# More on Sorting: Quick Sort and Heap Sort

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# Outline

- Another divide-and-conquer sorting algorithm
- The *heap*
- Heap sort

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# Using the Partitioning Algorithm

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- *Can we partition A* ***in place****?*

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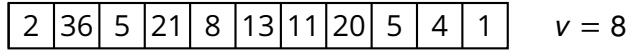


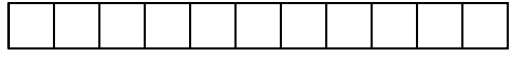

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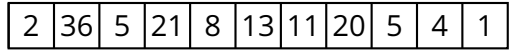


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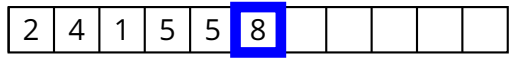

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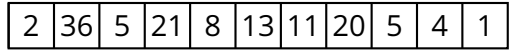


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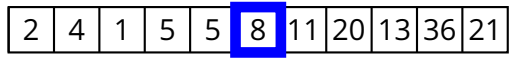

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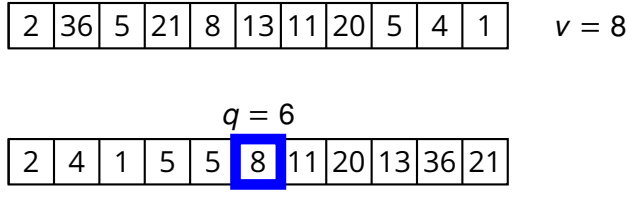

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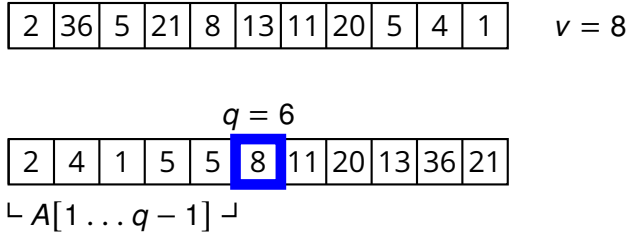

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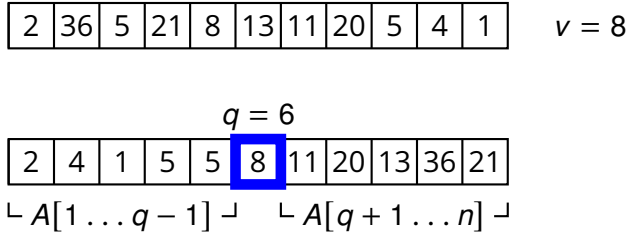

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```
QuickSort(A, begin, end)
1  if begin < end
2        q = Partition(A, begin, end)
3        QuickSort(A, begin, q − 1)
4        QuickSort(A, q + 1, end)
```

# Partition

- Start with $q = 1$
  - i.e., *assume all elements are greater than the pivot*
- Scan the array left-to-right, starting at position 2
- If an element $A[i]$ is less than or equal to pivot, then swap it with the current $q$ position and shift $q$ to the right

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  - $begin \leq k < q \Rightarrow A[k] \leq v$
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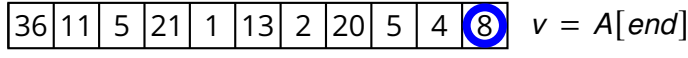

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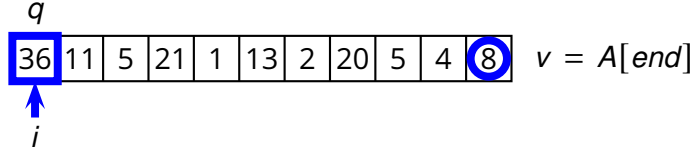

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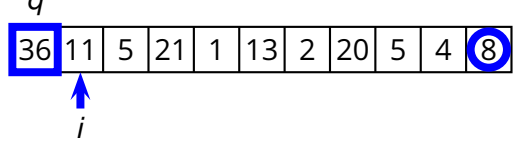

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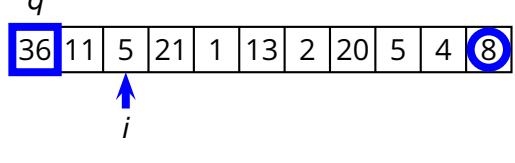

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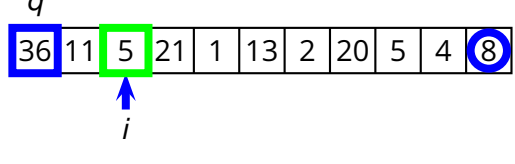

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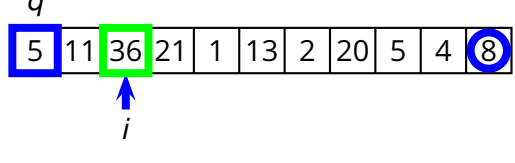

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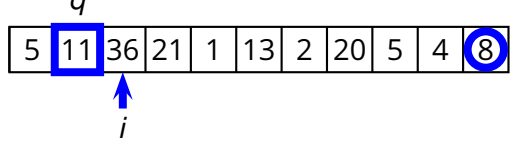

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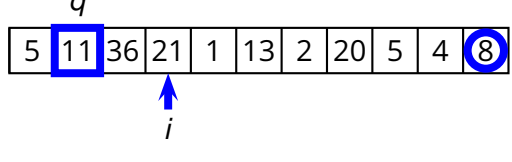

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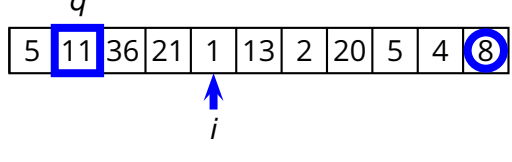

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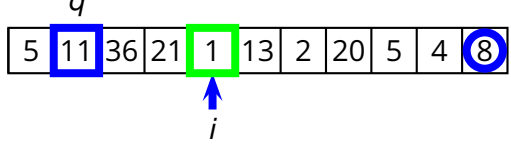

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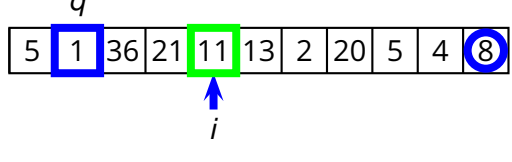

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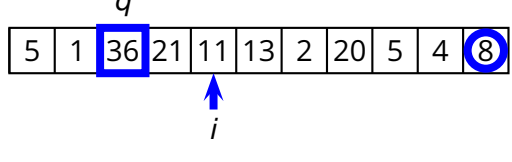

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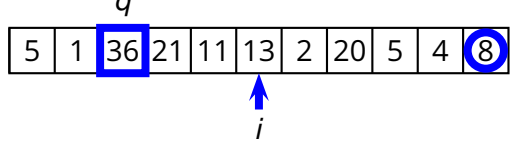

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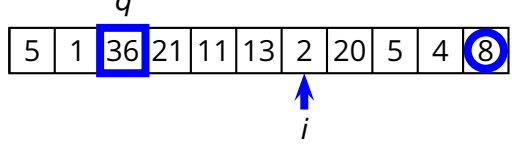

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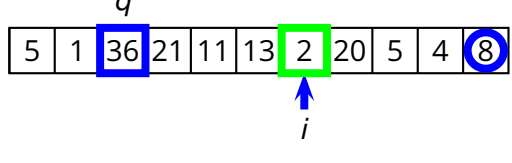

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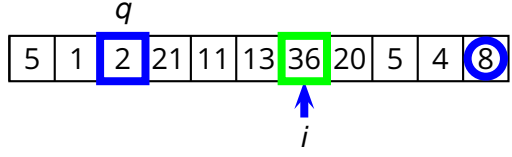

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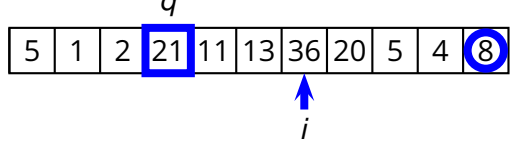

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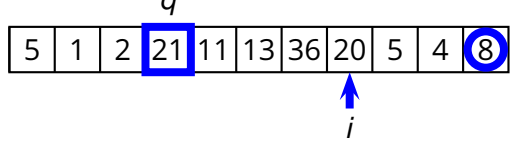

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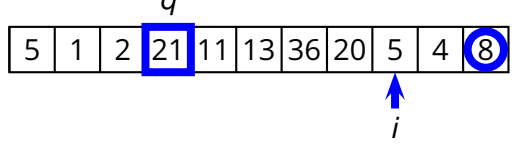

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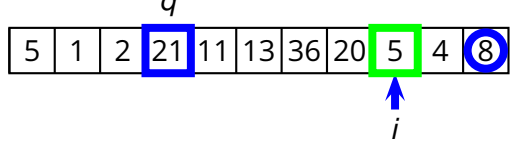

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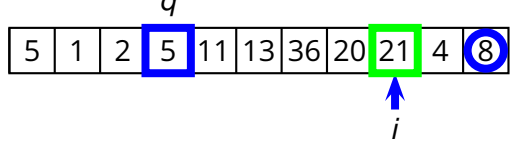

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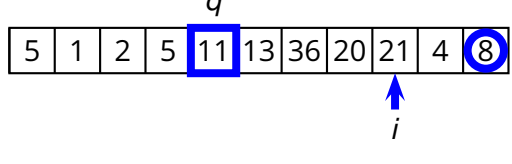

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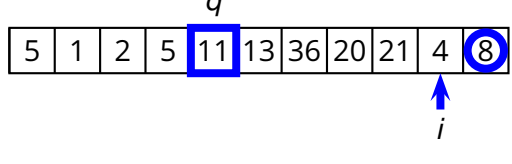

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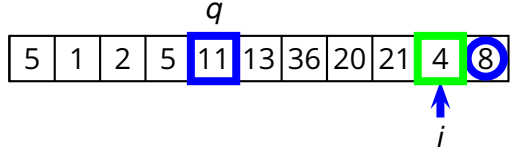

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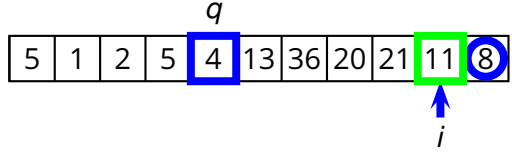

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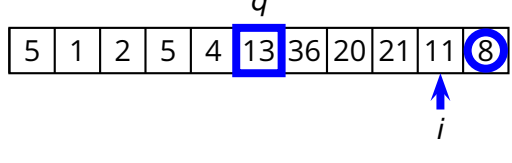

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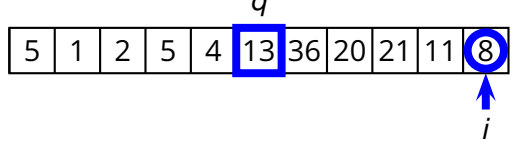

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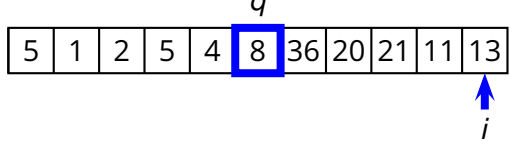

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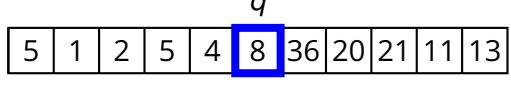

# Complete QuickSort Algorithm

```
Partition(A, begin, end)
1  q = begin
2  v = A[end]
3  for i = begin to end
4       if A[i] ≤ v
5            swap A[i] and A[q]
6            q = q + 1
7  return q − 1
```

```
QuickSort(A, begin, end)
1  if begin < end
2       q = Partition(A, begin, end)
3       QuickSort(A, begin, q − 1)
4       QuickSort(A, q + 1, end)
```

# Complexity of **Partition**

```
Partition(A, begin, end)
1  q = begin
2  v = A[end]
3  for i = begin to end
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# Complexity of Partition

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Partition(A, begin, end)
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2  v = A[end]
3  for i = begin to end
4       if A[i] ≤ v
5            swap A[i] and A[q]
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7  return q − 1
```

$$T(n) = \Theta(n)$$

# Complexity of QuickSort

```
QuickSort(A, begin, end)
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2         q = Partition(A, begin, end)
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- Worst case

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- Worst case
  - $q = begin$ or $q = end$

# Complexity of QuickSort

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QuickSort(A, begin, end)
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- Worst case
  - $q = begin$ or $q = end$
  - the partition transforms $P$ of size $n$ in $P$ of size $n - 1$

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$$T(n) = T(n-1) + \Theta(n)$$

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$$T(n) = T(n-1) + \Theta(n)$$

$$T(n) = \Theta(n^2)$$

# Complexity of QuickSort (2)

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QuickSort(A, begin, end)
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# Complexity of QuickSort (2)

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QuickSort(A, begin, end)
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- Best case

# Complexity of QuickSort (2)

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QuickSort(A, begin, end)
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```

- Best case
  - $q = \lceil n/2 \rceil$

# Complexity of QuickSort (2)

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- Best case
  - $q = \lceil n/2 \rceil$
  - the partition transforms $P$ of size $n$ into *two* problems $P$ of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil - 1$, respectively

# Complexity of QuickSort (2)

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QuickSort(A, begin, end)
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$$T(n) = 2T(n/2) + \Theta(n)$$

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$$T(n) = \Theta(n \log n)$$

# Sorting Algorithms Seen So Far

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| Algorithm | Complexity | | | In place? |
|---|---|---|---|---|
| | *worst* | *average* | *best* | |
| **Insertion-Sort** | $\Theta(n^2)$ | $\Theta(n^2)$ | $\Theta(n)$ | yes |
| **Selection-Sort** | $\Theta(n^2)$ | $\Theta(n^2)$ | $\Theta(n^2)$ | yes |
| **Bubble-Sort** | $\Theta(n^2)$ | $\Theta(n^2)$ | $\Theta(n^2)$ | yes |
| **Merge-Sort** | $\Theta(n \log n)$ | $\Theta(n \log n)$ | $\Theta(n \log n)$ | no |

# Sorting Algorithms Seen So Far

| Algorithm | Complexity | | | In place? |
|---|---|---|---|---|
| | *worst* | *average* | *best* | |
| **Insertion-Sort** | $\Theta(n^2)$ | $\Theta(n^2)$ | $\Theta(n)$ | yes |
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| **Merge-Sort** | $\Theta(n \log n)$ | $\Theta(n \log n)$ | $\Theta(n \log n)$ | no |
| **QuickSort** | | | | |

# Sorting Algorithms Seen So Far

| Algorithm | Complexity | | | In place? |
|---|---|---|---|---|
| | *worst* | *average* | *best* | |
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| **Bubble-Sort** | $\Theta(n^2)$ | $\Theta(n^2)$ | $\Theta(n^2)$ | yes |
| **Merge-Sort** | $\Theta(n \log n)$ | $\Theta(n \log n)$ | $\Theta(n \log n)$ | no |
| **QuickSort** | $\Theta(n^2)$ | $\Theta(n \log n)$ | $\Theta(n \log n)$ | yes |

# Sorting Algorithms Seen So Far

| Algorithm | Complexity | | | In place? |
|---|---|---|---|---|
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| ***??*** | $\Theta(n \log n)$ | | | ***yes*** |

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- Our first real ***data structure***

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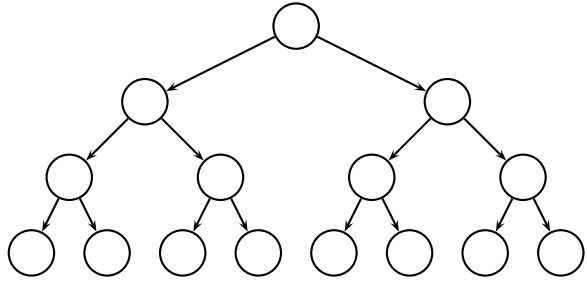

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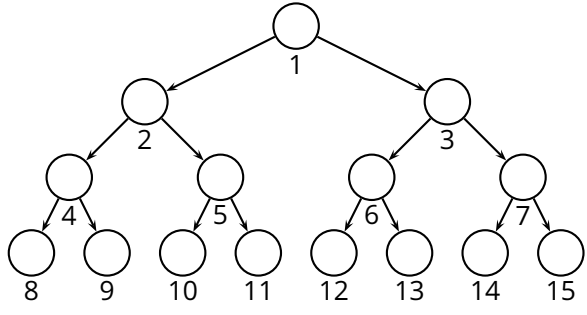

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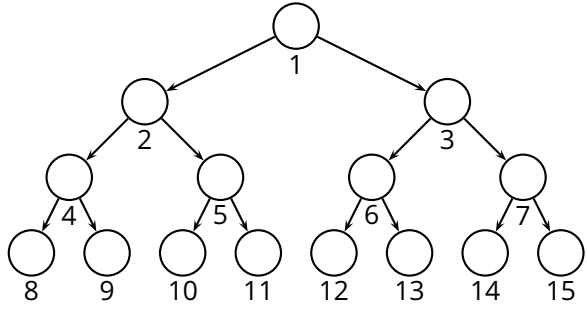


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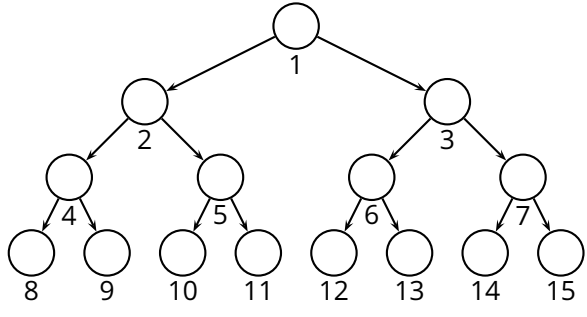


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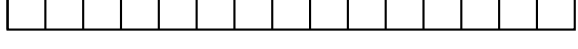

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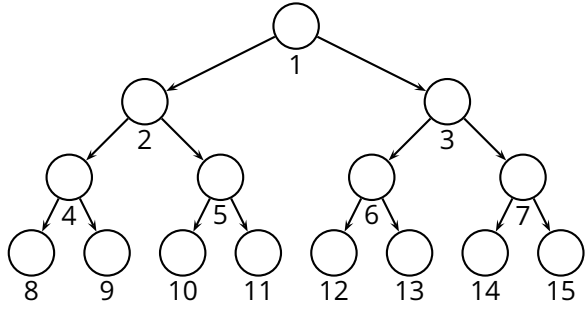


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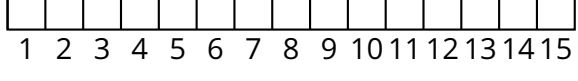

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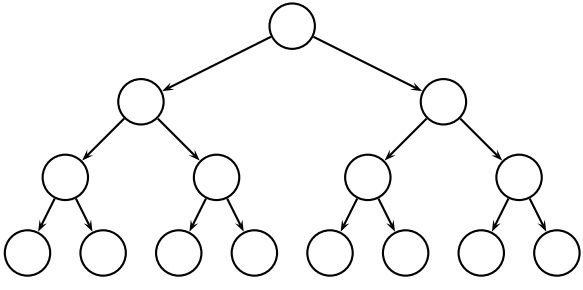

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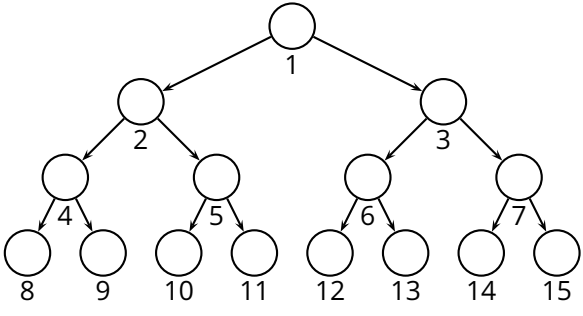

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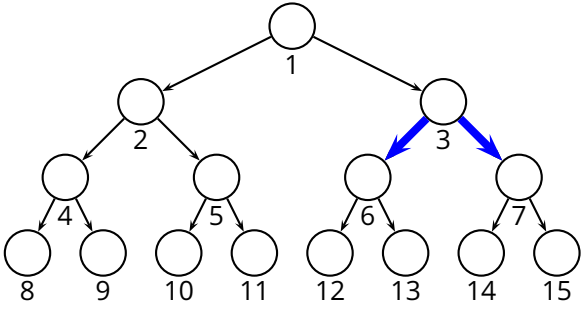

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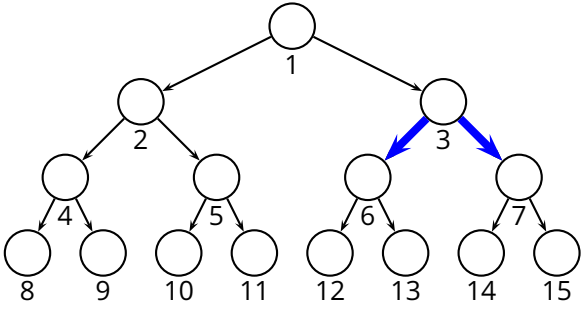


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    **return** $\lfloor i/2 \rfloor$

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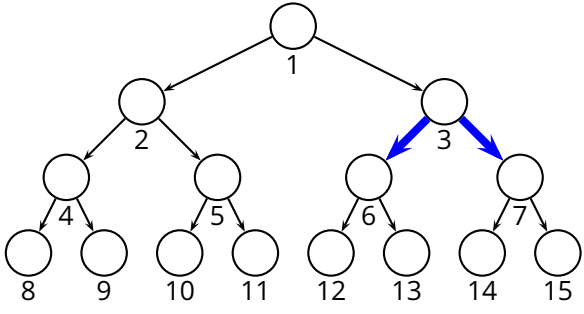


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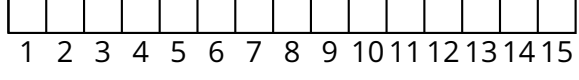

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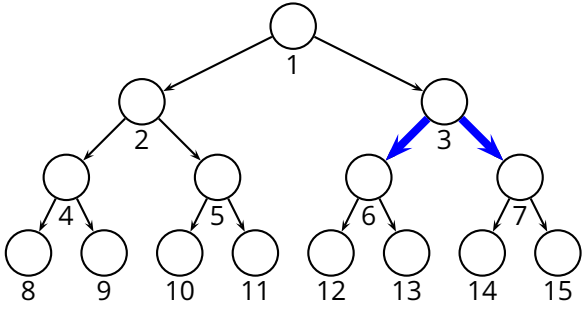


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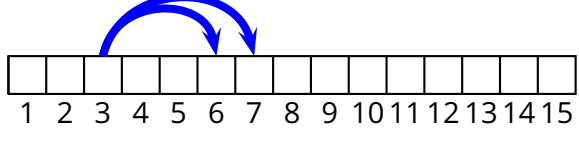

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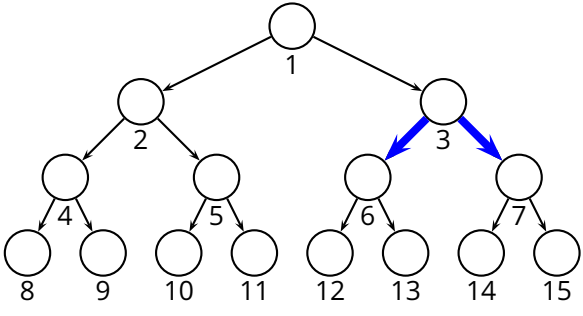


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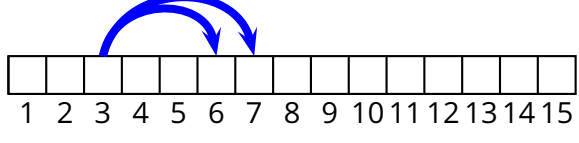


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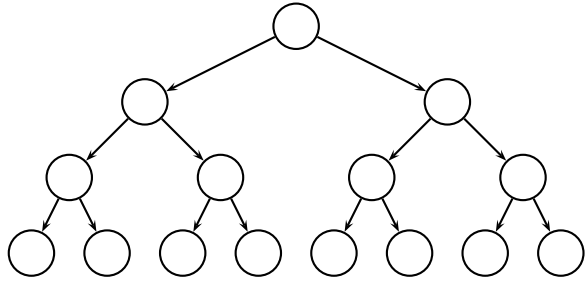

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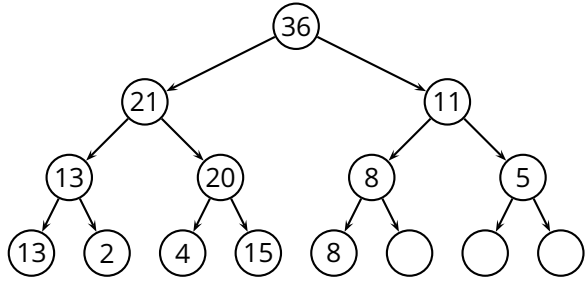

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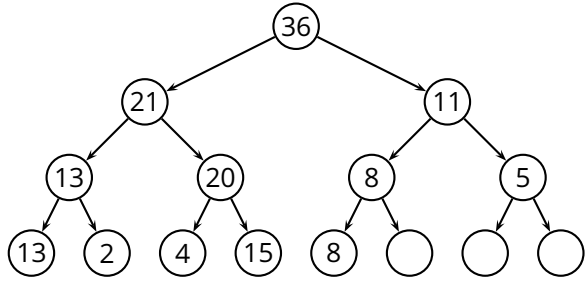


- Where is the max element?

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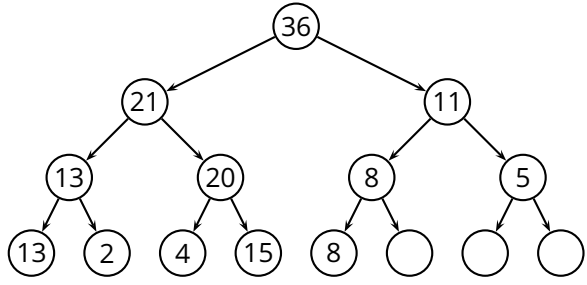


- Where is the max element?
- How can we implement **Heap-Extract-Max**?

# Heap-Extract-Max

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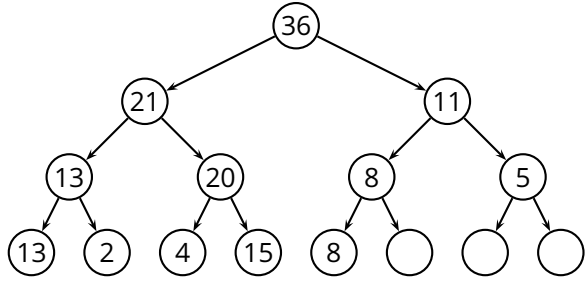

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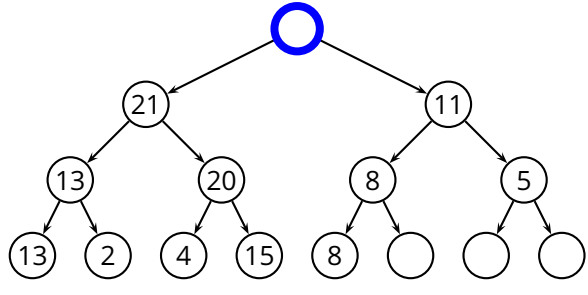

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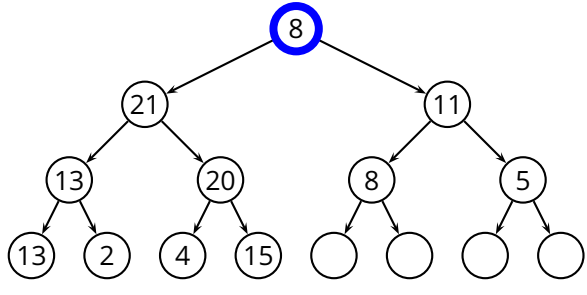

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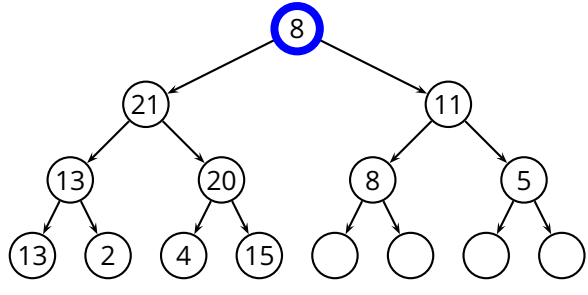


- Now we have two subtrees where the *max-heap property* holds

- **Max-Heapify**$(A, i)$ procedure
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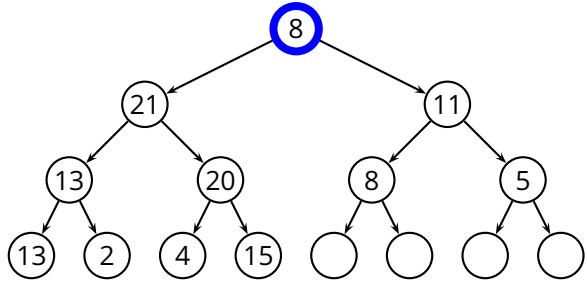

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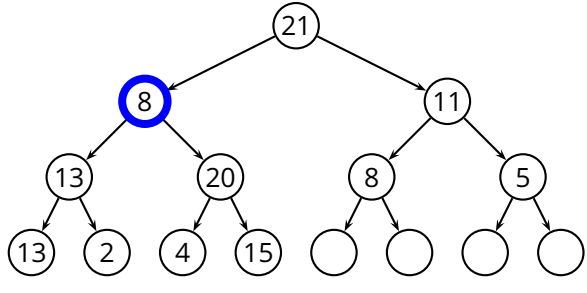

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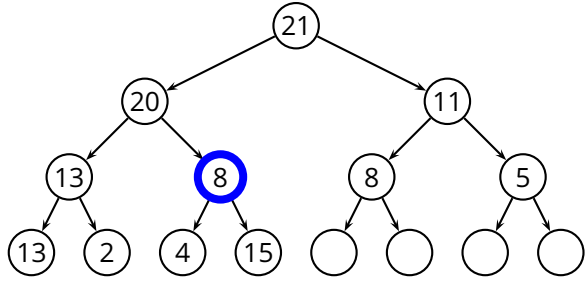

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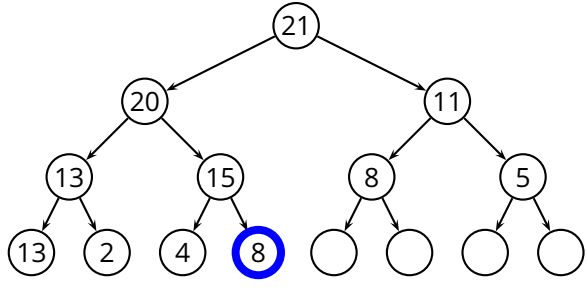

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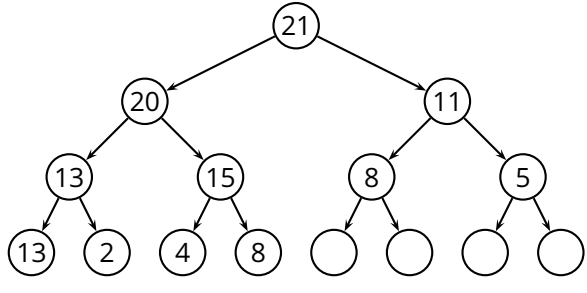

# Max-Heapify

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Max-Heapify(A, i)
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 3  if l ≤ A.heap-size and A[l] > A[i]
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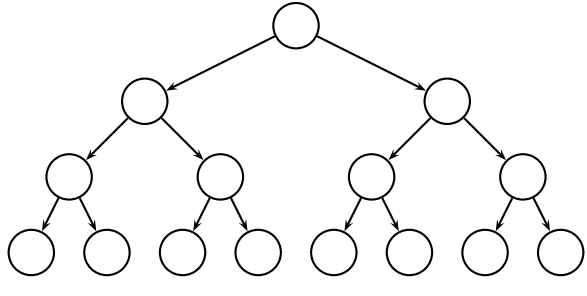

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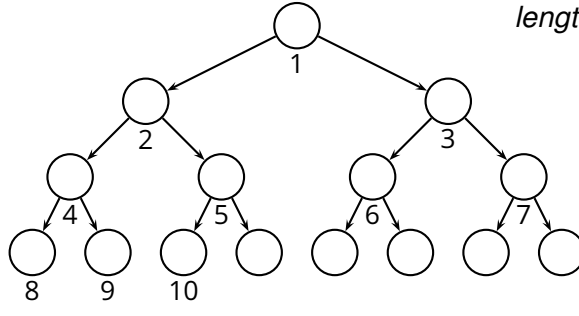

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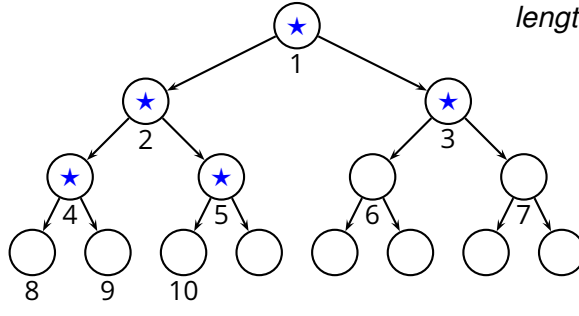


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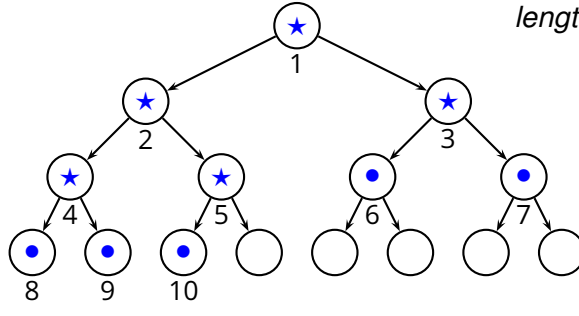


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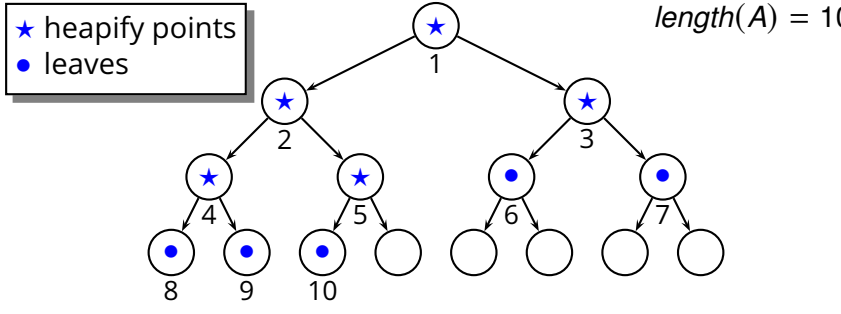

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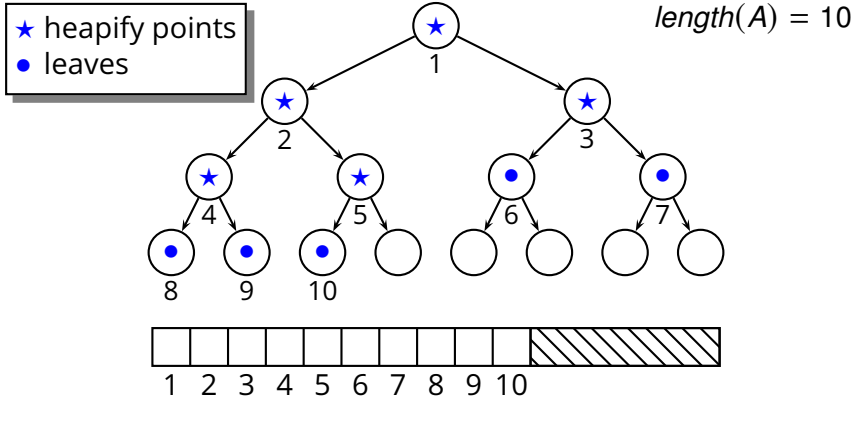

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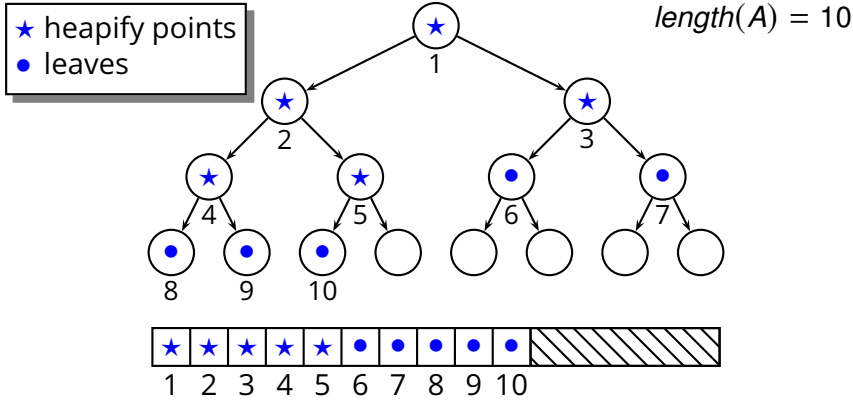

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- Benefits
  - in-place sorting; worst-case is $\Theta(n \log n)$

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