More on Sorting: Quick Sort and Heap Sort

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Outline

- Another divide-and-conquer sorting algorithm
- The *heap*
- Heap sort

Sorting Algorithms Seen So Far

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Algorithm	Complexity			In place?
	worst	*average*	*best*	

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Insertion-Sort	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n)$	yes
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Merge-Sort				

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	worst	*average*	*best*	
INSERTION-SORT	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n)$	yes
SELECTION-SORT	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$	yes
MERGE-SORT	$\Theta(n \log n)$	$\Theta(n \log n)$	$\Theta(n \log n)$	no

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Insertion-Sort	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n)$	yes
Selection-Sort	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$	yes
Merge-Sort	$\Theta(n \log n)$	$\Theta(n \log n)$	$\Theta(n \log n)$	no
??		$\Theta(n \log n)$		***yes***
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Using the Partitioning Algorithm

- *Basic step:* partition A in three parts based on a *chosen value* $v \in A$
  - A_L contains the set of elements that are *less than* v
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- *Can we partition A* ***in place****?*

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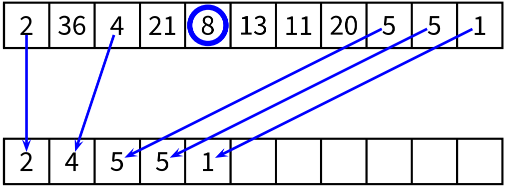

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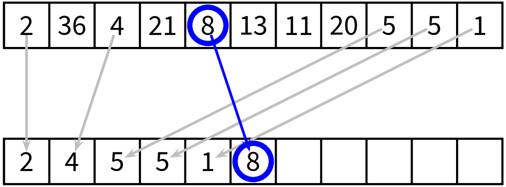

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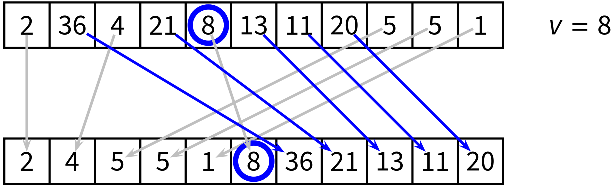

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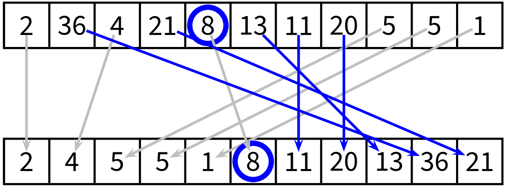

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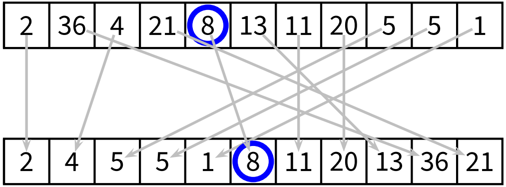

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$q = 6$

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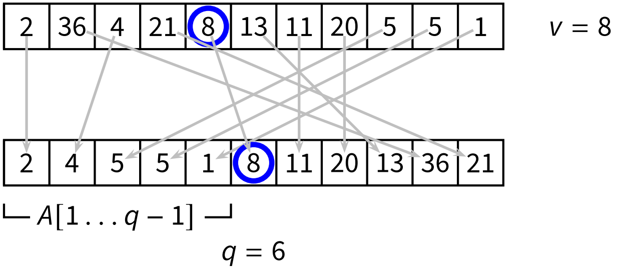

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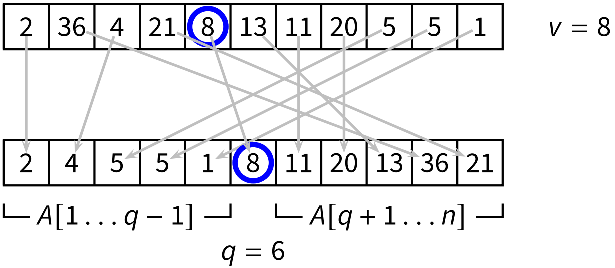

Another Divide-and-Conquer for Sorting

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- ***Combine:***

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```
QuickSort(A, begin, end)
if begin < end
        q = Partition(A, begin, end)
        QuickSort(A, begin, q − 1)
        QuickSort(A, q + 1, end)
```

Partition

- Start with $q = 1$
  - i.e., *assume all elements are greater than the pivot*
- Scan the array left-to-right, starting at position 2
- If an element $A[i]$ is less than or equal to pivot, then swap it with the current q position and shift q to the right

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- *Loop invariant*
  - $begin \leq k < q \Rightarrow A[k] \leq v$
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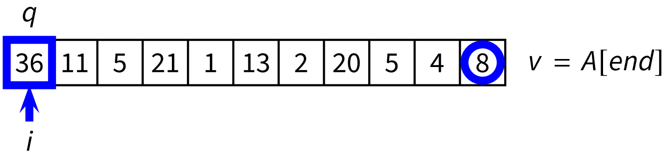

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q

$\boxed{36}$	11	5	21	1	13	2	20	5	4	8

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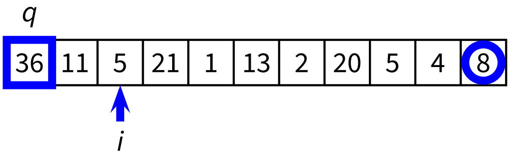

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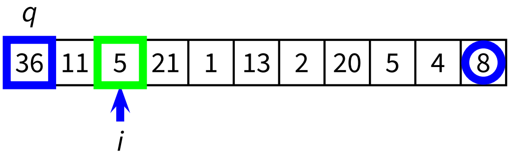

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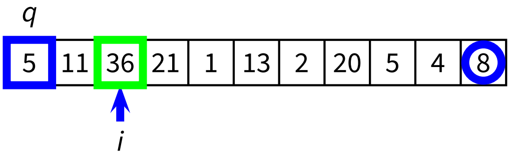

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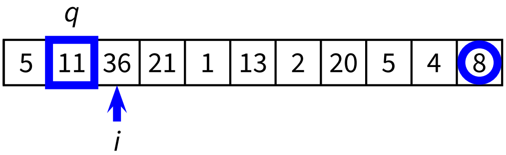

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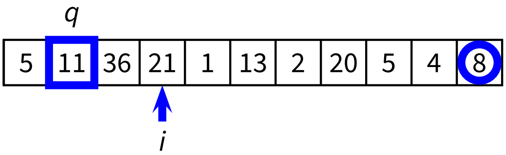

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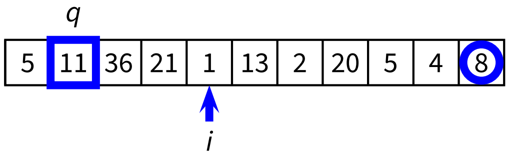

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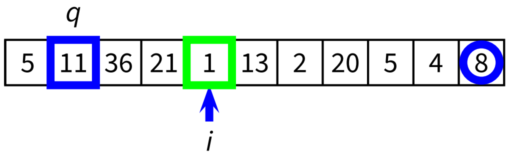

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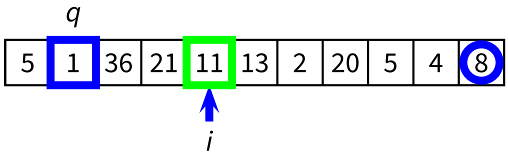

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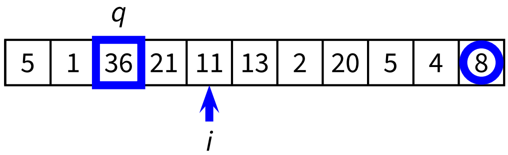

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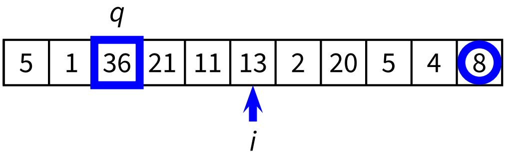

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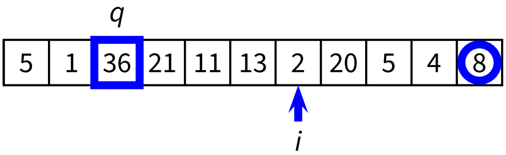

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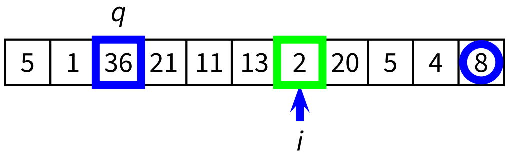

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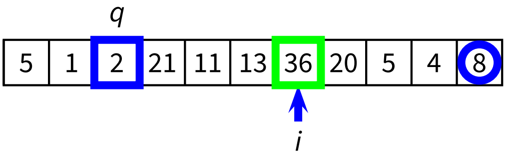

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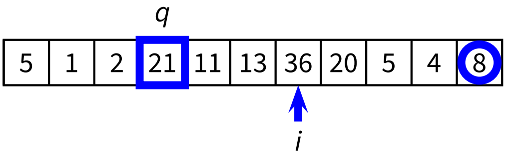

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			q							
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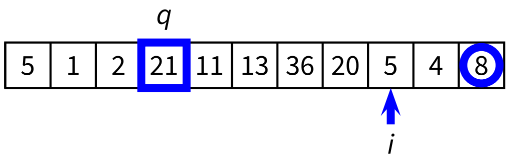

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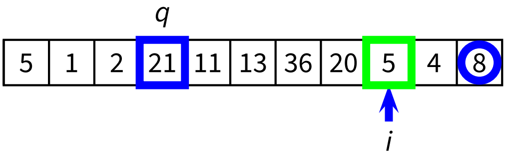

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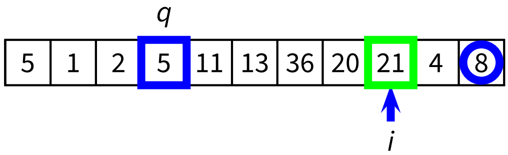

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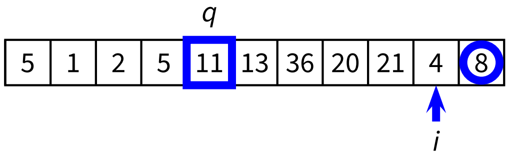

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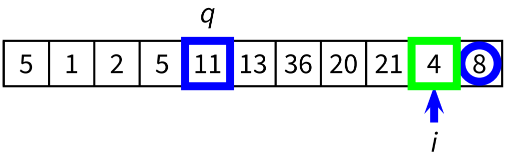

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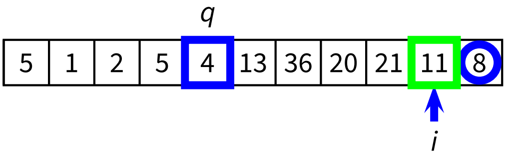

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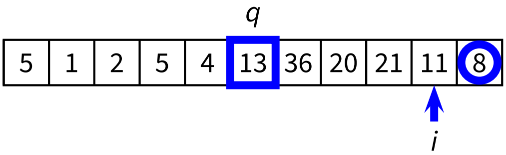

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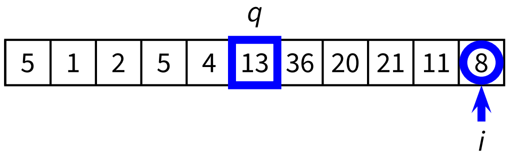

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					q					
5	1	2	5	4	**8**	36	20	21	11	13

Complete QuickSort Algorithm

```
Partition(A, begin, end)
q = begin
v = A[end]
for i = begin to end
    if A[i] ≤ v
        swap A[i] and A[q]
        q = q + 1
return q − 1
```

```
QuickSort(A, begin, end)
if begin < end
    q = Partition(A, begin, end)
    QuickSort(A, begin, q − 1)
    QuickSort(A, q + 1, end)
```

Complexity of PARTITION

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```

$$T(n) = \Theta(n)$$

Complexity of QuickSort

```
QuickSort(A, begin, end)
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```

```
QUICKSORT(A, begin, end)
if begin < end
        q = PARTITION(A, begin, end)
        QUICKSORT(A, begin, q − 1)
        QUICKSORT(A, q + 1, end)
```

- Worst case

```
QUICKSORT(A, begin, end)
1  if begin < end
2      q = PARTITION(A, begin, end)
3      QUICKSORT(A, begin, q - 1)
4      QUICKSORT(A, q + 1, end)
```

- Worst case
  - $q = \mathit{begin}$ or $q = \mathit{end}$

Complexity of QuickSort

```
QuickSort(A, begin, end)
if begin < end
        q = Partition(A, begin, end)
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```

- Worst case
  - $q = begin$ or $q = end$
  - the partition transforms P of size n in P of size $n - 1$

Complexity of QuickSort

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QuickSort(A, begin, end)
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- Worst case
  - $q = begin$ or $q = end$
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$$T(n) = T(n-1) + \Theta(n)$$

Complexity of QuickSort

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$$T(n) = \Theta(n^2)$$

Complexity of QuickSort (2)

QuickSort$(A, begin, end)$

```
if begin < end
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        QuickSort(A, q + 1, end)
```

QuickSort(A, $begin$, end)
```
if begin < end
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```

- Best case

```
QuickSort(A, begin, end)
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- Best case
  - $q = \lceil n/2 \rceil$

```
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$$T(n) = 2T(n/2) + \Theta(n)$$

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$$T(n) = \Theta(n \log n)$$

Sorting Algorithms Seen So Far

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Algorithm	Complexity			In place?
	worst	*average*	*best*	
Insertion-Sort	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n)$	yes
Selection-Sort	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$	yes
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??	$\Theta(n \log n)$			***yes***

Binary Heap

- Our first real ***data structure***

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- Interface

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  - **BUILD-MAX-HEAP**(A) rearranges A into a max-heap
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Binary Heap: Structure

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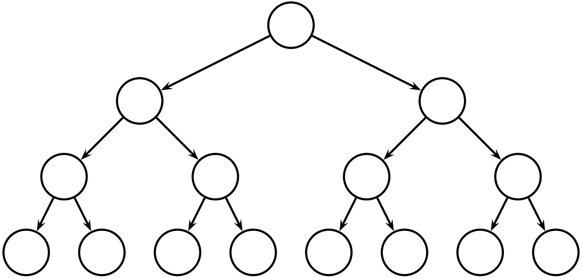

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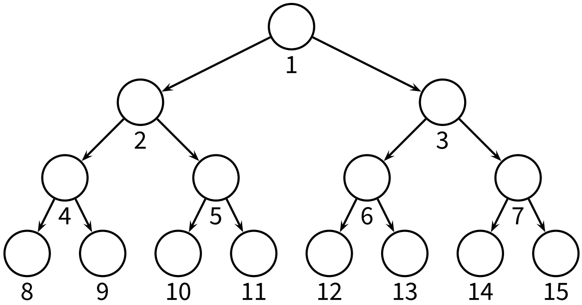

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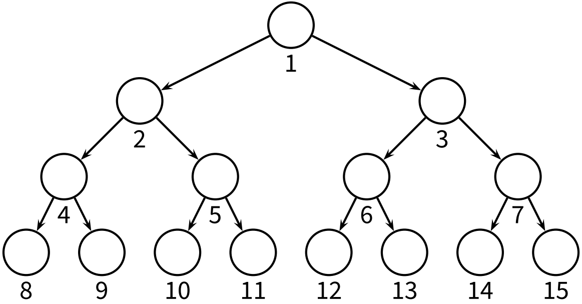

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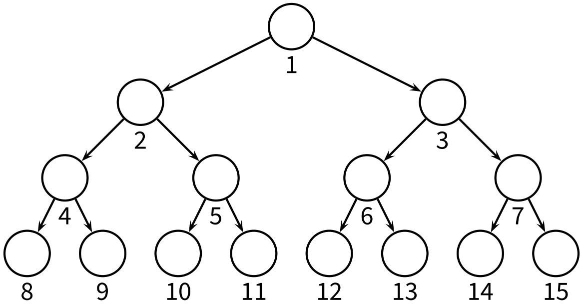

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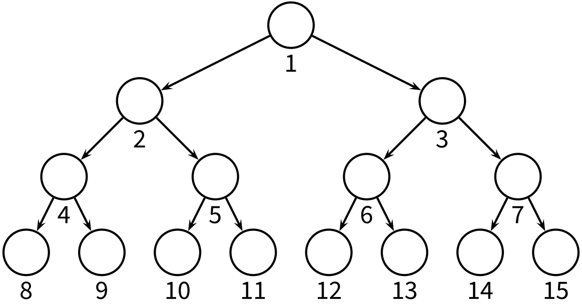

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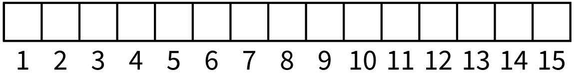

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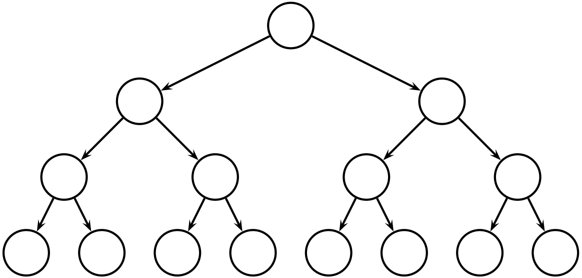

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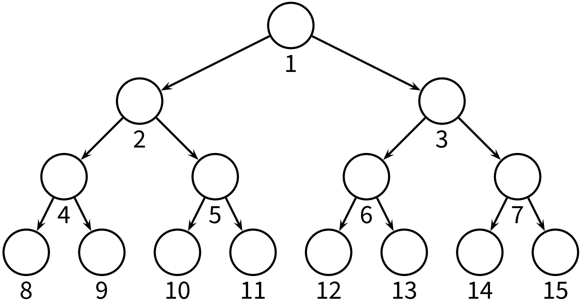

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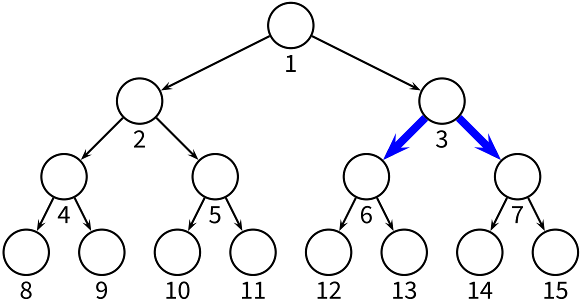

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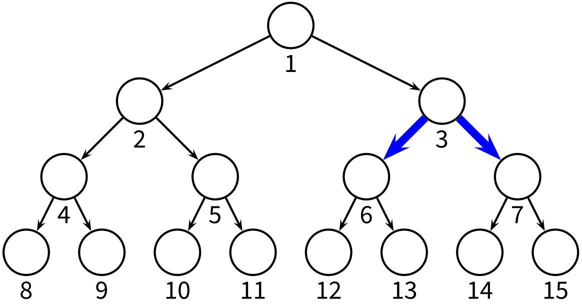

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Right(i)

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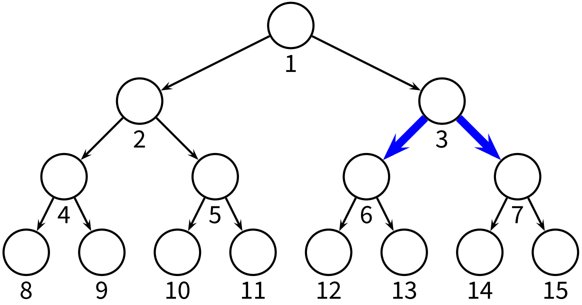

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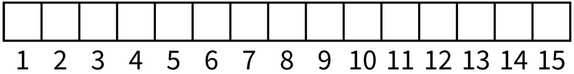

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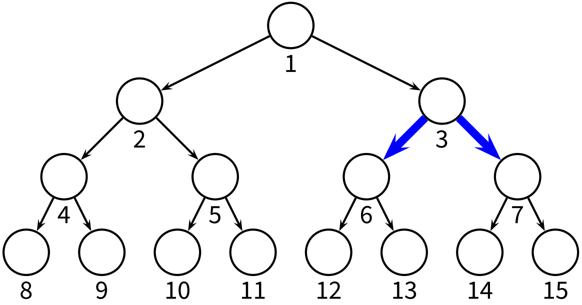

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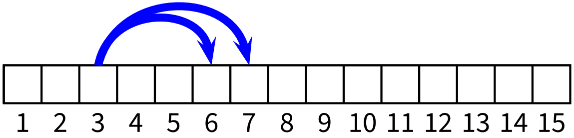
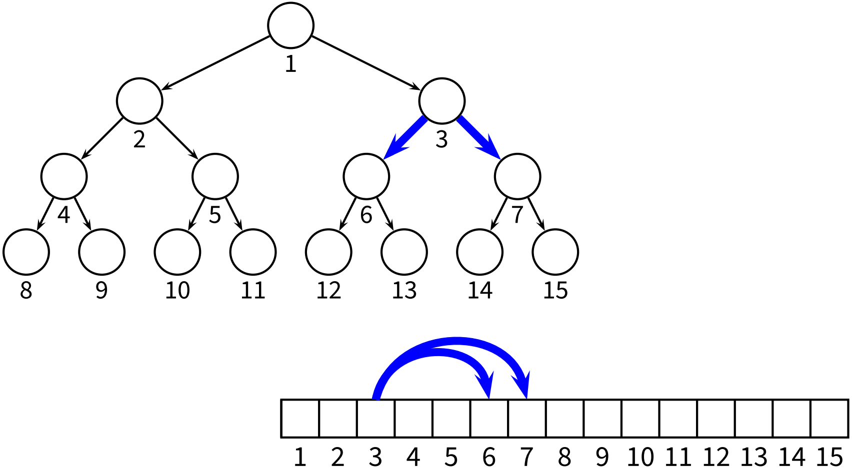
Right(i)

  return $2i + 1$



Binary Heap: Properties



PARENT(i)

  return $\lfloor i/2 \rfloor$

LEFT(i)

  return $2i$

RIGHT(i)

  return $2i + 1$

- ***Max-heap property:*** for all $i > 1$ $\boxed{A[\textsc{Parent}(i)] \geq A[i]}$

Example

- *Max-heap property:* for all $i > 1$ $\boxed{A[\textbf{Parent}(i)] \geq A[i]}$

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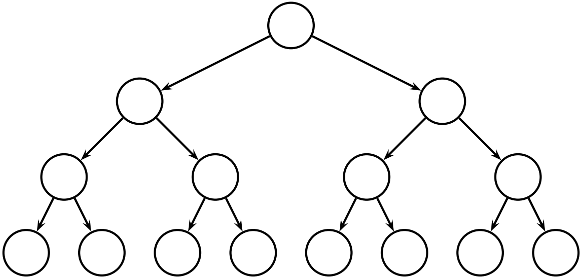

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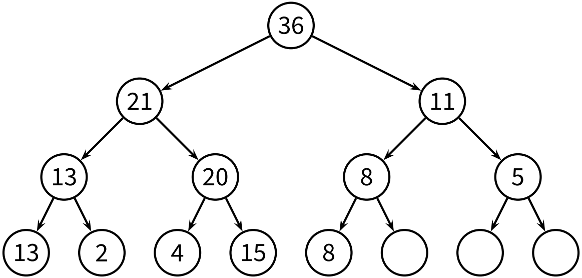

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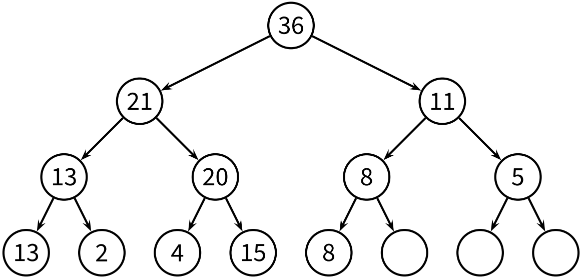

- Where is the max element?

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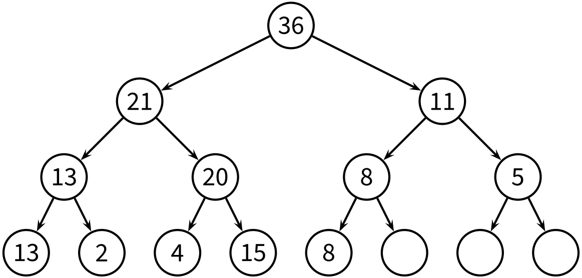

- Where is the max element?

- How can we implement **Heap-Extract-Max**?

- **Heap-Extract-Max** procedure
  - extract the max key
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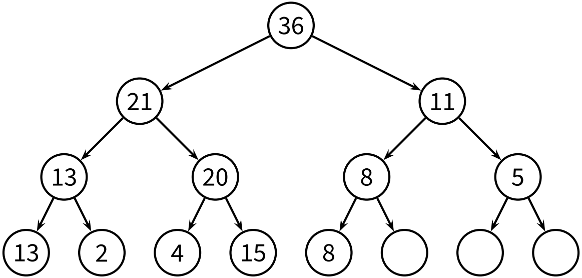

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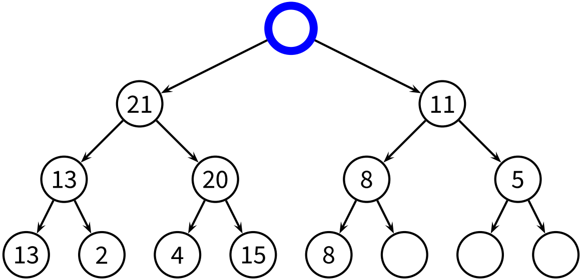

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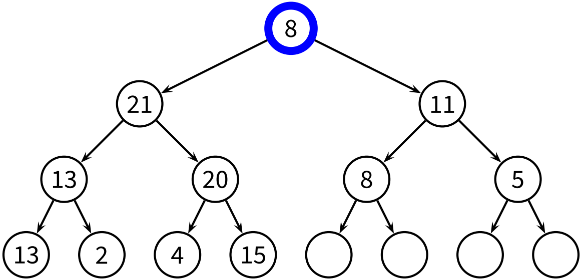

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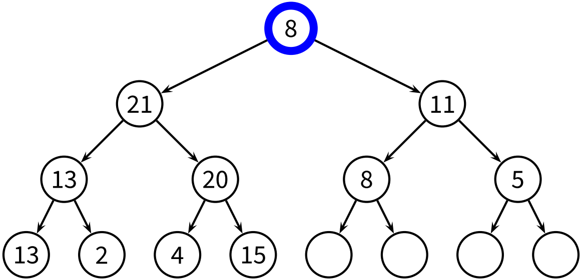

- Now we have two subtrees where the *max-heap property* holds

- **Max-Heapify**(A, i) procedure
  - *assume:* the *max-heap property* holds in the subtrees of node i
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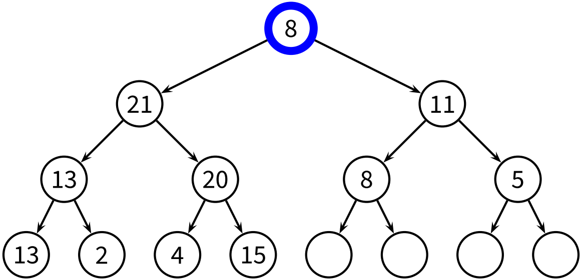

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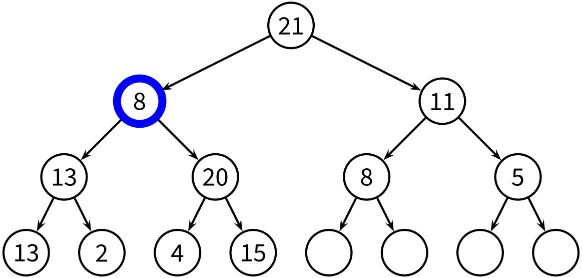

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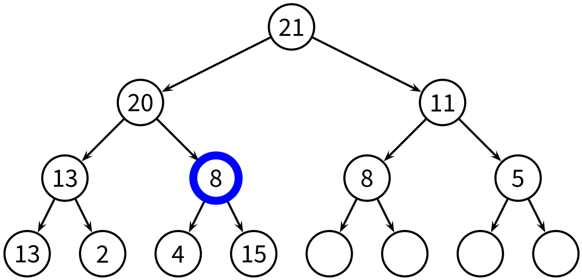

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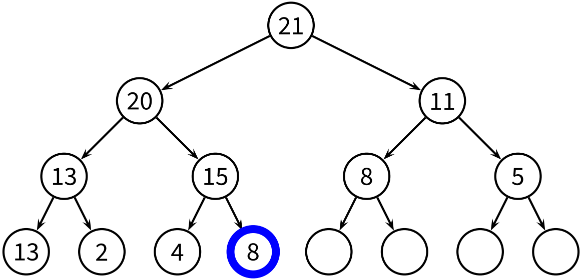

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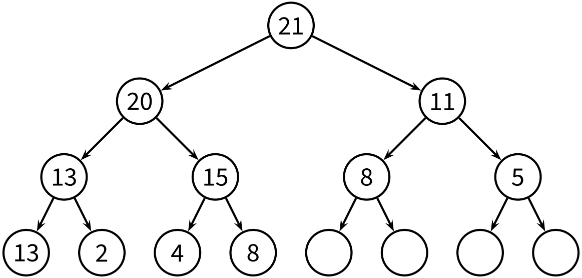

Max-Heapify

```
MAX-HEAPIFY(A, i)
l = LEFT(i)
r = RIGHT(i)
if l ≤ A.heap-size and A[l] > A[i]
    largest = l
else largest = i
if r ≤ A.heap-size and A[r] > A[largest]
    largest = r
if largest ≠ i
    swap A[i] and A[largest]
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$$\boxed{T(n) = \Theta(\log n)}$$

Building a Heap

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BUILD-MAX-HEAP(A)
A.heap-size = length(A)
for i = ⌊length(A)/2⌋ downto 1
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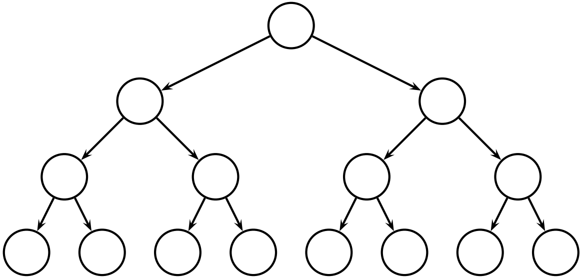

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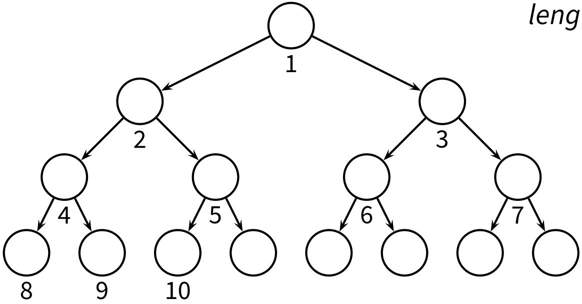

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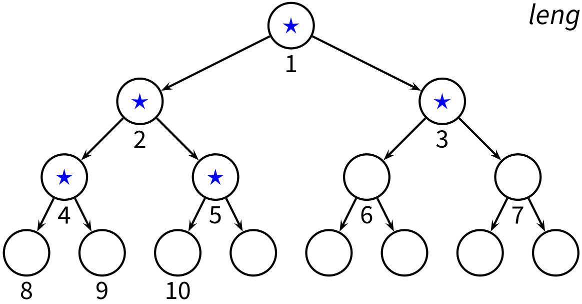

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BUILD-MAX-HEAP(A)

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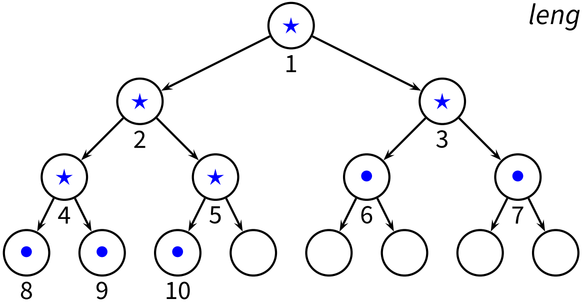
2 **for** $i = \lfloor length(A)/2 \rfloor$ **downto** 1
3 **MAX-HEAPIFY**(A, i)



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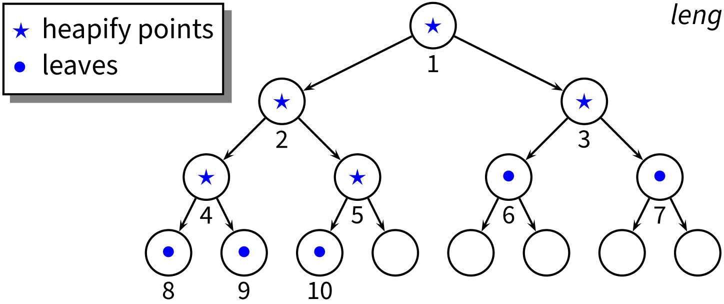

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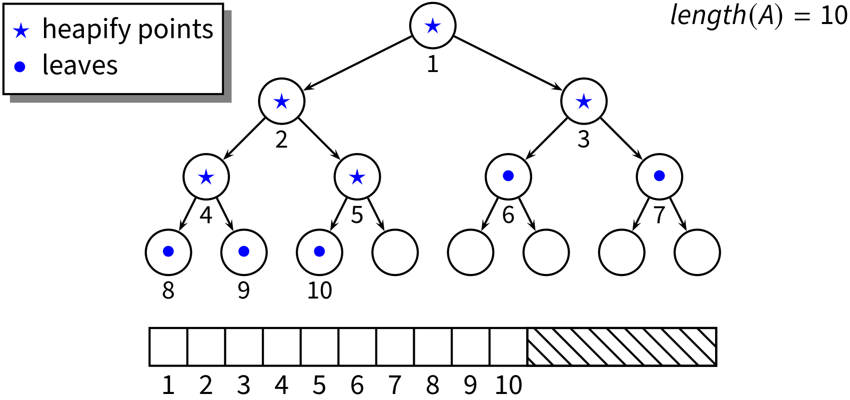

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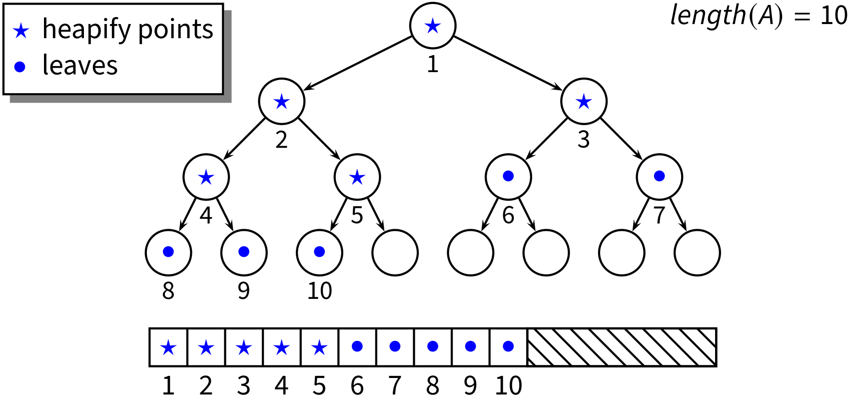

- Idea: we can use a heap to sort an array

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```
HEAP-SORT(A)
  BUILD-MAX-HEAP(A)
  for i = length(A) downto 1
      swap A[i] and A[1]
      A.heap-size = A.heap-size − 1
      MAX-HEAPIFY(A, 1)
```

Heap Sort

- Idea: we can use a heap to sort an array

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- What is the complexity of **HEAP-SORT**?

$$\boxed{T(n) = \Theta(n \log n)}$$

- Benefits
  - in-place sorting; worst-case is $\Theta(n \log n)$

Summary of Sorting Algorithms

Algorithm		Complexity		In place?
	worst	*average*	*best*	
INSERTION-SORT				

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Selection-Sort	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$	yes
Merge-Sort				

Algorithm	Complexity			In place?
	worst	*average*	*best*	
INSERTION-SORT	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n)$	yes
SELECTION-SORT	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$	yes
MERGE-SORT	$\Theta(n \log n)$	$\Theta(n \log n)$	$\Theta(n \log n)$	no

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Merge-Sort	$\Theta(n \log n)$	$\Theta(n \log n)$	$\Theta(n \log n)$	no
Quick-Sort				

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