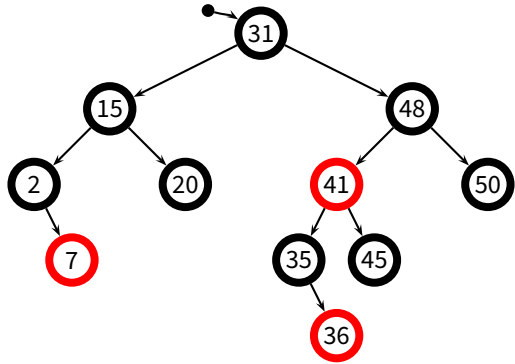


Red-Black Insertion (4)

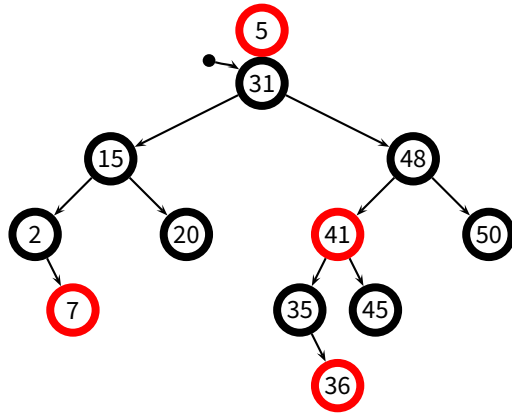


- An *in-line red-red* conflicts can be resolved with a rotation plus a color switch

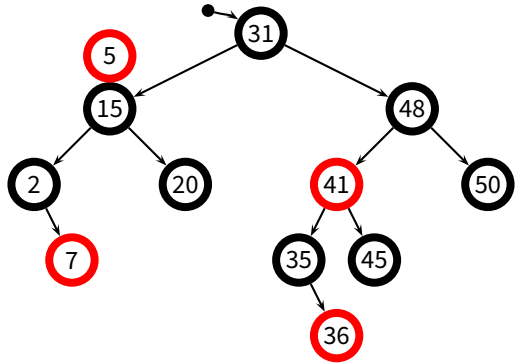
Red-Black Insertion (5)



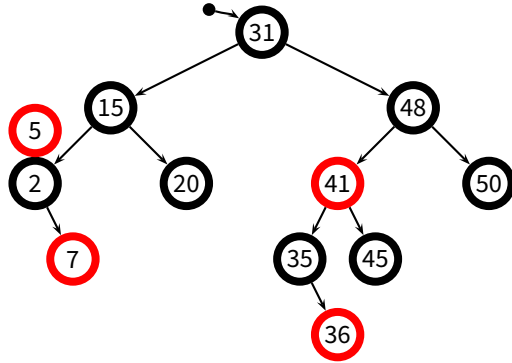
Red-Black Insertion (5)



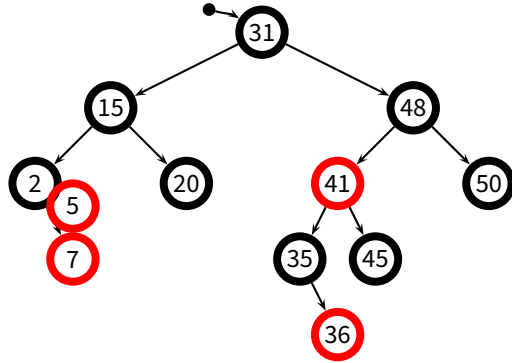
Red-Black Insertion (5)



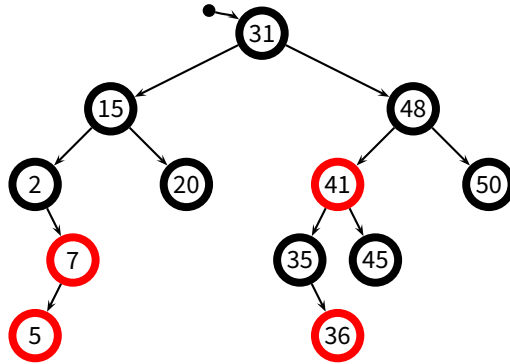
Red-Black Insertion (5)



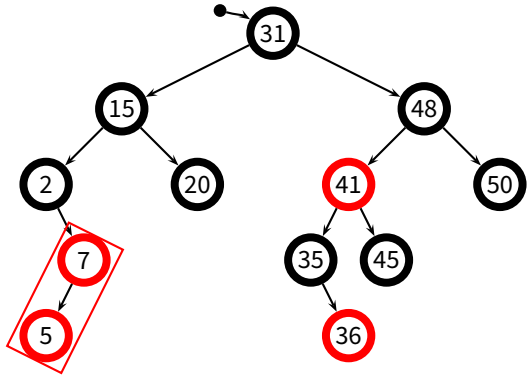
Red-Black Insertion (5)



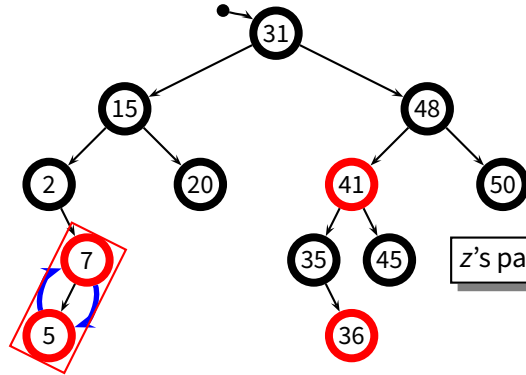
Red-Black Insertion (5)



Red-Black Insertion (5)

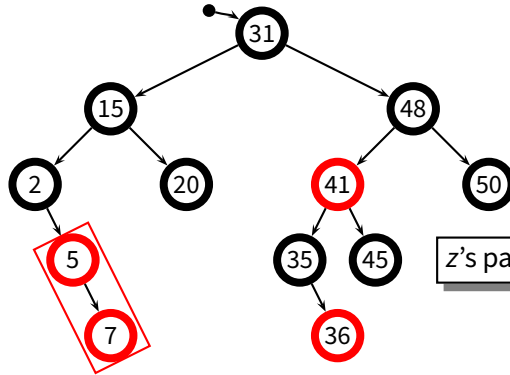


Red-Black Insertion (5)



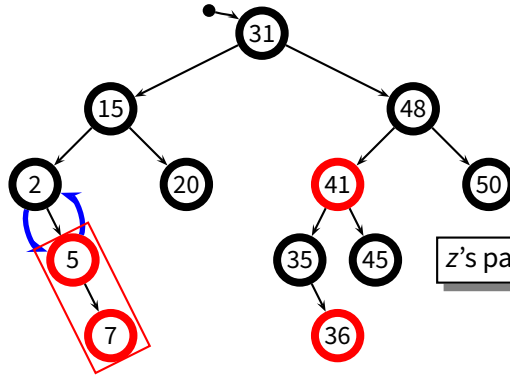
z's parent's sibling is **black**

Red-Black Insertion (5)

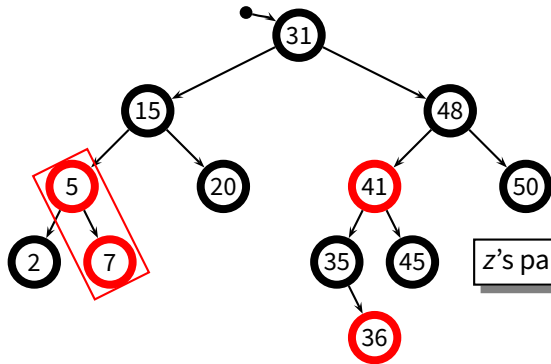


z's parent's sibling is **black**

Red-Black Insertion (5)

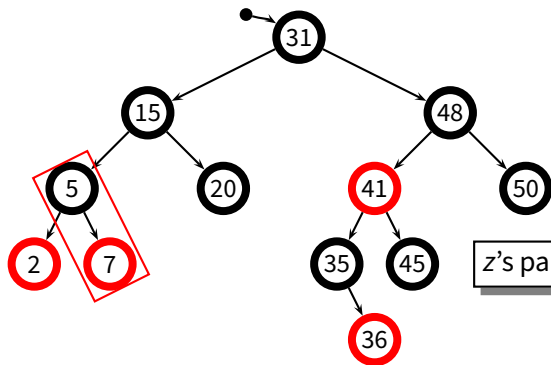


Red-Black Insertion (5)

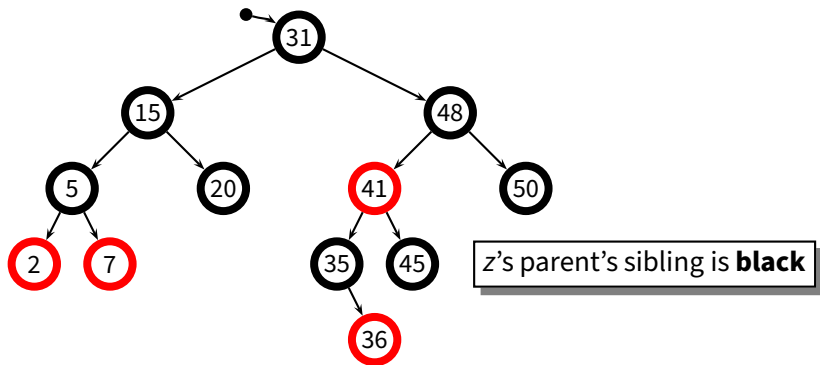


z's parent's sibling is **black**

Red-Black Insertion (5)



Red-Black Insertion (5)



- A zig-zag **red-red** conflicts can be resolved with a rotation to turn it into an *in-line* conflict, and then a rotation plus a color switch